# Basic Game Theory

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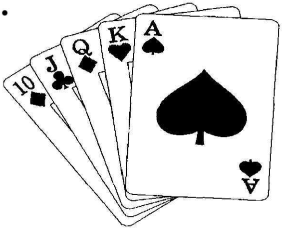
# Outline

# What is game theory?



The study of games!

- Bluffing in poker
- What move to make in chess
- How to play Rock-Scissors-Paper
- 


Also study of auction design, strategic deterrence, election laws, coaching decisions, routing protocols,...

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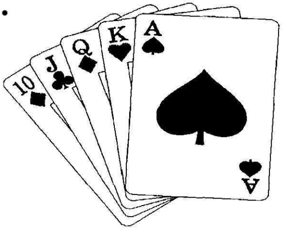


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Game theory is a formal way to analyze **interactions** among a **group** of **rational** agents who behave **strategically**.

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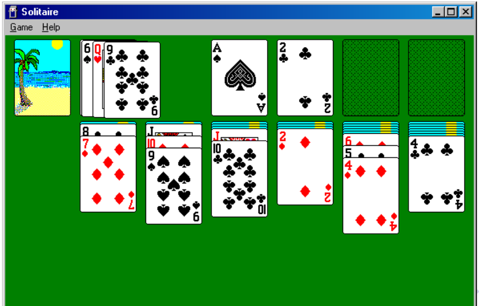
Game theory is a formal way to analyze **interactions** among a **group** of **rational** agents who behave **strategically**.

**Group:** Must have more than one decision maker

- Otherwise you have a decision problem, not a game




**Solitaire is not a game.**

# What is game theory?



Game theory is a formal way to analyze **interactions** among a **group** of **rational** agents who behave **strategically**.

**Interaction:** What one agent does directly affects at least one other agent

**Strategic:** Agents take into account that their actions influence the game

**Rational:** An agent chooses its best action (maximizes its expected utility)

# Example

Pretend that the entire class is going to go for lunch:

1. Everyone pays their own bill
2. Before ordering, everyone agrees to split the bill equally

Which situation is a game?

# Normal Form



A normal form game is defined by

- Finite set of agents (or players) $N$, $|N| = n$
- Each agent $i$ has an action space $A_i$
  - $A_i$ is non-empty and finite
- Outcomes are defined by action profiles ($a = (a_1, \ldots, a_n)$) where $a_i$ is the action taken by agent $i$
- Each agent has a utility function $u_i : A_1 \times \ldots \times A_n \mapsto \mathbb{R}$

# Examples

**Prisoners' Dilemma**

| | C | D |
|---|---|---|
| C | a,a | b,c |
| D | c,b | d,d |

$c > a > d > b$

**Pure coordination game**

$\forall$ action profiles $a \in A_1 \times \ldots \times A_n$ and $\forall i, j$, $u_i(a) = u_j(a)$.

| | L | R |
|---|---|---|
| L | 1,1 | 0,0 |
| R | 0,0 | 1,1 |

Agents do not have conflicting interests. There sole challenge is to coordinate on an action which is good for all.

# Zero-sum games



$\forall a \in A_1 \times A_2, u_1(a) + u_2(a) = 0$. That is, one player gains at the other player's expense.

**Matching Pennies**

| | H | T |
|---|---|---|
| H | 1,-1 | -1, 1 |
| T | -1,1 | 1,-1 |

| | H | T |
|---|---|---|
| H | 1 | -1 |
| T | -1 | 1 |

Given the utility of one agent, the other's utility is known.

# More Examples

Most games have elements of both cooperation and competition.

**BoS**

| | H | S |
|---|---|---|
| H | 2,1 | 0,0 |
| S | 0,0 | 1,2 |

**Hawk-Dove**

| | D | H |
|---|---|---|
| D | 3,3 | 1,4 |
| H | 4,1 | 0,0 |

# Strategies



**Notation:** Given set $X$, let $\Delta X$ be the set of all probability distributions over $X$.

**Definition**

*Given a normal form game, the set of mixed strategies for agent $i$ is*

$$S_i = \Delta A_i$$

The set of mixed strategy profiles is $S = S_1 \times \ldots \times S_n$.

