Mechanism Design

Kate Larson

University of Waterloo

January 16, 2013

Outline

Introduction

Game Theory

- Given a game we are able to analyse the strategies agents will follow

Social Choice

- Given a set of agents' preferences we can choose some outcome

Introduction

Today **Mechanism Design**

- Game Theory + Social Choice
- Goal of Mechanism Design is to
  - Obtain some outcome (function of agents' preferences)
  - But agents are rational
    - They may lie about their preferences

Goal

Define the rules of a game so that in equilibrium the agents do what we want.

Fundamentals

- Set of possible outcomes O
- Set of agents N, $|N| = n$
  - Each agent i has type $\theta_i \in \Theta_i$
  - Type captures all private information that is relevent to the agent's decision making
- Utility $u_i(o, \theta_i)$ over outcome $o \in O$
- Recall: goal is to implement some system wide solution
  - Captured by a social choice function

$$f : \Theta_1 \times \ldots \times \Theta_n \rightarrow O$$

where $f(\theta_1, \ldots, \theta_n) = o$ is a collective choice

Examples of Social Choice Functions

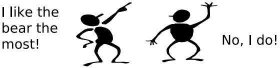
- **Voting:**
  - Choose a candidate among a group
- **Public project:**
  - Decide whether to build a swimming pool whose cost must be funded by the agents themselves
- **Allocation:**
  - Allocate a single, indivisible item to one agent in a group

Mechanisms

Recall that we want to implement a social choice function

- Need to know agents' preferences
- They may not reveal them to us truthfully

Example:



Mechanism Design Problem

- By having agents interact through an institution we might be able to solve the problem
- Mechanism:

$$M = (S_1, \ldots, S_n, g(\cdot))$$

where

  - S_i is the strategy space of agent i
  - $g : S_1 \times \ldots \times S_n \rightarrow O$ is the outcome function

Implementation

Definition

A mechanism $M = (S_1, \ldots, S_n, g(\cdot))$ **implements** *social choice function* $f(\Theta)$ *if there is an equilibrium strategy profile*

$$s^* = (s_1^*(\theta_1, \ldots, s_n^*(\theta_n))$$

of the game induced by M such that

$$g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = f(\theta_1, \ldots, \theta_n)$$

for all

$$(\theta_1, \ldots, \theta_n) \in \Theta_1 \times \ldots \times \Theta_n$$

Implementation

We did not specify the type of equilibrium in the definition

- Nash

$$u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i) \geq u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*$$

- Bayes-Nash

$$E[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)] \geq E[u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)]$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*$$

- Dominant

$$u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i) \geq u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*, \forall s_{-i}$$

Implementation

We did not specify the type of equilibrium in the definition

- Nash

$$u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i) \geq u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*$$

- Bayes-Nash

$$E[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)] \geq E[u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)]$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*$$

- Dominant

$$u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i) \geq u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*, \forall s_{-i}$$

Implementation

We did not specify the type of equilibrium in the definition

- Nash

$$u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i) \geq u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*$$

- Bayes-Nash

$$E[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)] \geq E[u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)]$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*$$

- Dominant

$$u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i) \geq u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)$$

$$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*, \forall s_{-i}$$

Properties for Mechanisms

- Efficiency
  - Select the outcome that maximizes total utility
- Fairness
  - Select outcome that minimizes the variance in utility
- Revenue maximization
  - Select outcome that maximizes revenue to a seller (or, utility to one of the agents)
- Budget-balanced
  - Implement outcomes that have balanced transfers across agents
- Pareto Optimal
  - Only implement outcomes o^* for which for all $o' \neq o^*$ either $u_i(o', \theta_i) = u_i(o^*, \theta_i) \forall i$ or $\exists i \in N$ with $u_i(o', \theta_i) < u_i(o^*, \theta_i)$

Participation Constraints

We can not force agents to participate in the mechanism. Let $\hat{u}_i(\theta_i)$ denote the (expected) utility to agent i with type θ_i of its outside option.

