Constraint Satisfaction

CS 486/686: Introduction to Artificial Intelligence

Outline

- What are Constraint Satisfaction Problems (CSPs)?
- Standard Search and CSPs
- Improvements
  - Backtracking
  - Backtracking + heuristics
  - Forward Checking

Introduction

Standard search

State is a “black box”: arbitrary data structure

Goal test: any function over states

Successor function: anything that lets you move from one state to another

Constraint satisfaction problems (CSPs)

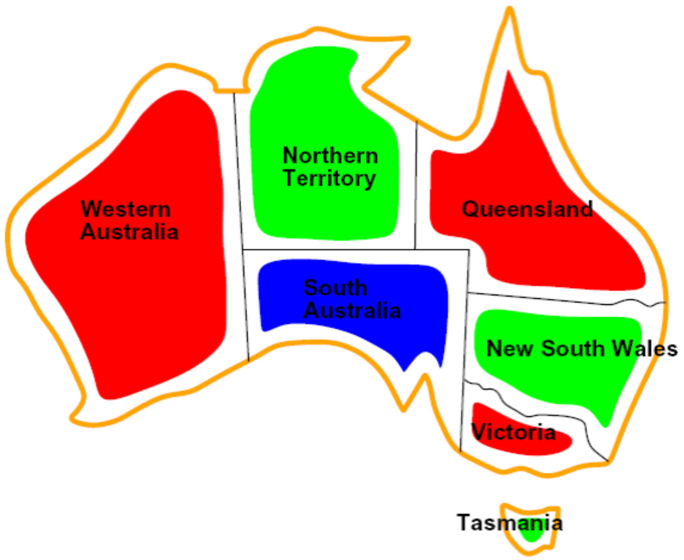
A special subset of search problems

States are defined by ***variables*** X_i with values from ***domains*** D_i

Goal test is a ***set of constraints*** specifying allowable combinations of values for subsets of variables

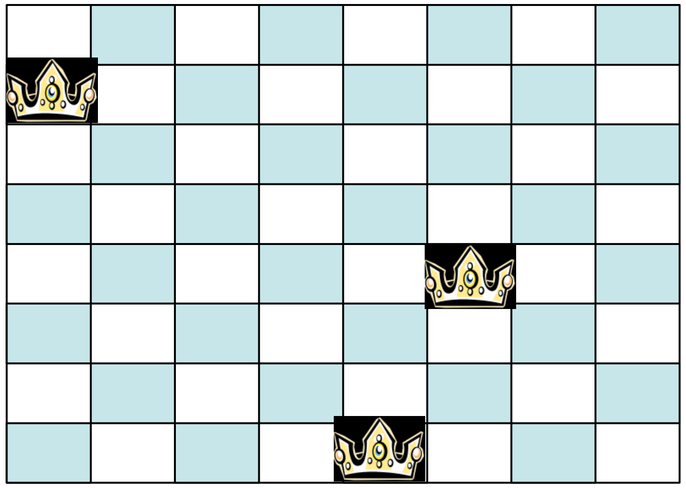
Example: Map Colouring

- **Variables**
  - V={T, V, NSW, Q, NT, WA, SA}



- **Domains**
  - D={red, blue, green}
- **Constraints**: adjacent regions must have different colours
  - Implicit: WA≠NT
  - Explicit: (WA, NT)∈ {(red, blue), (red, green), (blue, red)…}
- **Solution** is an assignment satisfying all constraints
  - {WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}

N Queens Problem



- **Variables**: $X_{i,j}$
- **Domains**: {0,1}
- **Constraints**:

$$\forall i, j, k \;\; (X_{ij}, X_{ik}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{kj}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0),(0,1),(1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0),(0,1),(1,0)\}$$

N Queens Problem

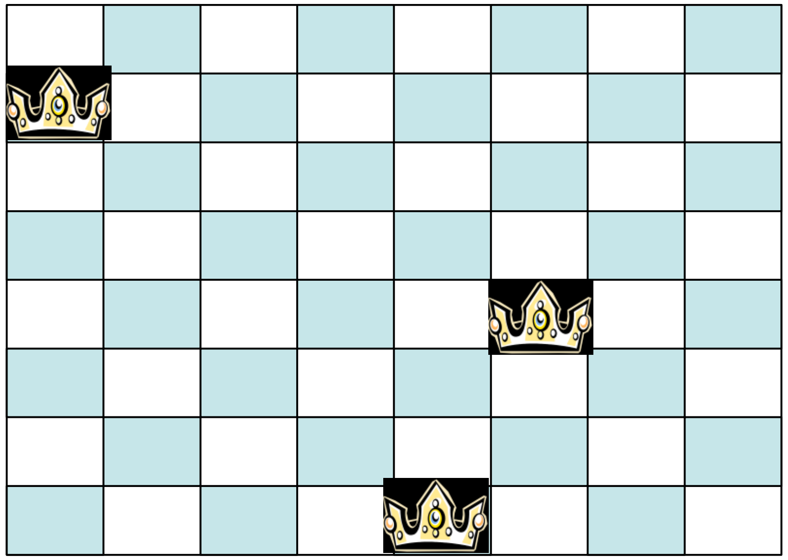
- **Variables**: Q_i
- **Domains**: {1,2,…,N}
- **Constraints**:
  - Implicit:

$$\forall i, j \text{ non-threatening}(Q_i, Q_j)$$

  - Explicit:

$$(Q_1, Q_2) \in \{(1,3),(1,4),\ldots\}$$

$$\ldots$$



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- **Variables**: $V_1, \ldots, V_n$
- **Domains**: {0,1}
- **Constraints**:
  - K constraints of the form $V_i^* \vee V_j^* \vee V_k^*$ V_i^* where V_i^* is either V_i or $\neg V_i$

$$A \neg B \vee \neg C$$
$$\neg A \vee B \vee D$$
$$D \vee B \vee E$$
$$\neg A \vee \neg B \vee C$$

A canonical NP-complete problem

Types of CSPs

- **Discrete Variables**
  - **Finite domains**
    - If domain has size d, then there are $O(d^n)$ complete assignments
    - Boolean CSPs (including 3-SAT)
  - **Infinite domains** (e.g. integers)
    - Constraint languages
    - Linear constraints are solvable but non-linear are undecidable
- **Continuous Variables**
  - Linear programming (linear constraints solvable in polynomial time)

Types of CSPs

- **Varieties of Constraints**
  - **Unary constraints:** involve a single variable
    - NSW≠red
  - **Binary constraints:** involve a pair of variables
    - NSW≠Q
  - **Higher-order constraints**: involve more than two variables
    - AllDiff($V_1,\ldots,V_n$)

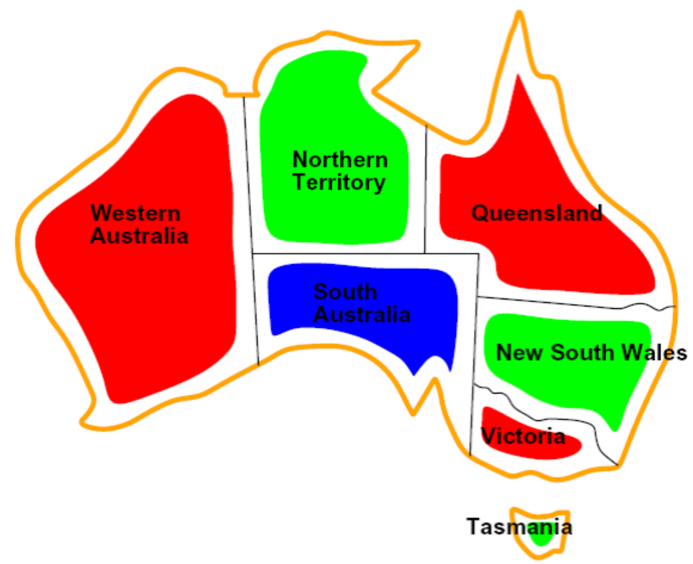
- **Soft Constraints** (**preferences**)
  - red "is better than" green
  - Constrained optimization problems