**Definition**

*A strategy $s_i$ is a probability distribution over $A_i$. $s_i(a_i)$ is the probability action $a_i$ will be played by mixed strategy $s_i$.*

# Strategies



**Definition**

*The support of a mixed strategy $s_i$ is*

$$\{a_i | s_i(a_i) > 0\}$$

**Definition**

*A pure strategy $s_i$ is a strategy such that the support has size 1, i.e.*

$$|\{a_i | s_i(a_i) > 0\}| = 1$$

*A pure strategy plays a single action with probability 1.*

# Expected Utility



The expected utility of agent $i$ given strategy profile $s$ is

$$u_i(s) = \sum_{a \in A} u_i(a) \Pi_{j=1}^{n} s_j(a_j)$$

**Example**

| | C | D |
|---|---|---|
| C | -1,-1 | -4,0 |
| D | 0, -4 | -3,-3 |

Given strategy profile $s = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{10}, \frac{9}{10}))$

$$u_1 = -1(\frac{1}{2})(\frac{1}{10}) - 4(\frac{1}{2})(\frac{9}{10}) - 3(\frac{1}{2})(\frac{9}{10}) = -3.2$$

$$u_2 = -1(\frac{1}{2})(\frac{1}{10}) - 4(\frac{1}{2})(\frac{1}{10}) - 3(\frac{1}{2})(\frac{9}{10}) = -1.6$$

# Best-response

Given a game, what strategy should an agent choose?
We first consider only pure strategies.

## Definition

*Given $a_{-i}$, the best-response for agent $i$ is $a_i \in A_i$ such that*

$$u_i(a_i^*, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \in A_i$$

*Note that the best response may not be unique.*
*A best-response set is*

$$B_i(a_{-i}) = \{a_i \in A_i | u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \in A_i\}$$

# Nash Equilibrium



**Definition**

*A profile $a^*$ is a Nash equilibrium if $\forall i$, $a_i^*$ is a best response to $a_{-i}^*$. That is*

$$\forall i u_i(a_i^*, a_{-i}^*) \geq u_i(a_i', a_{-i}^*) \; \forall a_i' \in A_i$$

*Equivalently, $a^*$ is a Nash equilibrium if $\forall i$*

$$a_i^* \in B(a_{-i}^*)$$

# Examples

PD

| | C | D |
|---|---|---|
| C | -1,-1 | -4,0 |
| D | 0,-4 | -3,-3 |

BoS

| | H | T |
|---|---|---|
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Matching Pennies

| | H | T |
|---|---|---|
| H | 1,-1 | -1,1 |
| T | -1,1 | 1,-1 |

# Nash Equilibria



We need to extend the definition of a Nash equilibrium. Strategy profile $s^*$ is a Nash equilibrium is for all $i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \; \forall s_i' \in S_i$$

Similarly, a best-response set is

$$B(s_{-i}) = \{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \forall s_i' \in S_i\}$$

# Examples

# Characterization of Mixed Nash Equilibria



$s^*$ is a (mixed) Nash equilibrium if and only if

- the expected payoff, given $s^*_{-i}$, to every action to which $s^*_i$ assigns positive probability is the same, and
- the expected payoff, given $s^*_{-i}$ to every action to which $s^*_i$ assigns zero probability is at most the expected payoff to any action to which $s^*_i$ assigns positive probability.

# Existence

**Theorem (Nash, 1950)**

*Every finite normal form game has a Nash equilibrium.*

**Proof:** Beyond scope of course.

**Basic idea:** Define set $X$ to be all mixed strategy profiles. Show that it has nice properties (compact and convex).

Define $f : X \mapsto 2^X$ to be the best-response set function, i.e. given $s$, $f(s)$ is the set all strategy profiles $s' = (s'_1, \ldots, s'_n)$ such that $s'_i$ is $i$'s best response to $s'_{-i}$.

