Social Choice
(Preference Aggregation)

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Social choice theory

• Study of decision problems in which a group has to make the decision
• The decision affects all members of the group
  – Their opinions should count!
• Applications:
  – Political elections
  – Other elections
  – Note that outcomes can be vectors
    • Allocation of money among agents, allocation of goods, tasks, resources...
Social choice theory

• CS applications:
  – Multiagent planning [Ephrati & Rosenschein]
  – Computerized elections [Cranor & Cytron]
    • Note: this is not the same as electronic voting
  – Accepting a joint project, rating Web articles [Avery, Resnick & Zeckhauser]
  – Rating CDs...
Assumptions

1. Agents have preferences over alternatives
   - Agents can rank order the outcomes
     - $a > b > c = d$ is read as “$a$ is preferred to $b$ which is preferred to $c$ which is equivalent to $d$”

2. Voters are sincere
   - They truthfully tell the center their preferences

3. Outcome is enforced on all agents
Formal model

- Set of agents $N=\{1,2,\ldots,n\}$
- Set of outcomes $O$
- Set of strict total orders on $O$, $L$

- **Social choice function** $C:L^n \rightarrow O$

- **Social welfare function** $C:L^n \rightarrow L^-$ where $L^-$ is the set of weak total orders on $O$
The problem

• Majority decision:
  – If more agents prefer a to b, then a should be chosen

• Two outcome setting is easy
  – Choose outcome with more votes!

• What happens if you have 3 or more possible alternatives?
Case 1: Agents specify their top preference

Ballot

X
Canadian Election System

• Plurality Voting
  – One name is ticked on a ballot
  – One round of voting
  – One candidate is chosen

Is this a “good” system?

What do we mean by good?
Example: Plurality

• 3 candidates
  – Lib, NDP, C

• 21 voters with the preferences
  – 10 Lib>NDP>C
  – 6 NDP>C>Lib
  – 5 C>NDP>Lib

• Result: Lib 10, NDP 6, C 5
  – But a majority of voters (11) prefer all other parties more than the Libs!
What can we do?

• Majority system
  – Works well when there are 2 alternatives
  – Not great when there are more than 2 choices

• Proposal:
  – Organize a series of votes between 2 alternatives at a time
  – How this is organized is called an agenda
    • Or a cup (often in sports)
Agendas

- 3 alternatives \{a, b, c\}
- Agenda a, b, c

Majority vote between a and b

Chosen alternative
Agenda paradox

- Binary protocol (majority rule) = cup
- Three types of agents:
  1. $x > z > y$ (35%)
  2. $y > x > z$ (33%)
  3. $z > y > x$ (32%)

- Power of agenda setter (e.g. chairman)
- Vulnerable to irrelevant alternatives ($z$)
Another problem: Pareto dominated winner paradox

Agents:
1. $x > y > b > a$
2. $a > x > y > b$
3. $b > a > x > y$

\[ x \quad a \quad b \quad y \]

BUT
Everyone prefers $x$ to $y$!
Case 2: Agents specify their complete preferences

Maybe the problem was with the ballots!

Ballot

X > Y > Z

Now have more information
Condorcet

• Proposed the following
  – Compare each pair of alternatives
  – Declare “a” is socially preferred to “b” if more voters strictly prefer a to b

• Condorcet Principle: If one alternative is preferred to all other candidates then it should be selected
Example: Condorcet

• 3 candidates
  – Lib, NDP, C

• 21 voters with the preferences
  – 10 Lib>NDP>C
  – 6 NDP>C>Lib
  – 5 C>NDP>Lib

• Result:
  – NDP win! (11/21 prefer them to Lib, 16/21 prefer them to C)
A Problem

- 3 candidates
  - Lib, NDP, C
- 3 voters with the preferences
  - Lib > NDP > C
  - NDP > C > Lib
  - C > Lib > NDP
- Result:
  - No Condorcet Winner
Borda Count

• Each ballot is a list of ordered alternatives
• On each ballot compute the rank of each alternative
• Rank order alternatives based on decreasing sum of their ranks

\[ A > B > C \quad A: 4 \]
\[ A > C > B \quad B: 8 \]
\[ C > A > B \quad C: 6 \]
Borda Count