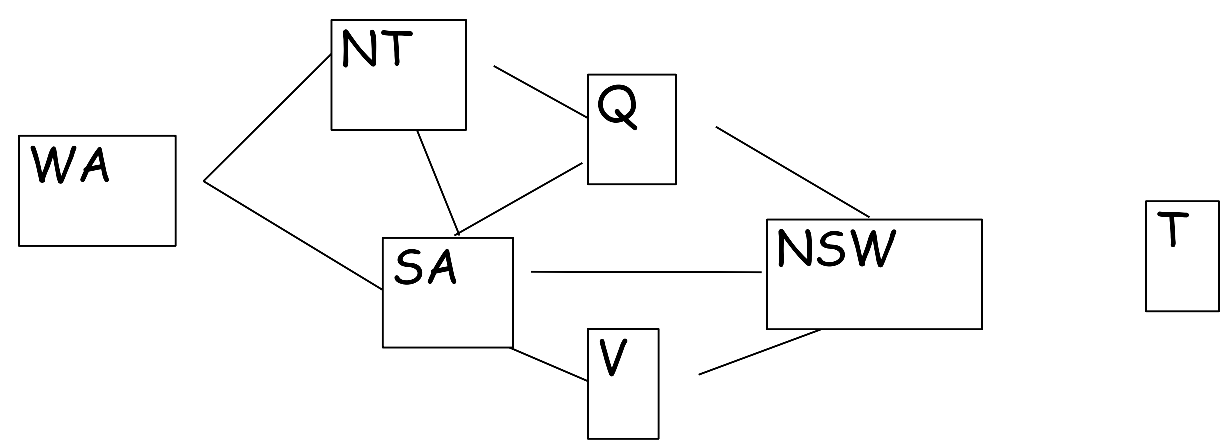
Constraint Graphs

You can represent binary constraints with a **constraint graph**

Nodes are variables

Edges are constraints





CSPs and Search

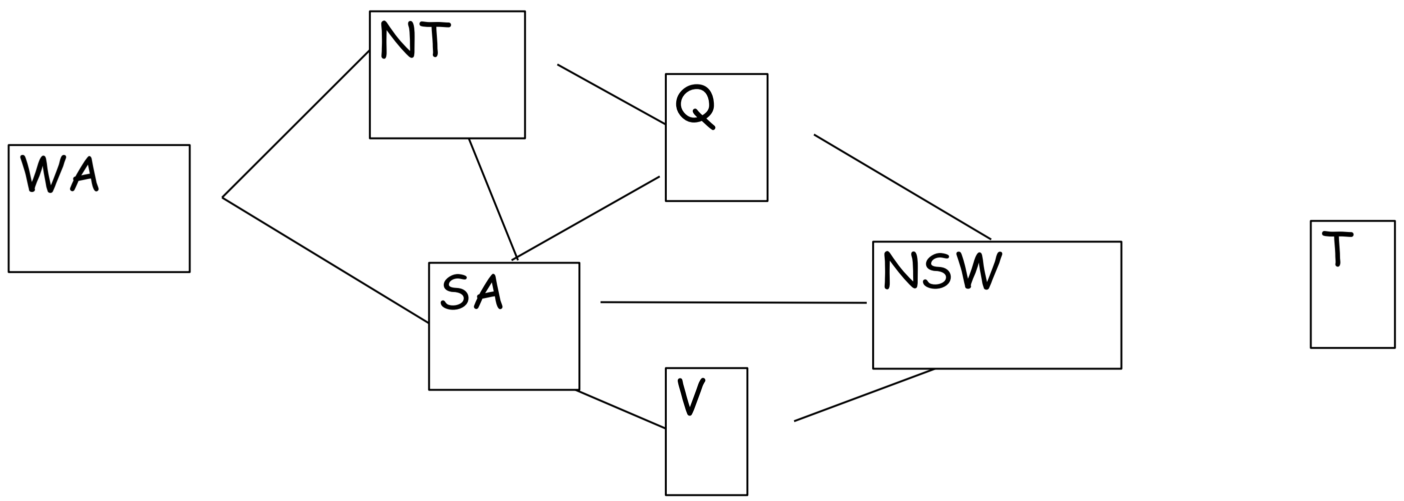
We can use standard search to solve CSPs

States:

Initial State:

Successor Function:

Goal Test:



CSPs and Search

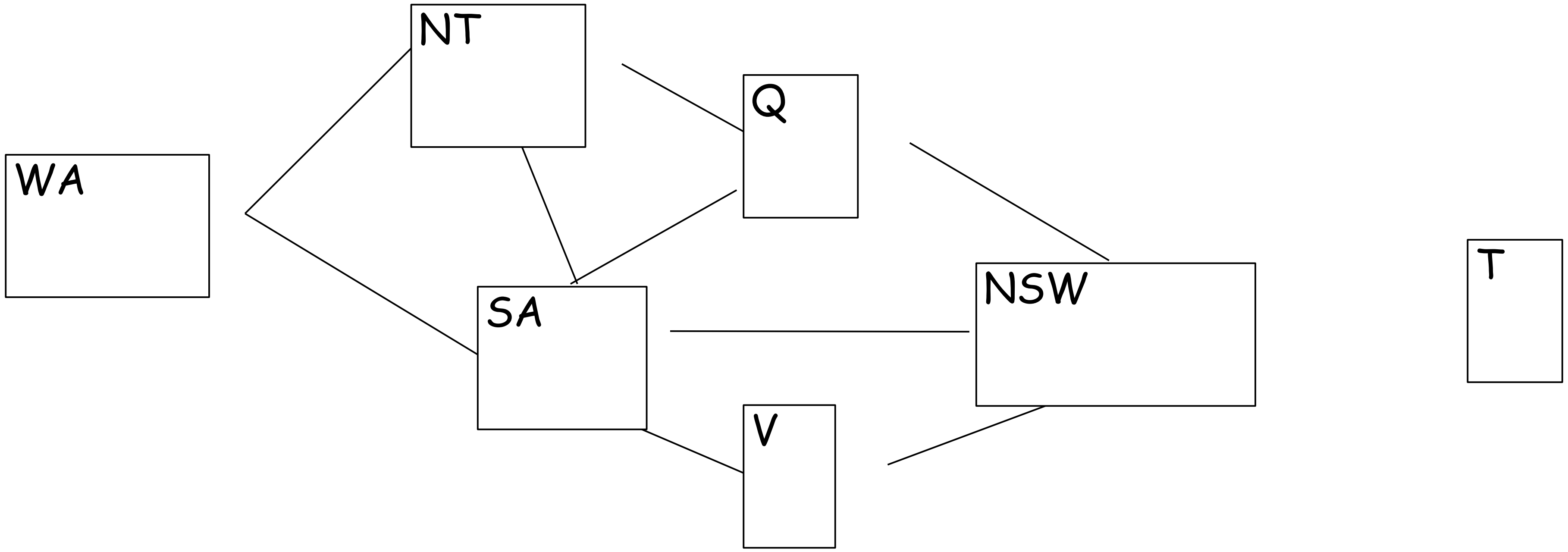
States:

Initial State:

Successor Function:

Goal Test:

What happens if we run something like BFS?