Show that $f$ satisfies required properties of a fixed point theorem (Kakutani's or Brouwer's).

Then, $f$ has a fixed point, i.e. there exists $s$ such that $f(s) = s$. This $s$ is mutual best-response – NE!

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# Interpretations of Nash Equilibria



- Consequence of rational inference
- Focal point
- Self-enforcing agreement
- Stable social convention
- ...

# Dominant and Dominated Strategies



For the time being, let us restrict ourselves to pure strategies.

**Definition**

*Strategy $s_i$ is a strictly dominant strategy if for all $s_i' \neq s_i$ and for all $s_{-i}$*

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

Prisoner's Dilemma

| | C | D |
|---|---|---|
| C | -1,-1 | -4,0 |
| D | 0, -4 | -3,-3 |

Dominant-strategy equilibria

# Dominated Strategies



**Definition**

*A strategy $s_i$ is strictly dominated if there exists another strategy $s_i'$ such that for all $s_{-i}$*

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

**Definition**

*A strategy $s_i$ is weakly dominated if there exists another strategy $s_i'$ such that for all $s_{-i}$*

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

*with strict inequality for some $s_{-i}$.*

# Example



| | L | R |
|---|---|---|
| U | 1,-1 | -1,1 |
| M | -1,1 | 1,-1 |
| D | -2,5 | -3,2 |

D is strictly dominated

| | L | R |
|---|---|---|
| U | 5,1 | 4,0 |
| M | 6,0 | 3,1 |
| D | 6,4 | 4,4 |

U and M are weakly dominated

# Iterated Deletion of Strictly Dominated Strategies



Algorithm

- Let $R_i$ be the removed set of strategies for agent $i$
- $R_i = \emptyset$
- Loop
  - Choose $i$ and $s_i$ such that $s_i \in A_i \setminus R_i$ and there exists $s_i'$ such that
  $$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})\ \forall s_{-i}$$
  - Add $s_i$ to $R_i$
  - Continue

# Example

| | R | C | L |
|---|---|---|---|
| U | 3,-3 | 7,-7 | 15, -15 |
| D | 9,-9 | 8,-8 | 10,-10 |

# Some Results

**Theorem**

*If a unique strategy profile $s^*$ survives iterated deletion then it is a Nash equilibrium.*

**Theorem**

*If $s^*$ is a Nash equilibrium then it survives iterated elimination.*

Weakly dominated strategies cause some problems.

# Domination and Mixed Strategies

The definitions of domination (both strict and weak) can be easily extended to mixed strategies in the obvious way.

**Theorem**

*Agent $i$'s pure strategy $s_i$ is strictly dominated if and only if there exists another (mixed) strategy $\sigma_i$ such that*

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$$

*for all $s_{-i}$.*

# Example

| | L | R |
|---|---|---|
| U | 10,1 | 0,4 |
| M | 4,2 | 4,3 |
| D | 0,5 | 10,2 |

Strategy $(\frac{1}{2}, 0, \frac{1}{2})$ strictly dominates pure strategy $M$.

**Theorem**

*If pure strategy $s_i$ is strictly dominated, then so is any (mixed) strategy that plays $s_i$ with positive probability.*

# Maxmin and Minmax Strategies



- A **maxmin strategy** of player $i$ is one that maximizes its worst case payoff in the situation where the other agent is playing to cause it the greatest harm

$$\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- A **minmax strategy** is the one that minimizes the maximum payoff the other player can get

$$\arg\min_{s_i} \max s_{-i} u_{-i}(s_i, s_{-i})$$

# Example



In 2-player games, maxmin value of one player is equal to the minmax value of the other player.

| | L | R |
|---|---|---|
| U | 2,3 | 5,4 |
| D | 0,1 | 1,2 |

Calculate maxmin and minmax values for each player (you can restrict to pure strategies).

# Zero-Sum Games



- The maxmin value of one player is equal to the minmax value of the other player
- For both players, the set of maxmin strategies coincides with the set of minmax strategies
- Any maxmin outcome is a Nash equilibrium. These are the only Nash equilibrium.