• Simple
• Always a Borda Winner
• BUT does not always choose Condorcet winner!
• 3 voters
  – 2: b > a > c > d
  – 1: a > c > d > b

Borda scores: a: 5, b: 6, c: 8, d: 11
Therefore a wins
BUT b is the Condorcet winner
Inverted-order paradox

- Borda rule with 4 alternatives
  - Each agent gives 1 points to best option, 2 to second best...
- Agents:
  1. $x > c > b > a$
  2. $a > x > c > b$
  3. $b > a > x > c$
  4. $x > c > b > a$
  5. $a > x > c > b$
  6. $b > a > x > c$
  7. $x > c > b > a$

- $x=13$, $a=18$, $b=19$, $c=20$
- Remove $x$: $c=13$, $b=14$, $a=15$
Borda rule vulnerable to irrelevant alternatives

• Three types of agents:

1. \( x > z > y \) (35%)
2. \( y > x > z \) (33%)
3. \( z > y > x \) (32%)

• Borda winner is \( x \)
• Remove \( z \): Borda winner is \( y \)
Desirable properties for a voting protocol

- **Universality**
  - It should work with any set of preferences

- **Transitivity**
  - It should produce an ordered list of alternatives
  - That is, we work with social welfare function

- **Pareto efficient**
  - If all all agents prefer \( x \) to \( y \) then in the outcome \( x \) should be preferred to \( y \)
  - \( \text{SWF} \ W \) is pareto efficient if for any \( o_1, o_2 \in O, \forall i \ o_1 \succ_i o_2 \) implies that \( o_1 \succ_W o_2 \)
Desirable properties for a voting protocol

Independence of Irrelevant Alternatives (IIA)

- Comparison of two alternatives depends only on their standings among agents’ preferences, not on the ranking of other alternatives
- SWF $W$ is IIA if for any $o_1, o_2 \in O$, and two preference profiles $\succ_i, \succ''_i$, $\forall i \; o_1 \succ_i o_2 \iff o_1 \succ'_i o_2$ implies that $o_1 \succ_{W(\succ')} o_2 \iff o_1 \succ_{W(\succ'')} o_2$

- **No dictators**
  - SWF $W$ has no dictator if
    $\neg \exists I \; \forall o_1, o_2 \; (o_1 \succ_i o_2 \Rightarrow o_1 \succ_{W} o_2)$
Arrow’s Theorem (1951)

- If there are 3 or more alternatives and a finite number of agents then there is no protocol which satisfies the 5 desired properties
Is there anything that can be done?

- Can we relax the properties?
- No dictator
  - Fundamental for a voting protocol
- Paretian
  - Also seems to be pretty desirable
- Transitivity
  - Maybe you only need to know the top ranked alternative
    - Stronger form of Arrow’s theorem says that you are still in trouble
- Independence
- Universality
  - Some hope here (1 dimensional preferences, spacial preferences)
Take-home Message

• Despair?
  – No ideal voting method
  – That would be boring!

• A group is more complex than an individual
• Weigh the pro’s and con’s of each system and understand the setting they will be used in

• Do not believe anyone who says they have the best voting system out there!
Proof of Arrow’s theorem (slide 1 of 3)