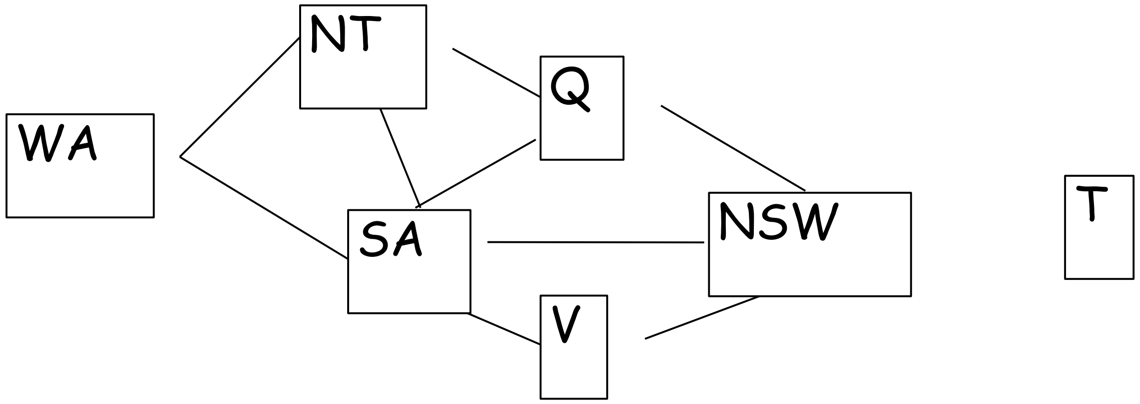


Commutativity

Key Insight:

- CSPs are **commutative**
  - Order of actions taken does not effect outcome
  - Can assign variables in any order

- CSP algorithms take advantage of this
  - Consider possible assignments for a **single variable at each node** in the search tree



{WA=red, NT=blue} is equivalent to {NT=blue, WA=red}

Backtracking Search

Backtracking search is the basic algorithm for CSPs

- Select unassigned variable X
- For each value $\{x_1,\ldots,x_n\}$ in domain of X
  - If value satisfies constraints, assign $X=x_i$ and exit loop
- If an assignment is found
  - Move to next variable
- If no assignment found
  - **Back up** to preceding variable and try a different assignment for it