- **ex ante individual-rationality**: agents choose to participate before they know their own type

$$E_{\theta \in \Theta}[u_i(f(\theta), \theta_i)] \geq E_{\theta_i \in \Theta_i} \hat{u}_i(\theta_i)$$

- **interim individual-rationality**: agents can withdraw once they know their own type

$$E_{\theta_{-i} \in \Theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i)] \geq \hat{u}_i(\theta_i)$$

- **ex-post individual-rationality**: agents can withdraw from the mechanism at the end

$$u_i(f(\theta), \theta_i) \geq \hat{u}_i(\theta_i)$$

Participation Constraints

We can not force agents to participate in the mechanism. Let $\hat{u}_i(\theta_i)$ denote the (expected) utility to agent i with type θ_i of its outside option.

- **ex ante individual-rationality**: agents choose to participate before they know their own type

$$E_{\theta\in\Theta}[u_i(f(\theta),\theta_i)] \geq E_{\theta_i\in\Theta_i}\hat{u}_i(\theta_i)$$

- **interim individual-rationality**: agents can withdraw once they know their own type

$$E_{\theta_{-i}\in\Theta_{-i}}[u_i(f(\theta_i,\theta_{-i}),\theta_i)] \geq \hat{u}_i(\theta_i)$$

- **ex-post individual-rationality**: agents can withdraw from the mechanism at the end

$$u_i(f(\theta),\theta_i) \geq \hat{u}_i(\theta_i)$$

Participation Constraints

We can not force agents to participate in the mechanism. Let $\hat{u}_i(\theta_i)$ denote the (expected) utility to agent i with type θ_i of its outside option.

- **ex ante individual-rationality**: agents choose to participate before they know their own type

$$E_{\theta \in \Theta}[u_i(f(\theta), \theta_i)] \geq E_{\theta_i \in \Theta_i} \hat{u}_i(\theta_i)$$

- **interim individual-rationality**: agents can withdraw once they know their own type

$$E_{\theta_{-i} \in \Theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i)] \geq \hat{u}_i(\theta_i)$$

- **ex-post individual-rationality**: agents can withdraw from the mechanism at the end

$$u_i(f(\theta), \theta_i) \geq \hat{u}_i(\theta_i)$$

Participation Constraints

We can not force agents to participate in the mechanism. Let $\hat{u}_i(\theta_i)$ denote the (expected) utility to agent i with type θ_i of its outside option.

- **ex ante individual-rationality**: agents choose to participate before they know their own type

$$E_{\theta\in\Theta}[u_i(f(\theta),\theta_i)] \geq E_{\theta_i\in\Theta_i}\hat{u}_i(\theta_i)$$

- **interim individual-rationality**: agents can withdraw once they know their own type

$$E_{\theta_{-i}\in\Theta_{-i}}[u_i(f(\theta_i,\theta_{-i}),\theta_i)] \geq \hat{u}_i(\theta_i)$$

- **ex-post individual-rationality**: agents can withdraw from the mechanism at the end

$$u_i(f(\theta),\theta_i) \geq \hat{u}_i(\theta_i)$$

Direct Mechanisms

Definition

A **direct mechanism** *is a mechanism where*

$$S_i = \Theta_i \text{ for all } i$$

and

$$g(\theta) = f(\theta) \text{ for all } \theta \in \Theta_1 \times \ldots \times \Theta_n$$

Incentive Compatibility

Definition

A direct mechanism is **incentive compatible** *if it has an equilibrium s^* where*

$$s_i^*(\theta_i) = \theta_i$$

for all $\theta_i \in \Theta_i$ and for all i. That is, truth-telling by all agents is an equilibrium.