# Solving Zero-Sum Games



Let $U_i^*$ be unique expected utility for player $i$ in equilibrium. Recall that $U_1^* = -U_2^*$.

$$
\begin{array}{lll}
\text{minimize} & U_1^* & \\
\text{subject to} & \sum_{a_k \in A_2} u_1(a_j, a_k) s_2(a_k) \leq U_1^* & \forall a_j \in A_1 \\
& \sum_{a_k \in A_2} s_2(a_k) = 1 & \\
& s_2(a_k) \geq 0 & \forall a_k \in A_2
\end{array}
$$

LP for 2's mixed strategy in equilibrium.

# Solving Zero-Sum Games



Let $U_i^*$ be unique expected utility for player $i$ in equilibrium. Recall that $U_1^* = -U_2^*$.

$$
\begin{array}{lll}
\text{maximize} & U_1^* & \\
\text{subject to} & \sum_{a_j \in A_1} u_1(a_j, a_k) s_1(a_j) \geq U_1^* & \forall a_k \in A_2 \\
& \sum_{a_j \in A_1} s_1(a_j) = 1 & \\
& s_1(a_j) \geq 0 & \forall a_j \in A_1
\end{array}
$$

LP for 1's mixed strategy in equilibrium.

# Two-Player General-Sum Games

LP formulation does not work for general-sum games since agents' interests are no longer diametrically opposed.

**Linear Complementarity Problem** (LCP)
Find any solution that satisfies

$$
\begin{array}{lr}
\sum_{a_k \in A_2} u_1(a_j, a_k) s_2(a_k) + r_1(a_j) = U_1^* & \forall a_j \in A_1 \\
\sum_{a_j \in A_1} u_2(a_j, a_k) s_1(a_j) + r_2(a_k) = U_2^* & \forall a_k \in A_2 \\
\sum_{a_j \in A_1} s_1(a_j) = 1 \;\; \sum_{a_k \in A_2} s_2(a_k) = 1 & \\
s_1(a_j) \geq 0, s_2(a_k) \geq 0 & \forall a_j \in A_1, a_k \in A_2 \\
r_1(a_j) \geq 0, r_2(a_k) \geq 0 & \forall a_j \in A_1, a_k \in A_2 \\
r_1(a_j) s_1(a_j) = 0, r_2(a_k) s_2(a_k) = 0 & \forall a_j \in A_1, a_k \in A_2
\end{array}
$$

For $n \geq 3$-player games, formulate a non-linear complementarity problem.

# Complexity of Finding a NE

- Characterization is tricky since we do not have a decision problem (i.e. every game has at least one Nash Equilibrium)
- NE is in PPAD: Polynomial parity argument, directed version
  - Given an exponential-size directed graph, with every node having in-degree and out-degree at most one described by a polynomial-time computable function $f(v)$ that outputs the predecessor and successor of $v$, and a vertex $s$ with a successor but no predecessors, find a $t \neq s$ that either has no successors or predecessors.

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# Extensive Form Games

## aka Dynamic Games, aka Tree-Form Games

- Extensive form games allows us to model situations where agents take actions over time
- Simplest type is the perfect information game

# Perfect Information Game



**Perfect Information Game:** $G = (N, A, H, Z, \alpha, \rho, \sigma, u)$

- $N$ is the player set $|N| = n$
- $A = A_1 \times \ldots \times A_n$ is the action space
- $H$ is the set of non-terminal choice nodes
- $Z$ is the set of terminal nodes
- $\alpha : H \to 2^A$ action function, assigns to a choice node a set of possible actions
- $\rho : H \to N$ player function, assigns a player to each non-terminal node (player who gets to take an action)
- $\sigma : H \times A \to H \cup Z$, successor function that maps choice nodes and an action to a new choice node or terminal node where

  $$\forall h_1, h_2 \in H \text{ and } a_1, a_2 \in A \text{ if } h_1 \neq h_2 \text{ then } \sigma(h_1, a_1) \neq \sigma(h_2,$$