One variable at a time

Check constraints as you go

Backtracking Example

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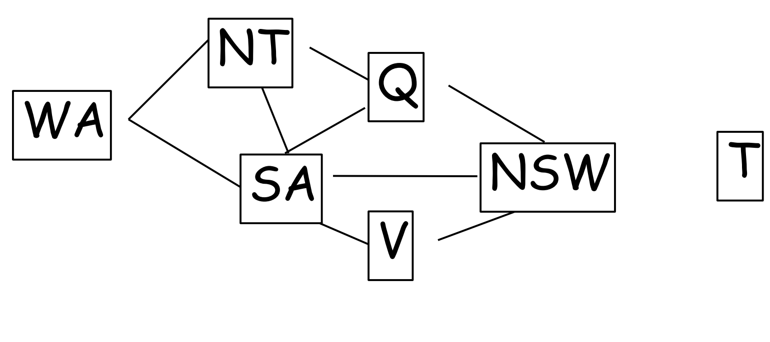
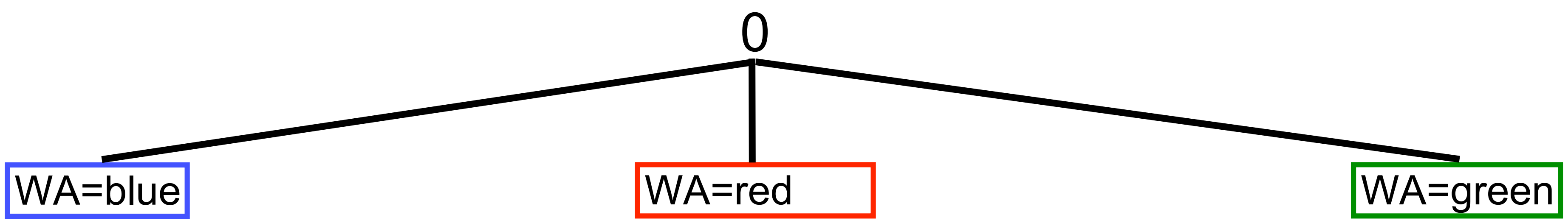
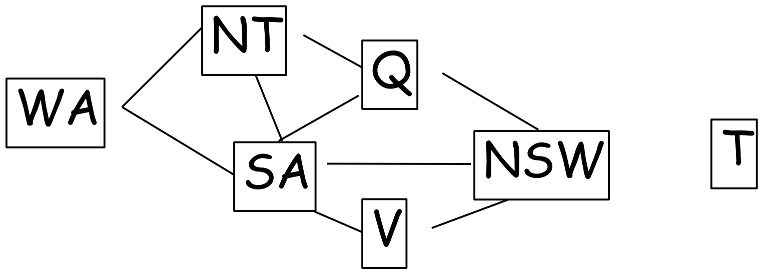
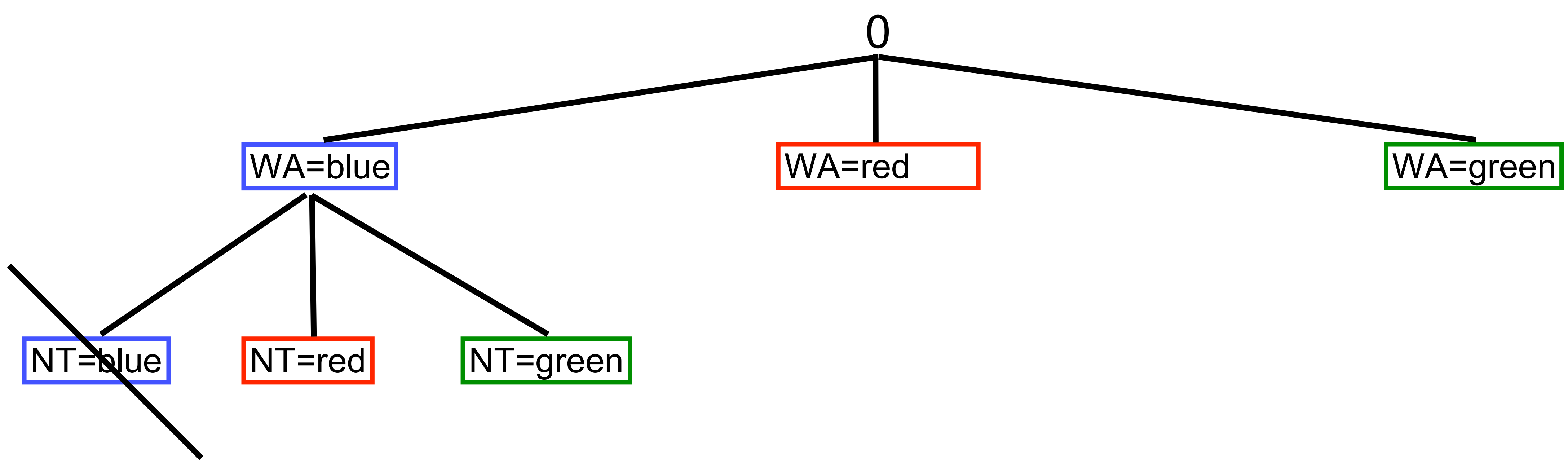