Definition

A direct mechanism is **strategy-proof** *if it is incentive compatible and the equilibrium is a dominant strategy equilibrium.*

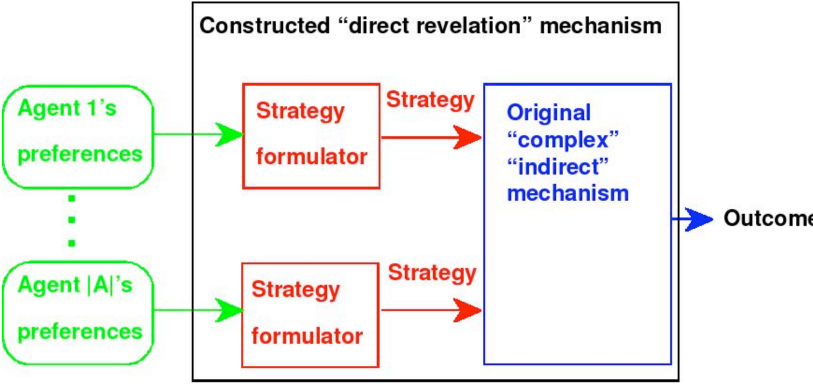
Revelation Principle

Theorem

Suppose there exists a mechanism $M = (S_1, \ldots, S_n, g(\cdot))$ that implements social choice function f in dominant strategies. Then there is a direct strategy-proof mechanism M' which also implements f.
[Gibbard 73; Green & Laffont 77; Myerson 79]

"The computations that go on within the mind of any bidder in the nondirect mechanism are shifted to become part of the mechanism in the direct mechanism."
[McAfee & McMillan 87]

Revelation Principle: Intuition



Theoretical Implications

- **Literal interpretation:** Need only study direct mechanisms
  - A modeler can limit the search for an optimal mechanism to the class of direct IC mechanisms
  - If no direct mechanism can implement social choice function f then no mechanism can
  - Useful because the space of possible mechanisms is huge

Practical Implications

- Incentive-compatibility is “free”
  - Any outcome implemented by mechanism M can be implemented by incentive-compatible mechanism M'
- “Fancy” mechanisms are unneccessary
  - Any outcome implemented by a mechanism with complex strategy space S can be implemented by a direct mechanism

BUT Lots of mechanisms used in practice are not direct and incentive-compatible!

Practical Implications

- Incentive-compatibility is “free”
  - Any outcome implemented by mechanism M can be implemented by incentive-compatible mechanism M'
- “Fancy” mechanisms are unneccessary
  - Any outcome implemented by a mechanism with complex strategy space S can be implemented by a direct mechanism

BUT Lots of mechanisms used in practice are not direct and incentive-compatible!

Quick Review

We now know

- What a mechanism is
- What it means for a SCF to be dominant-strategy implementable
- Revelation Principle

We do not yet know

- What types of SCF are dominant-strategy implementable

Gibbard-Satterthwaite Impossibility

Theorem

Assume that

- O *is finite and* $|O| \geq 3$,
- *each* $o \in O$ *can be achieved by SCF* f *for some* θ, *and*
- Θ *includes all possible strict orderings over* O.

Then f *is implementable in dominant strategies (strategy-proof) if and only if it is dictatorial.*

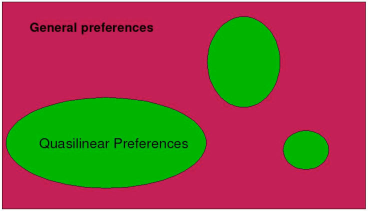
Definition

SCF f *is* **dictatorial** *if there is an agent* i *such that for all* θ

$$f(\theta) \in \{o \in O | u_i(o, \theta_i) \geq u_i(o', \theta_i) \forall o' \in O\}$$

Circumventing Gibbard-Satterthwaite

- Use a weaker equilibrium concept
- Design mechanisms where computing a beneficial manipulation is hard
- Randomization
- Restrict the structure of agents' preferences



1

Single-Peaked Preferences

- Define $A = [0, 1]$ be the outcome space
- Each agent $i \in N$ has a preference $\succeq_i$ over A such that $\exists p_i \in A$ such that for all $\{x\} \in A \setminus \{p_i\}$ and for all $\lambda \in [0, 1)$, $(\lambda x + (1 - \lambda)p_i) \succeq_i x$.
  - political decisions
  - facility location
  - temperature settings
- The Median-Voter rule is strategy-proof.