- $u = (u_1, \ldots, u_n)$ where $u_i : Z \to \mathbb{R}$ is utility function for player $i$ over $Z$

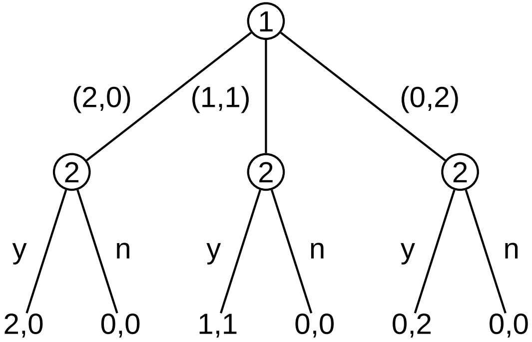
# Tree Representation



- The definition is really a tree description
- Each node is defined by its history (sequence of nodes leading from root to it)
- The descendents of a node are all choice and terminal nodes in the subtree rooted at the node.

# Example

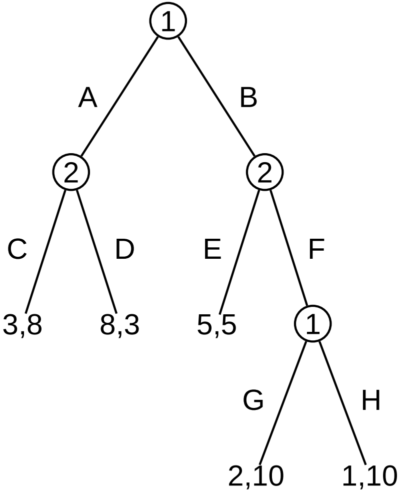
**Sharing two items**



# Strategies

- A strategy, $s_i$ of player $i$ is a function that assigns an action to each non-terminal history, at which the agent can move.
- Outcome: $o(s)$ of strategy profile $s$ is the terminal history that results when agents play $s$
- **Important:** The strategy definition requires a decision at each choice node, regardless of whether or not it is possible to reach that node given earlier moves

# Example



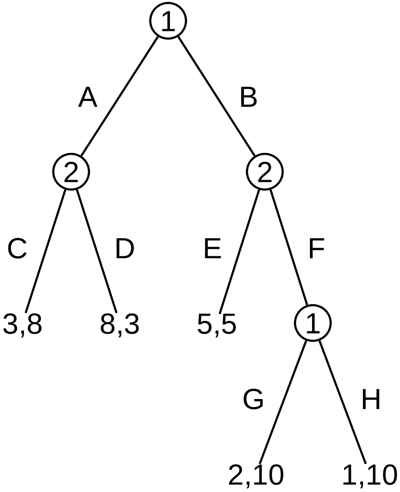

Strategy sets for the agents

$S_1 = \{(A, G), (A, H), (B, G), (B,$

$S_2 = \{(C, E), (C, F), (D, E), (D,$

# Example






Strategy sets for the agents

$$S_1 = \{(A,G),(A,H),(B,G),(B,$$

$$S_2 = \{(C,E),(C,F),(D,E),(D,$$

# Example



We can transform an extensive form game into a normal form game.

| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2, 10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

# Nash Equilibria



**Definition (Nash Equilibrium)**

*Strategy profile $s^*$ is a Nash Equilibrium in a perfect information, extensive form game if for all $i$*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \forall s_i'$$

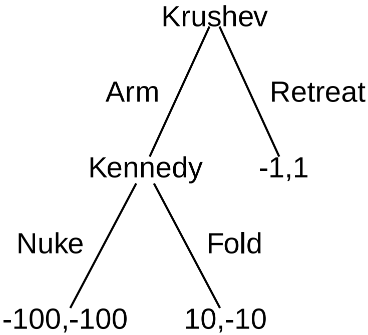
**Theorem**

*Any perfect information game in extensive form has a pure strategy Nash equilibrium.*

Intuition: Since players take turns, and everyone sees each move there is no reason to randomize.

# Example: Bay of Pigs






What are the NE?

