CS 798: Multiagent Systems
Computing Equilibria

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Outline

1. Dominant and Dominated Strategies
2. Maxmin and Minmax Strategies
3. Solving Games
Dominant and Dominated Strategies

For the time being, let us restrict ourselves to pure strategies.

**Definition**

Strategy $s_i$ is a strictly dominant strategy if for all $s'_i \neq s_i$ and for all $s_{-i}$

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

Prisoner’s Dilemma

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<th>C</th>
<th>D</th>
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<tbody>
<tr>
<td>C</td>
<td>-1, -1</td>
<td>-4, 0</td>
</tr>
<tr>
<td>D</td>
<td>0, -4</td>
<td>-3, -3</td>
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</tbody>
</table>

Dominant-strategy equilibria
Dominated Strategies

**Definition**

A strategy $s_i$ is strictly dominated if there exists another strategy $s_i'$ such that for all $s_{-i}$

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

**Definition**

A strategy $s_i$ is weakly dominated if there exists another strategy $s_i'$ such that for all $s_{-i}$

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

with strict inequality for some $s_{-i}$. 
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Example

<table>
<thead>
<tr>
<th></th>
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<th>R</th>
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<tbody>
<tr>
<td>U</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>M</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>D</td>
<td>-2,5</td>
<td>-3,2</td>
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D is strictly dominated

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<tbody>
<tr>
<td>U</td>
<td>5,1</td>
<td>4,0</td>
</tr>
<tr>
<td>M</td>
<td>6,0</td>
<td>3,1</td>
</tr>
<tr>
<td>D</td>
<td>6,4</td>
<td>4,4</td>
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</table>

U and M are weakly dominated
Iterated Deletion of Strictly Dominated Strategies

Algorithm

- Let $R_i$ be the removed set of strategies for agent $i$
- $R_i = \emptyset$
- Loop
  - Choose $i$ and $s_i$ such that $s_i \in A_i \setminus R_i$ and there exists $s_i'$ such that
    $$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \forall s_{-i}$$
  - Add $s_i$ to $R_i$
  - Continue
Example

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<tr>
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<tbody>
<tr>
<td>U</td>
<td>3,-3</td>
<td>7,-7</td>
<td>15,-15</td>
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<tr>
<td>D</td>
<td>9,-9</td>
<td>8,-8</td>
<td>10,-10</td>
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Some Results

Theorem

*If a unique strategy profile $s^*$ survives iterated deletion then it is a Nash equilibrium.*

Theorem

*If $s^*$ is a Nash equilibrium then it survives iterated elimination.*

Weakly dominated strategies cause some problems.
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Weakly dominated strategies cause some problems.
The definitions of domination (both strict and weak) can be easily extended to mixed strategies in the obvious way.

**Theorem**

*Agent i’s pure strategy \( s_i \) is strictly dominated if and only if there exists another (mixed) strategy \( \sigma_i \) such that*

\[ u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \]

*for all \( s_{-i} \).*
Example

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<td>4,3</td>
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Strategy \((\frac{1}{2}, 0, \frac{1}{2})\) strictly dominates pure strategy \(M\).

Theorem

If pure strategy \(s_i\) is strictly dominated, then so is any (mixed) strategy that plays \(s_i\) with positive probability.
Example

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Strategy \( \left( \frac{1}{2}, 0, \frac{1}{2} \right) \) strictly dominates pure strategy \( M \).

Theorem

*If pure strategy \( s_i \) is strictly dominated, then so is any (mixed) strategy that plays \( s_i \) with positive probability.*
Maxmin and Minmax Strategies

- A **maxmin strategy** of player $i$ is one that maximizes its worst case payoff in the situation where the other agent is playing to cause it the greatest harm

  $$\arg \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- A **minmax strategy** is the one that minimizes the maximum payoff the other player can get

  $$\arg \min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$
Example

In 2-player games, maxmin value of one player is equal to the minmax value of the other player.

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Calculate maxmin and minmax values for each player (you can restrict to pure strategies).
Zero-Sum Games

- The maxmin value of one player is equal to the minmax value of the other player.
- For both players, the set of maxmin strategies coincides with the set of minmax strategies.
- Any maxmin outcome is a Nash equilibrium. These are the only Nash equilibrium.
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- Any maxmin outcome is a Nash equilibrium. These are the only Nash equilibrium.
Solving Zero-Sum Games

Let $U_i^*$ be unique expected utility for player $i$ in equilibrium. Recall that $U_1^* = -U_2^*$.

minimize $U_1^*$
subject to $\sum_{a_k \in A_2} u_1(a_j, a_k)s_2(a_k) \leq U_1^* \quad \forall a_j \in A_1$
$\sum_{a_k \in A_2} s_2(a_k) = 1$
$s_2(a_k) \geq 0 \quad \forall a_k \in A_2$

LP for 2’s mixed strategy in equilibrium.
Let $U^*_i$ be unique expected utility for player $i$ in equilibrium. Recall that $U^*_1 = -U^*_2$.

maximize \[ U^*_1 \]
subject to \[ \sum_{a_j \in A_1} u_1(a_j, a_k) s_1(a_j) \geq U^*_1 \quad \forall a_k \in A_2 \]
\[ \sum_{a_j \in A_1} s_1(a_j) = 1 \]
\[ s_1(a_j) \geq 0 \quad \forall a_j \in A_1 \]

LP for 1’s mixed strategy in equilibrium.
Two-Player General-Sum Games

LP formulation does not work for general-sum games since agents’ interests are no longer diametrically opposed.

Linear Complementarity Problem (LCP)
Find any solution that satisfies

\[
\begin{align*}
\sum_{a_k \in A_2} u_1(a_j, a_k) s_2(a_k) + r_1(a_j) &= U_1^* & \forall a_j \in A_1 \\
\sum_{a_j \in A_1} u_2(a_j, a_k) s_1(a_j) + r_2(a_k) &= U_2^* & \forall a_k \in A_2 \\
\sum_{a_j \in A_1} s_1(a_j) &= 1 & \sum_{a_k \in A_2} s_2(a_k) &= 1 \\
s_1(a_j) &\geq 0, s_2(a_k) \geq 0 & \forall a_j \in A_1, a_k \in A_2 \\
r_1(a_j) &\geq 0, r_2(a_k) \geq 0 & \forall a_j \in A_1, a_k \in A_2 \\
r_1(a_j) s_1(a_j) &= 0, r_2(a_k) s_2(a_k) &= 0 & \forall a_j \in A_1, a_k \in A_2
\end{align*}
\]