Cooperative Game Theory

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Outline

Introduction

Today we discuss *cooperative game theory* (also known as *coalitional game theory*.

- Basic modelling unit is the *group*
  - Compared to the *individual* in non-cooperative game theory
- Agents are still self-interested.
- We model preferences of the agents, but not their individual actions
  - Instead we look at group capabilities

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Coalitional Games with Transferable Utility

A *coalitional game with transferable utility* is a pair (N, v) where

- N is a (finite) set of agents
- $v : 2^N \rightarrow \mathbb{R}$ is the *characteristic function*.
  - For each $S \subseteq N$, $v(S)$ is the value that the agents can share amongst themselves.
  - $v(\emptyset) = 0$

Questions studied by cooperative game theory

- Which coalitions will form?
- How should the coalitions divide its value among its members?

Examples: Voting game

- 4 political parties A, B, C, and D which have 45, 25, 15, and 15 representatives respectively
- To pass a \$100 billion spending bill, at least 51 votes are needed
- If passed, then the parties get to decide how the money should be allocated. If not passed, then everyone gets 0.

Game

- $N = A \cup B \cup C \cup D$
- $v : 2^N \to \mathbb{R}$ where

$$v(S) = \begin{cases} \$100 \text{ Billion} & \text{if } |S| \geq 51 \\ 0 & \text{otherwise} \end{cases}$$

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Examples: Treasure Game

- N gold prospectors and more than $2|N|$ gold pieces
- Two prospectors are required to carry a gold piece

Game

- N agents
- $v(S) = \lfloor \frac{|S|}{2} \rfloor$

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Types of Games: Superadditive

Definition

A game $G = (N, v)$ is superadditive if for all $S, T \subset N$, if $S \cap T \emptyset$ then $v(S \cup T) \geq v(S) + v(T)$.

- Superadditivity makes sense if coalitions can always work without interfering with one another.
- Superadditive implies that the *grand coalition* has the highest value among all coalitions.

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- Quite common in practice.

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Analyzing TU Games

The central question when analysing TU games is how to divide the value of the coalition among the members. We focus on the grand coalition.

- Payoff vector $x = (x_1, \ldots, x_n)$ where $n = |N|$.
- Desire
  - Feasibility: $\sum_{i \in N} x_i \leq v(N)$
  - Efficiency: $\sum_{i \in N} x_i = v(N)$
  - Individual Rationality: $x_i \geq v(\{i\})$

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Solution Concepts

Given a payoff vector, x, we are interested in understanding whether it is a *good* payoff vector.

- Stable: Would agents want to leave and form other coalitions? (Core)
- Fair: Does the payoff vector represent what each agent brings to the coalition? (Shapley value)

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The Core

Definition

A payoff vector is in the core of game (N, v) *if and only if*

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S)$$

Examples: Treasure Game

Examples: Voting Game

Existence of the Core: General characterization

Definition

A set of non-negative weights, λ, is balanced if

$$\forall i \in N, \sum_{S|i \in S} \lambda(S) = 1.$$

Theorem

A game (N, v) has a non-empty core if and only if for all balanced sets of weights, λ

$$v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S).$$

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Existence of the Core: Specific Results

- Convex games have a non-empty core.
- In simple games the core is empty if and only if there are no veto agents.
  - An agent i is a veto agent if $v(N \setminus \{i\}) = 0$.
- If there are veto agents then the core consists of all x such that $x_j = 0$ if j is not a veto-agent.

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Fairness

- **Interchangeable agents:** i and j are interchangeable if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all S such that $i, j \notin S$
  - **Symmetry:** Interchangeable agents should receive the same payments, $x_i = x_j$
- **Dummy agent:** i is a dummy agent if the amount it contributes to a coalition is exactly the amount that it could have achieved alone: $\forall S, i \notin S, v(S \cup \{i\}) - v(S) = v(\{i\})$
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- **Additivity:**

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Shapley Value

There is a unique payoff vector that satisfies our fairness properties.

Definition

Given a game (N, v) the Shapley value of player i is

$$\phi(i) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)![v(S \cup \{i\}) - v(S)].$$

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Example: Voting Game

Relation Between the Core and Shapley Value

- In general, there is none.
- For convex games, the Shapley value is in the core.

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Alternative Solution Concepts

- ϵ-core, least core
- Nucleolous
- Kernel

Compact Representations

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Power in Weighted Voting Games

- **Shapley-Shubik Index** : Let π be a permutation of the agents, and let $S_\pi(i)$ denote all agents j such that $\pi(j) < \pi(i)$

$$\phi(i) = \frac{1}{N!}\sum_{\pi}[v(S\pi(i) \cup \{i\}) - v(S\pi(i))]$$

- **Banzhaf Index**

$$\beta(i) = \frac{1}{2^{|N|-1}}\sum_{S}[v(S \cup \{i\} - v(S)]$$

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