# Subgame Perfect Equilibrium

Nash Equilibrium can sometimes be too weak a solution concept.

**Definition (Subgame)**

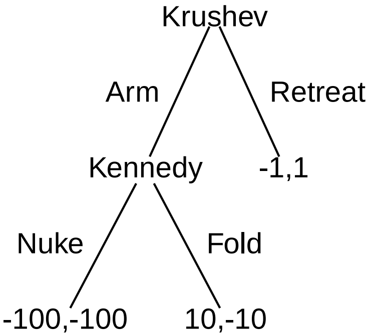
*Given a game $G$, the subgame of $G$ rooted at node $j$ is the restriction of $G$ to its descendents of $h$.*

**Definition (Subgame perfect equilibrium)**

*A strategy profile $s^*$ is a subgame perfect equilibrium if for all $i \in N$, and for all subgames of $G$, the restriction of $s^*$ to $G'$ ($G'$ is a subgame of $G$) is a Nash equilibrium in $G'$. That is*

$$\forall i, \forall G', u_i(s_i^*|_{G'}, s_{-i}^*|_{G'}) \geq u_i(s_i'|_{G'}, s_{-i}^*|_{G'}) \forall s_i'$$

# Example: Bay of Pigs




What are the SPE?

# Existence of SPE

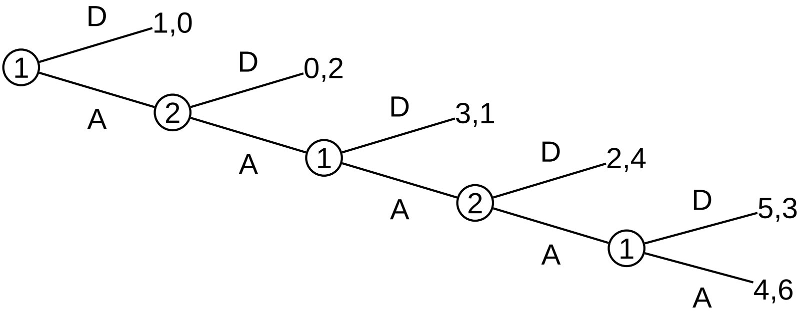
**Theorem (Kuhn's Thm)**

*Every finite extensive form game with perfect information has a SPE.*

You can find the SPE by backward induction.

- Identify equilibria in the bottom-most trees
- Work upwards

# Centipede Game



# Imperfect Information Games

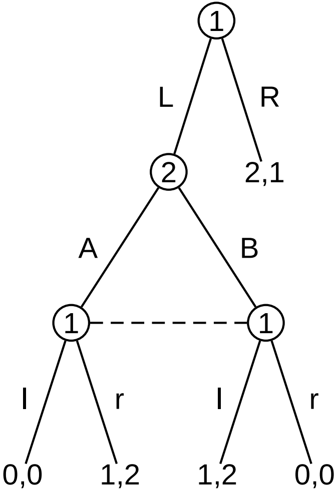


- Sometimes agents have not observed everything, or else can not remember what they have observed

**Imperfect information games**: Choice nodes $H$ are partitioned into *information sets*.

- If two choice nodes are in the same information set, then the agent can not distinguish between them.
- Actions available to an agent must be the same for all nodes in the same information set