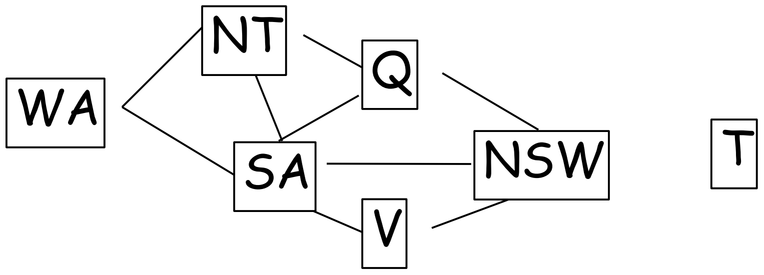
Backtracking Example



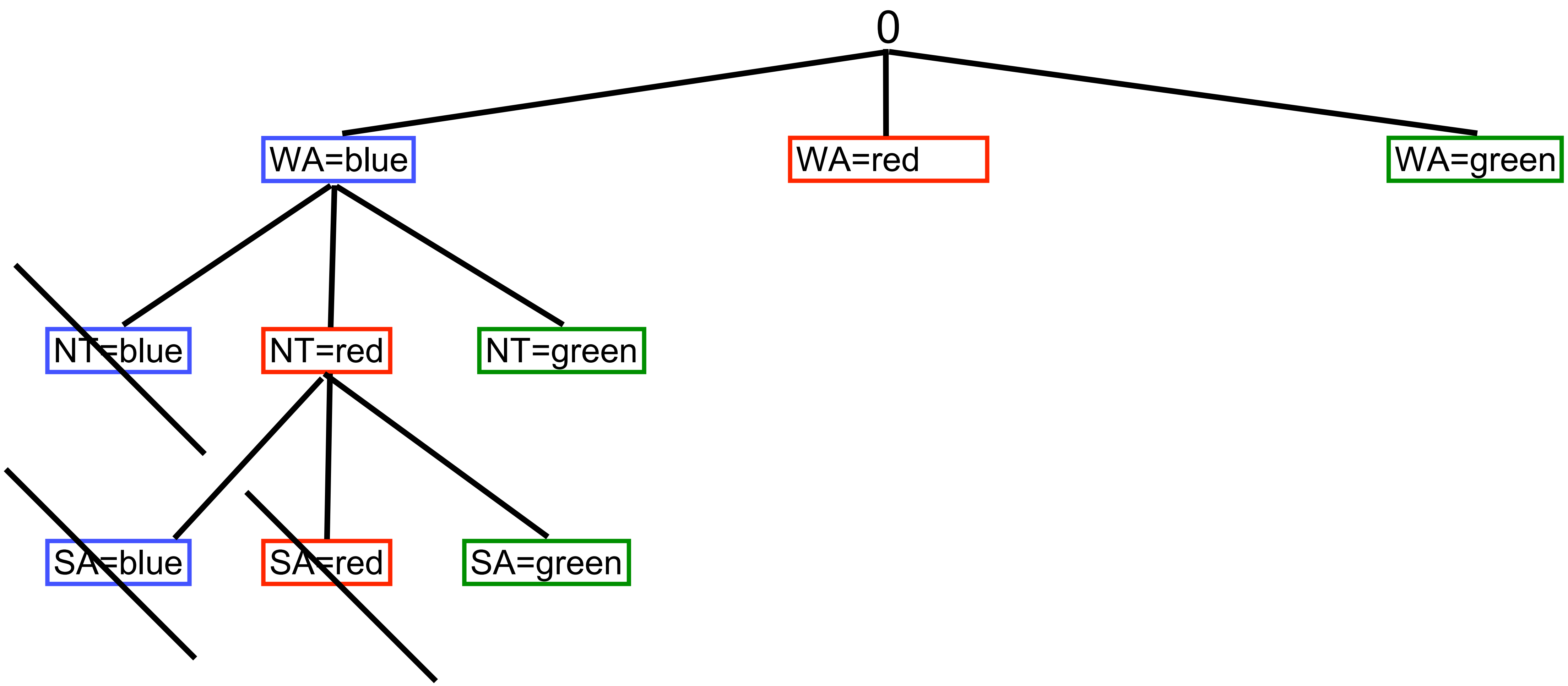


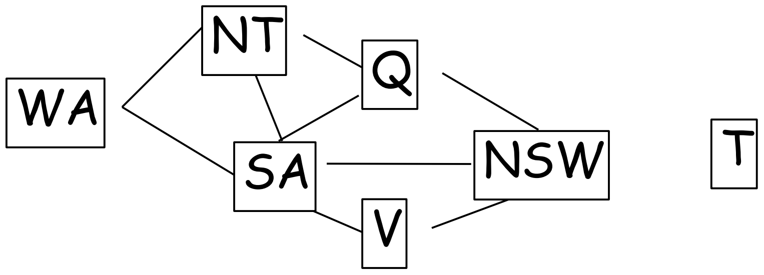
Backtracking Example





Backtracking Example





Backtracking and Efficiency

Note that backtracking search is basically DFS with some small improvements. Can we improve on it further?

Ordering:

- Which variables should be tried first?
- In what order should a variable's values be tried?

Filtering:

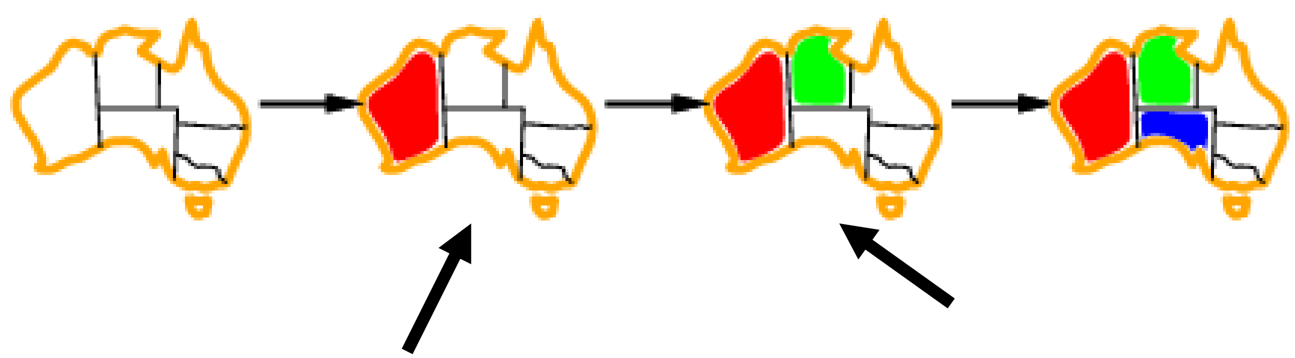
- Can we detect failure early?

Structure:

- Can we exploit the problem structure?

