Bayes Nash Equilibrium Example

This example is from Microeconomic Theory by Mas-Colell, Whinston, and Green.

There are two firms, 1 and 2. Each can develop a product, but once the product is developed it is shared by the firms. To develop a product costs a firm $c \in (0, 1)$. This is known to everyone. The benefit to each firm $i$ is known only by that firm. That is, each firm has a type $\theta_i$ that is independently drawn from uniform distribution over $[0, 1]$, and its benefit from the product if it’s type is $\theta_i$ is $(\theta_i)^2$. The timing of the game is as follows. First, each firm privately learns its type. Then, they each simultaneously choose either to develop the product or not.

Let $s_i(\theta_i) = 1$ mean that player $i$ develops the product when its type is $\theta_i$ and let $s_i(\theta_i) = 0$ mean that it does not develop the product. If firm $i$ develops the product then its utility is $(\theta_i)^2 - c$ no matter what firm $j$ does. If firm $i$ does not develop the product then its (expected) utility is $(\theta_i)^2 \text{Prob}(s_j(\theta_j) = 1)$. Therefore, firm $i$ will develop the product only if

$$(\theta_i)^2 - c \geq (\theta_i)^2 \text{Prob}(s_j(\theta_j) = 1)$$

or

$$\theta_i \geq \left[ \frac{c}{1 - \text{Prob}(s_j(\theta_j) = 1)} \right]^{\frac{1}{2}}$$

Note that the best-response strategy for agent $i$ takes the form of a cutoff rule. It is best off developing the product if its type $\theta_i$ is above a certain threshold (which depends on the strategy of firm $j$) and does not develop the product if its type is below this threshold.

Assume that $\hat{\theta}_1, \hat{\theta}_2 \in (0, 1)$ are the cutoff values for firms 1 and 2. Since $\text{Prob}(s_i(\theta_i) = 1) = 1 - \hat{\theta}_i$, we must have

$$(\hat{\theta}_1)^2 \hat{\theta}_2 = c$$
$$(\hat{\theta}_2)^2 \hat{\theta}_1 = c$$

Solving, we see

$$\hat{\theta}_1 = \hat{\theta}_2 = c^{\frac{1}{4}}$$