# Example



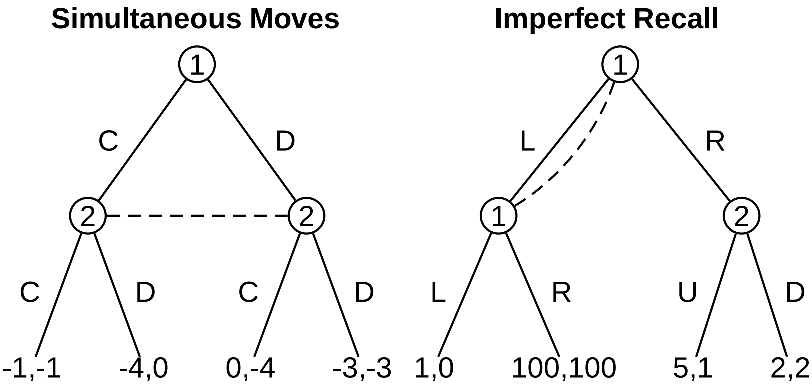

Information sets for agent 1

$$I_1 = \{\{\emptyset\}, \{(L, A), (L, B)\}\}$$

$$I_2 = \{\{L\}\}$$

# More Examples





# Strategies



- **Pure strategy:** a function that assigns an action in $A_i(I_i)$ to each information set $I_i \in \mathcal{I}_i$
- **Mixed strategy:** probability distribution over pure strategies
- **Behavorial strategy:** probability distribution over actions available to agent $i$ at each of its information sets (independent distributions)

# Behavioral Strategies

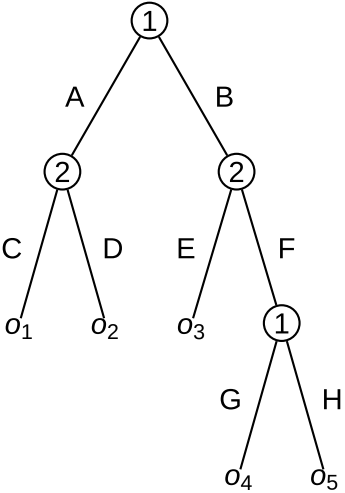


**Definition**

*Given extensive game G, a behavioral strategy for player i specifies, for every* $I_i \in \mathcal{I}_i$ *and action* $a_i \in A_i(I_i)$*, a probability* $\lambda_i(a_i, I_i) \geq 0$ *with*

$$\sum_{a_i \in A_i(I_i)} \lambda(a_i, I_i) = 1$$

# Example




**Mixed Strategy:**
(0.4(A,G), 0.6(B,H))

**Behavorial Strategy:**

- Play A with probability 0.5
- Play G with probability 0.3

# Mixed and Behavorial Strategies



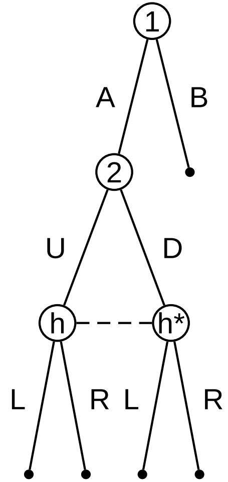
In general you can not compare the two types of strategies.

**But** for games with perfect recall

- Any mixed strategy can be replaced with a behavorial strategy
- Any behavorial strategy can be replaced with a mixed strategy

# Example






**Mixed Strategy:**
(<0.3(A,L)>,<0.2(A,R)>,
<0.5(B,L)>)

**Behavioral Strategy:**

- At $I_1$: (0.5, 0.5)
- At $I_2$: (0.6, 0.4)

# Bayesian Games



So far we have assumed that all players know what game they are playing

- Number of players
- Actions available to each player
- Payoffs associated with strategy profiles

| | L | R |
|---|---|---|
| U | 3,? | -2, ? |
| D | 0, ? | 6, ? |

**Bayesian games** (games of incomplete information) are used to represent uncertainties about the game being played

# Bayesian Games



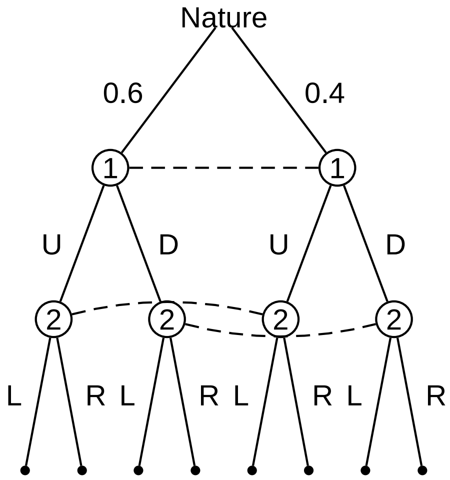
There are different possible representations.