- Follows [Mas-Colell, Whinston & Green, 1995]
- Assuming \( G \) is Paretian and independent of irrelevant alternatives, we show that \( G \) is dictatorial
- **Def.** Set \( S \subseteq A \) is decisive for \( x \) over \( y \) whenever
  - \( x >_i y \) for all \( i \in S \)
  - \( x <_i y \) for all \( i \in A-S \)
  - \( \Rightarrow x > y \)
- **Lemma 1.** If \( S \) is decisive for \( x \) over \( y \), then for any other candidate \( z \), \( S \) is decisive for \( x \) over \( z \) and for \( z \) over \( y \)
- **Proof.** Let \( S \) be decisive for \( x \) over \( y \). Consider: \( x >_i y >_i z \) for all \( i \in S \) and \( y >_i z >_i x \) for all \( i \in A-S \)
  - Since \( S \) is decisive for \( x \) over \( y \), we have \( x > y \)
  - Because \( y >_i z \) for every agent, by the Pareto principle we have \( y > z \)
  - Then, by transitivity, \( x > z \)
  - By independence of irrelevant alternatives (\( y \)), \( x > z \) whenever every agent in \( S \) prefers \( x \) to \( z \) and every agent not in \( S \) prefers \( z \) to \( x \). I.e., \( S \) is decisive for \( x \) over \( z \)
- To show that \( S \) is decisive for \( z \) over \( y \), consider: \( z >_i x >_i y \) for all \( i \in S \) and \( y >_i z >_i x \) for all \( i \in A-S \)
  - Then \( x > y \) since \( S \) is decisive for \( x \) over \( y \)
  - \( z > x \) from the Pareto principle and \( z > y \) from transitivity
  - Thus \( S \) is decisive for \( z \) over \( y \) \( \odot \)
Proof of Arrow’s theorem

(slide 2 of 3)

- Given that S is decisive for x over y, we deduced that S is decisive for x over z and z over y.
- Now reapply Lemma 1 with decision z over y as the hypothesis and conclude that
  - S is decisive for z over x
  - which implies (by Lemma 1) that S is decisive for y over x
  - which implies (by Lemma 1) that S is decisive for y over z
  - Thus: Lemma 2. If S is decisive for x over y, then for any candidates u and v, S is decisive for u over v (i.e., S is decisive)

- Lemma 3. For every S ⊆ A, either S or A-S is decisive (not both)
- Proof: Suppose x >_i y for all i ∈ S and y >_i x for all i ∈ A-S (only such cases need to be addressed, because otherwise the left side of the implication in the definition of decisiveness between candidates does not hold). Because either x > y or y > x, S is decisive or A-S is decisive
Proof of Arrow’s theorem (slide 3 of 3)

- **Lemma 4.** If $S$ is decisive and $T$ is decisive, then $S \cap T$ is decisive

  - **Proof.**
    - Let $S = \{ i: z_i > y_i, x \} \cup \{ i: x_i > z_i, y \}$
    - Let $T = \{ i: y_i > x_i, z \} \cup \{ i: x_i > z_i, y \}$
    - For $i \not\in S \cup T$, let $y_i > z_i, x$
    - Now, since $S$ is decisive, $z > y$
    - Since $T$ is decisive, $x > z$
    - Then by transitivity, $x > y$
    - So, by independence of irrelevant alternatives ($z$), $S \cap T$ is decisive for $x$ over $y$.
      - (Note that if $x_i > y$, then $i \in S \cap T$.)
    - Thus, by Lemma 2, $S \cap T$ is decisive

- **Lemma 5.** If $S = S_1 \cup S_2$ (where $S_1$ and $S_2$ are disjoint and exhaustive) is decisive, then $S_1$ is decisive or $S_2$ is decisive

  - **Proof.** Suppose neither $S_1$ nor $S_2$ is decisive. Then $\sim S_1$ and $\sim S_2$ are decisive. By Lemma 4, $\sim S_1 \cap \sim S_2 = \sim S$ is decisive. But we assumed $S$ is decisive. Contradiction

- **Proof of Arrow’s theorem**
  - Clearly the set of all agents is decisive. By Lemma 5 we can keep splitting a decisive set into two subsets, at least one of which is decisive. Keep splitting the decisive set(s) further until only one agent remains in any decisive set. That agent is a dictator. QED
Stronger version of Arrow’s theorem

- In Arrow’s theorem, social choice functional G outputs a ranking of the outcomes
- The impossibility holds even if only the highest ranked outcome is sought:
- **Thrm.** Let $|O| \geq 3$. If a social choice function $f: R \rightarrow$ outcomes is monotonic and Paretian, then $f$ is dictatorial
  - $f$ is monotonic if $[x = f(R)$ and $x$ maintains its position in $R'] \Rightarrow f(R') = x$
  - $x$ maintains its position whenever $x >_i y \Rightarrow x > _{i'} y$