Ordering: Most Constrained Variable

- Choose the variable which has the fewest “legal” moves
  - AKA **minimum remaining values** (MRV)



D_{NT}={green, blue}

D_{SA}={green, blue}

D_{others}={red, green, blue}

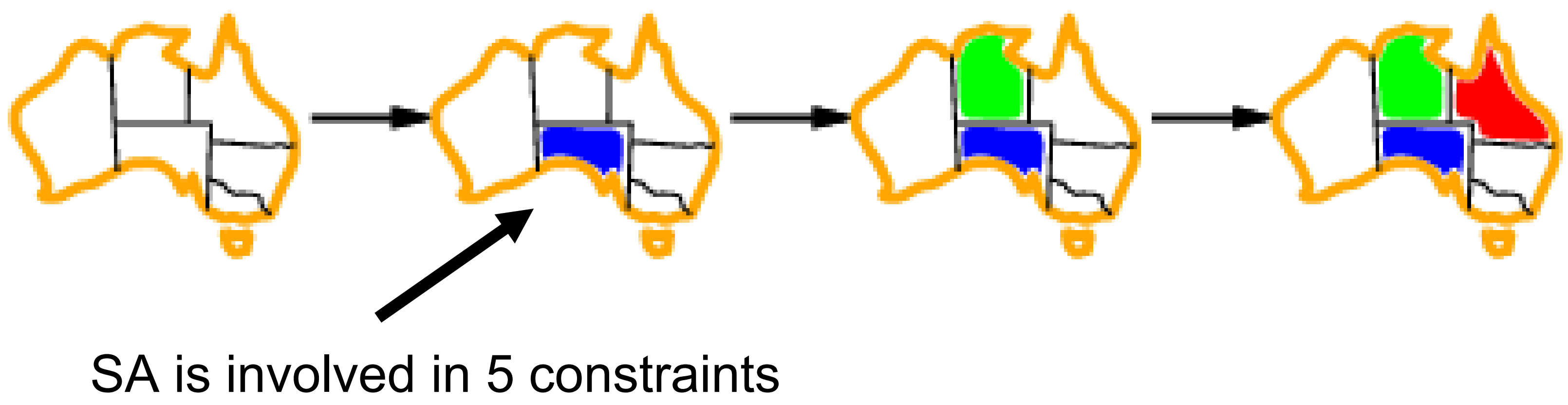
D_{SA}={blue}

D_{Q}={blue, red}

D_{others}={red,green,blue}

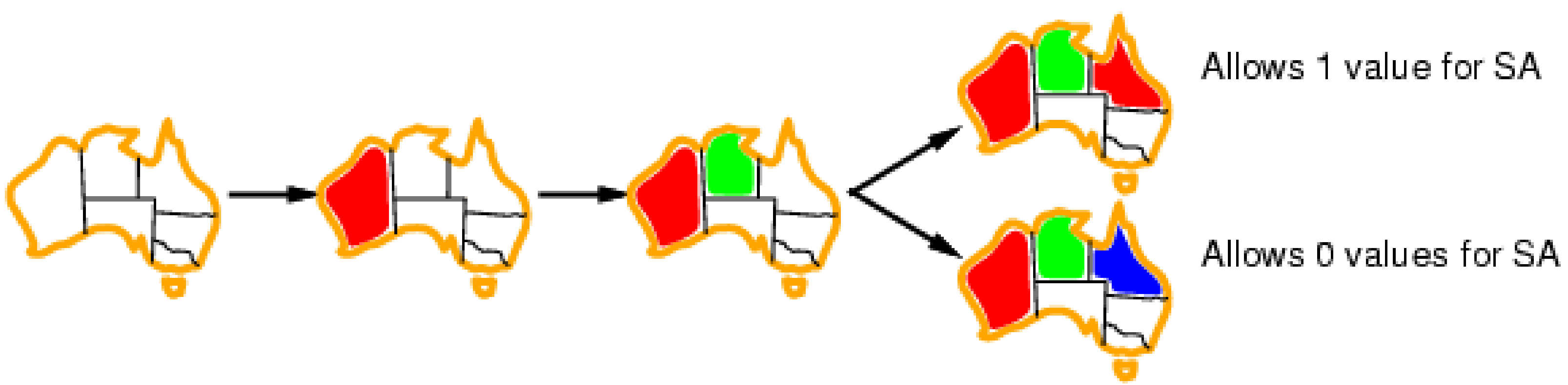
Ordering: Most Constraining Variable

- Most constraining variable:
  - Choose variable with most constraints on remaining variables
- Tie-breaker among most constrained variables



