CS 798: Multiagent Systems Normal Form Games

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January 11, 2010

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Outline







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Normal Form

A normal form game is defined by

- Finite set of agents (or players) N, |N| = n
- Each agent *i* has an action space A_i
 - A_i is non-empty and finite
- Outcomes are defined by action profiles, *a* = (*a*₁,..., *a_n*), where *a_i* is the action taken by agent *i*
- Each agent has a utility function $u_i : A_1 \times \ldots \times A_n \mapsto \mathbb{R}$

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Examples

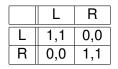
Prisoners' Dilemma

	С	D
С	a,a	b,c
D	c,b	d,d

c > a > d > b

Pure coordination game

 \forall action profiles $a \in A_1 \times \ldots \times A_n$ and $\forall i, j, u_i(a) = u_j(a)$.



Agents do not have conflicting interests. There sole challenge is to coordinate on an action which is good for all.

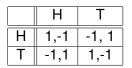
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Zero-sum games

 $\forall a \in A_1 \times A_2$, $u_1(a) + u_2(a) = 0$. That is, one player gains at the other player's expense.

Matching Pennies





Given the utility of one agent, the other's utility is known.

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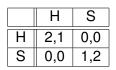
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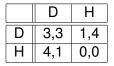
More Examples

Most games have elements of both cooperation and competition.

BoS

Hawk-Dove





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We have defined some games, but so far we have no way of discussing what a *good* outcome of a game is.

- Sometimes one outcome *o* is at least as good for every agent as another outcome *o*' and there is some agent who strictly prefers *o* to *o*'.
 - It seem reasonable to say that o is better than o'
 - We say that o Pareto dominates o'.
- An outcome *o** is *Pareto-optimal* if there is no other outcome that Pareto-dominates it.

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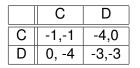
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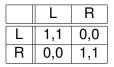
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Examples

Prisoners' Dilemma

Pure coordination game





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Examples

Matching Pennies

	Н	Т
Η	1, -1	-1, 1
Т	-1, 1	1, -1

BoS

	H	S
Η	2, 1	0,0
S	0,0	1,2

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Strategies

Notation: Given set *X*, let ΔX be the set of all probability distributions over *X*.

Definition

Given a normal form game, the set of mixed strategies for agent i is

$$S_i = \Delta A_i$$

The set of mixed strategy profiles is $S = S_1 \times \ldots \times S_n$.

Definition

A strategy s_i is a probability distribution over A_i . $s_i(a_i)$ is the probability action a_i will be played by mixed strategy s_i .

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Strategies

Definition

The support of a mixed strategy s_i is

 $\{a_i|s_i(a_i)>0\}$

Definition

A pure strategy s_i is a strategy such that the support has size 1, *i.e.*

$$|\{a_i|s_i(a_i)>0\}|=1$$

A pure strategy plays a single action with probability 1.

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Expected Utility

The expected utility of agent *i* given strategy profile *s* is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^n s_j(a_j)$$

Example

	С	D
С	-1,-1	-4,0
D	0, -4	-3,-3

Given strategy profile

$$s = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{10}, \frac{9}{10}))$$

$$u_1 = -1(\frac{1}{2})(\frac{1}{10}) - 4(\frac{1}{2})(\frac{9}{10}) - 3(\frac{1}{2})(\frac{9}{10}) = -3.2$$

$$u_2 = -1(\frac{1}{2})(\frac{1}{10}) - 4(\frac{1}{2})(\frac{1}{10}) - 3(\frac{1}{2})(\frac{9}{10}) = -1.6$$

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Best-response

Given a game, what strategy should an agent choose? We first consider only pure strategies.

Definition

Given a_{-i} , the best-response for agent i is $a_i \in A_i$ such that

$$u_i(a_i^*,a_{-i}) \geq u_i(a_i',a_{-i}) orall a_i' \in A_i$$

Note that the best response may not be unique. A best-response set is

$$B_i(a_{-i}) = \{a_i \in A_i | u_i(a_i, a_{-i}) \ge u_i(a'_i, a_{-i}) \forall a'_i \in A_i\}$$

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Nash Equilibrium

Definition

A profile a^* is a Nash equilibrium if $\forall i, a_i^*$ is a best response to a_{-i}^* . That is

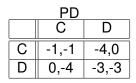
$$\forall iu_i(a_i^*, a_{-i}^*) \geq u_i(a_i', a_{-i}^*) \ \forall a_i' \in A_i$$

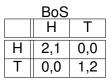
Equivalently, a^* is a Nash equilibrium if $\forall i$

$$a_i^* \in B(a_{-i}^*)$$

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Examples





Matching Pennies

	H	Т
Η	1,-1	-1,1
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Nash Equilibria

We need to extend the definition of a Nash equilibrium. Strategy profile s^* is a Nash equilibrium is for all *i*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \ \forall s_i' \in S_i$$

Similarly, a best-response set is

$$B(\boldsymbol{s}_{-i}) = \{\boldsymbol{s}_i \in \boldsymbol{S}_i | u_i(\boldsymbol{s}_i, \boldsymbol{s}_{-i}) \geq u_i(\boldsymbol{s}_i', \boldsymbol{s}_{-i}) \forall \boldsymbol{s}_i' \in \boldsymbol{S}_i\}$$

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Examples

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Characterization of Mixed Nash Equilibria

 s^* is a (mixed) Nash equilibrium if and only if

- the expected payoff, given s^{*}_{-i}, to every action to which s^{*}_i assigns positive probability is the same, and
- the expected payoff, given s^{*}_{-i} to every action to which s^{*}_i assigns zero probability is at most the expected payoff to any action to which s^{*}_i assigns positive probability.

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Theorem (Nash, 1950)

Every finite normal form game has a Nash equilibrium.

Proof: Beyond scope of course. **Basic idea:** Define set X to be all mixed strategy profiles. Define $f: X \mapsto 2^X$ to be the best-response set function, i.e.

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What does it mean to play a mixed strategy?

- Randomize to confuse your opponent
- Randomize when you are uncertain about the other's action
- Mixed strategies are a description of what might happen in repeated play
- Mixed strategies describe population dynamics

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Interpretations of Nash Equilibria

- Consequence of rational inference
- Focal point
- Self-enforcing agreement
- Stable social convention

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