Single-Peaked Preferences

- Define $A = [0, 1]$ be the outcome space
- Each agent $i \in N$ has a preference $\succeq_i$ over A such that $\exists p_i \in A$ such that for all $\{x\} \in A \setminus \{p_i\}$ and for all $\lambda \in [0, 1)$, $(\lambda x + (1 - \lambda)p_i) \succeq_i x$.
  - political decisions
  - facility location
  - temperature settings
- The Median-Voter rule is strategy-proof.

Quasi-linear preferences

- Outcome $o = (x, t_1, \ldots, t_n)$
  - x is a "project choice"
  - $t_i \in \mathbb{R}$ are transfers (money)
- Utility function of agent i

$$u_i(o, \theta_i) = v_i(x, \theta_i) - t_i$$

- Quasi-linear mechanism

$$M = (S_1, \ldots, S_n, g(\cdot))$$

where

$$g(\cdot) = (x(\cdot), t_1(\cdot), \ldots, t_n(\cdot))$$

Social Choice Functions and Quasi-linearity

- SCF is **efficient** if for all θ

$$\sum_{i=1}^{n} v_i(x(\theta), \theta_i) \geq \sum_{i=1}^{n} v_i(x'(\theta), \theta_i) \forall x'(\theta)$$

  This is also known as **social welfare maximizing**

- SCF is **budget-balanced** if

$$\sum_{i=1}^{n} t_i(\theta) = 0$$

  Weakly budget-balanced if

$$\sum_{i=1}^{n} t_i(\theta) \geq 0$$

Groves Mechanisms [Groves 73]

A **Groves mechanism** $M = (S_1, \ldots, S_n, (x, t_1, \ldots, t_n))$ is defined by

- Choice rule

$$x^*(\theta) = \arg\max_x \sum_i v_i(x, \theta_i)$$

- Transfer rules

$$t_i(\theta) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(x^*(\theta), \theta_j)$$

where $h_i(\cdot)$ is an (arbitrary) function that does not depend on the reported type θ_i' of agent i.

Groves Mechanisms

Theorem

Groves mechanisms are strategy-proof and efficient.

We have gotten around Gibbard-Satterthwaite.

Proof

Agent i's utility for strategy $\hat{\theta}_i$, given $\hat{\theta}_{-i}$ from agents $j \neq i$ is

$$\begin{aligned} u_i(\hat{\theta}_i) &= v_i(x^*(\hat{\theta}, \theta_i) - t_i(\hat{\theta}) \\ &= v_i(x^*(\hat{\theta}, \theta_i) + \sum_{j \neq i} v_j(x^*(\hat{\theta}, \hat{\theta}_j) - h_i(\hat{\theta}_{-i}) \end{aligned}$$

Ignore $h_i(\hat{\theta}_{-i})$ and notice $x^*(\hat{\theta}) = \arg\max_x \sum_i v_i(x, \hat{\theta}_i)$ i.e it maximizes the sum of reported values. Therefore, agent i should announce $\hat{\theta}_i = \theta_i$ to maximize its own payoff.

Thm: Groves mechanisms are unique (up to $h_i(\theta_{-i})$).

Vickrey-Clarke-Groves Mechanism
aka Clarke mechansism, aka Pivotal mechanism

- Implement efficient outcome

$$x^* = \arg\max_x \sum_i v_i(x, \theta_i)$$

- Compute transfers

$$t_i(\theta) = \sum_{j \neq i} v_j(x^{-i}, \theta_j) - \sum_{j \neq i} v_j(x^*, \theta_j)$$

where $x^{-i} = \arg\max_x \sum_{j \neq i} v_j(x, \theta_j)$

VCG are efficient and strategy-proof.

VCG Mechanism

Agent's equilibrium utility is

$$\begin{aligned} u_i((x^*, t), \theta_i) &= v_i(x^*, \theta_i) - \left[\sum_{j \neq i} v_j(x^{-i}, \theta_j) - \sum_{j \neq i} v_j(x^*, \theta_j)\right] \\ &= \sum_{j=1}^{n} v_j(x^*, \theta_j) - \sum_{j \neq i} v_j(x^{-i}, \theta_j) \\ &= \text{marginal contribution to the welfare of the sys} \end{aligned}$$

Examples