**Information Sets**

- $N$ set of agents
- $G$ set of games
  - Same strategy sets for each game and agent
- $\Pi(G)$ is the set of all probability distributions over $G$
  - $P(G) \in \Pi(G)$ common prior
- $I = (I_1, \ldots, I_n)$ are information sets (partitions over games)

# Extensive Form With Chance Moves

A special player, Nature, makes probabilistic moves.



# Epistemic Types

Epistemic types captures uncertainty directly over a game's utility functions.

- $N$ set of agents
- $A = (A_1, \ldots, A_n)$ actions for each agent
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$ where $\Theta_i$ is *type space* of each agent
- $p : \Theta \to [0, 1]$ is common prior over types
- Each agent has utility function $u_i : A \times \Theta \to \mathbb{R}$

# Example



**BoS**

- 2 agents
- $A_1 = A_2 = \{\text{soccer, hockey}\}$
- $\Theta = (\Theta_1, \Theta_2)$ where $\Theta_1 = \{\text{H, S}\}$, $\Theta_2 = \{\text{H, S}\}$
- Prior: $p_1(H) = 1$, $p_2(H) = \frac{2}{3}$, $p_2(S) = \frac{1}{3}$

Utilities can be captured by matrix-form

$\theta_2 = H$

| | H | S |
|---|---|---|
| H | 2,2 | 0,0 |
| S | 0,0 | 1,1 |

$\theta_2 = S$

| | H | S |
|---|---|---|
| H | 2,1 | 0,0 |
| S | 0,0 | 1,2 |

# Strategies and Utility

- A strategy $s_i(\theta_i)$ is a mapping from $\Theta_i$ to $A_i$. It specifies what action (or what distribution of actions) to take for each type.

**Utility:** $u_i(s|\theta_i)$

- *ex-ante* EU (know nothing about types)

$$EU = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s_i|\theta_i)$$

- *interim* EU (know own type)

$$EU = EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \Pi_{j \in N} s_j(a_j, \theta_j)) u_i(a, \theta_-$$

- *ex-post* EU (know everyones type)

# Example



- 2 firms, 1 and 2, competing to create some product.
- If one makes the product then it has to share with the other.
- Product development cost is $c \in (0, 1)$
- Benefit of having the product is known only to each firm
  - Type $\theta_i$ drawn uniformly from $[0, 1]$
  - Benefit of having product is $\theta_i^2$

## Bayes Nash Equilibrium



**Definition (BNE)**

*Strategy profile $s^*$ is a Bayes Nash equilibrium if $\forall i$, $\forall \theta_i$*

$$EU(s_i^*, s_{-i}^* | \theta_i) \geq EU(s_i', s_{-i}^* | \theta_i) \forall s_i' \neq s_i^*$$

# Example Continued



- Let $s_i(\theta_i) = 1$ if $i$ develops product, and 0 otherwise.
- If $i$ develops product

$$u_i = \theta_i^2 - c$$

If it does not then

$$u_i = \theta_i^2 Pr(s_j(\theta_j) = 1)$$

- Thus, develop product if and only if

$$\theta_i^2 - c \geq \theta_i^2 Pr(s_j(\theta_j) = 1) \Rightarrow \theta_i \geq \sqrt{\frac{c}{1 - Pr(s_j(\theta_j) = 1)}}$$

# Example Continued



Suppose $\hat{\theta}_1, \hat{\theta}_2 \in (0,1)$ are cutoff values in BNE.

- If so, then $Pr(s_j(\theta_j) = 1) = 1 - \hat{\theta}_j$
- We must have

$$\hat{\theta}_i \geq \sqrt{\frac{c}{\hat{\theta}_j}} \Rightarrow \hat{\theta}_i^2 \hat{\theta}_j = c$$

and

$$\hat{\theta}_j^2 \hat{\theta}_i = c$$

- Therefore

$$\hat{\theta}_i^2 \hat{\theta}_j = \hat{\theta}_j^2 \hat{\theta}_i$$

and so

$$\hat{\theta}_i = \hat{\theta}_j = \theta^* = c^{\frac{1}{3}}$$