# CS 798: Multiagent Systems

## Extensive Form Games

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# Outline

# Extensive Form Games

## aka Dynamic Games, aka Tree-Form Games

- Extensive form games allows us to model situations where agents take actions over time
- Simplest type is the perfect information game

# Perfect Information Game: $G = (N, A, H, Z, \alpha, \rho, \sigma, u)$

- $N$ is the player set $|N| = n$
- $A = A_1 \times \ldots \times A_n$ is the action space
- $H$ is the set of non-terminal choice nodes
- $Z$ is the set of terminal nodes
- $\alpha : H \to 2^A$ action function, assigns to a choice node a set of possible actions
- $\rho : H \to N$ player function, assigns a player to each non-terminal node (player who gets to take an action)
- $\sigma : H \times A \to H \cup Z$, successor function that maps choice nodes and an action to a new choice node or terminal node where

  $$\forall h_1, h_2 \in H \text{ and } a_1, a_2 \in A \text{ if } h_1 \neq h_2 \text{ then } \sigma(h_1, a_1) \neq \sigma(h_2, a_2)$$

- $u = (u_1, \ldots, u_n)$ where $u_i : Z \to \mathbb{R}$ is utility function for player $i$ over $Z$

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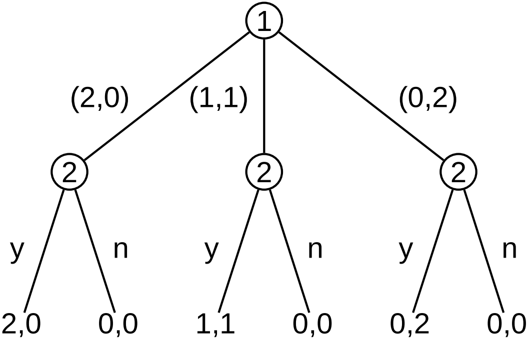
# Tree Representation

- The definition is really a tree description
- Each node is defined by its history (sequence of nodes leading from root to it)
- The descendents of a node are all choice and terminal nodes in the subtree rooted at the node.

# Example

**Sharing two items**



# Strategies

- A strategy, $s_i$ of player $i$ is a function that assigns an action to each non-terminal history, at which the agent can move.
- Outcome: $o(s)$ of strategy profile $s$ is the terminal history that results when agents play $s$
- **Important:** The strategy definition requires a decision at each choice node, regardless of whether or not it is possible to reach that node given earlier moves

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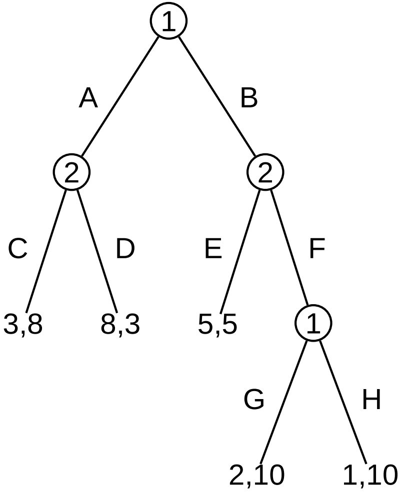


Strategy sets for the agents

$$S_1 = \{(A, G), (A, H), (B, G), (B, H)\}$$

$$S_2 = \{(C, E), (C, F), (D, E), (D, F)\}$$

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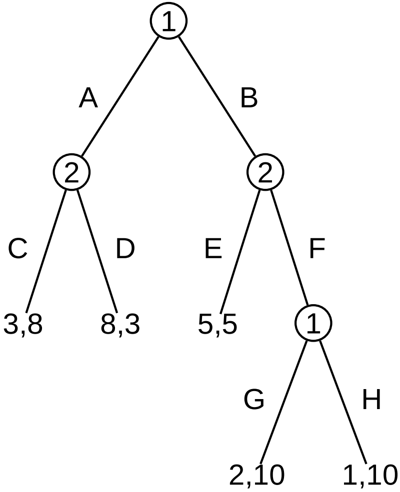


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# Example

We can transform an extensive form game into a normal form game.

| | (C,E) | (C,F) | (D,E) | (D,F) |
|---|---|---|---|---|
| (A,G) | 3,8 | 3,8 | 8,3 | 8,3 |
| (A,H) | 3,8 | 3,8 | 8,3 | 8,3 |
| (B,G) | 5,5 | 2,10 | 5,5 | 2, 10 |
| (B,H) | 5,5 | 1,0 | 5,5 | 1,0 |

# Nash Equilibria

**Definition (Nash Equilibrium)**

*Strategy profile $s^*$ is a Nash Equilibrium in a perfect information, extensive form game if for all $i$*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \forall s_i'$$

**Theorem**

*Any perfect information game in extensive form has a pure strategy Nash equilibrium.*

Intuition: Since players take turns, and everyone sees each move there is no reason to randomize.

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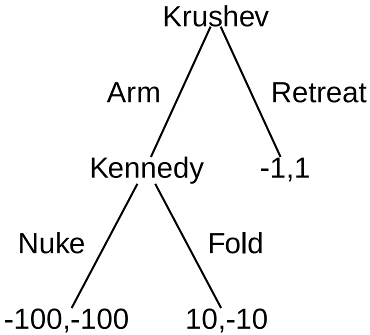
# Example: Bay of Pigs




What are the NE?

# Subgame Perfect Equilibrium

Nash Equilibrium can sometimes be too weak a solution concept.

**Definition (Subgame)**

*Given a game $G$, the subgame of $G$ rooted at node $j$ is the restriction of $G$ to its descendents of $h$.*

**Definition (Subgame perfect equilibrium)**

*A strategy profile $s^*$ is a subgame perfect equilibrium if for all $i \in N$, and for all subgames of $G$, the restriction of $s^*$ to $G'$ ($G'$ is a subgame of $G$) is a Nash equilibrium in $G'$. That is*

$$\forall i, \forall G', u_i(s_i^*|_{G'}, s_{-i}^*|_{G'}) \geq u_i(s_i'|_{G'}, s_{-i}^*|_{G'}) \forall s_i'$$

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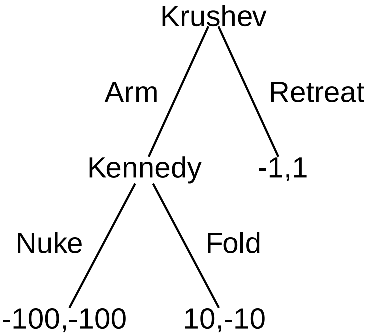


What are the SPE?

# Existence of SPE

**Theorem (Kuhn's Thm)**

*Every finite extensive form game with perfect information has a SPE.*

You can find the SPE by backward induction.

- Identify equilibria in the bottom-most trees
- Work upwards

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