Ordering: Least-Constraining Value

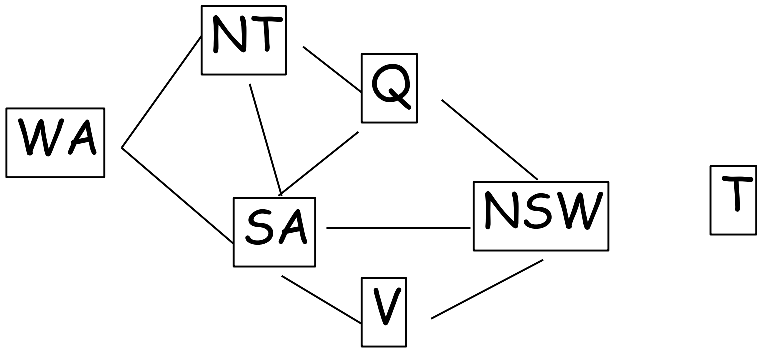
- Given a variable, choose the least constraining value:
  - The one that rules out the fewest values in the remaining variables



Filtering: Forward Checking

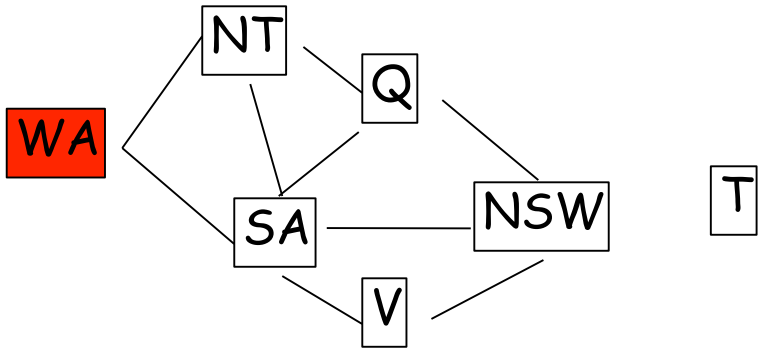
- Forward checking:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values

Example: Forward Checking



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB

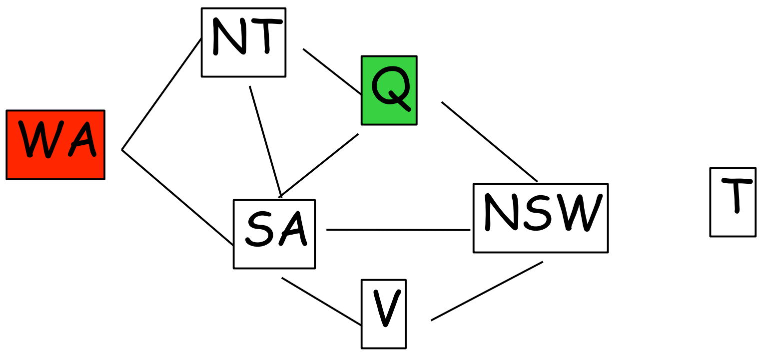
Example: Forward Checking



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB

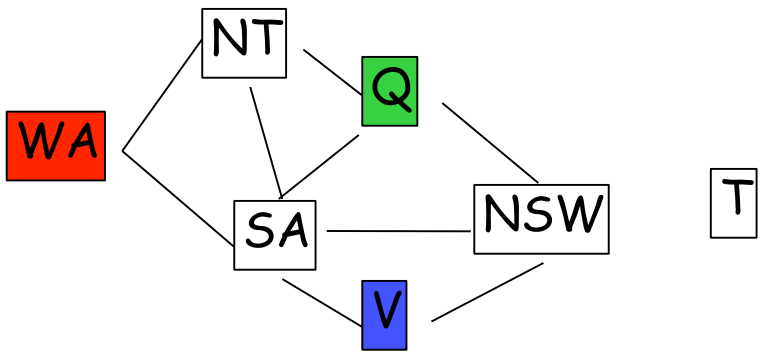
Forward checking removes the value Red of NT and of SA

Example: Forward Checking



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB

Example: Forward Checking



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	RB	B	B	RGB

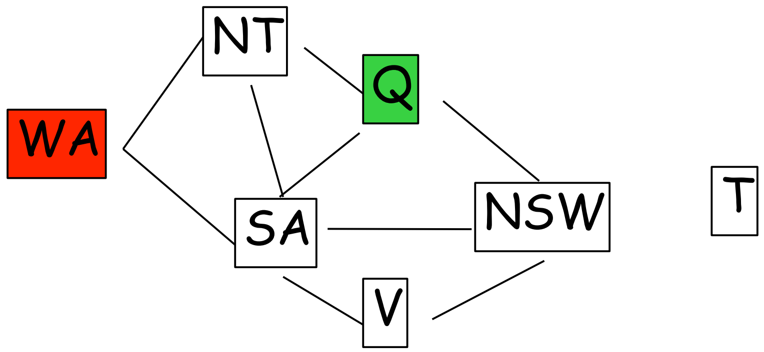
Example: Forward Checking

Empty set: the current assignment
{(WA ← R), (Q ← G), (V ← B)}
does not lead to a solution

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	RB	B	B	RGB

Filtering: Arc Consistency

Forward checking propagates information from assigned to unassigned variables, but it can not detect all future failures early



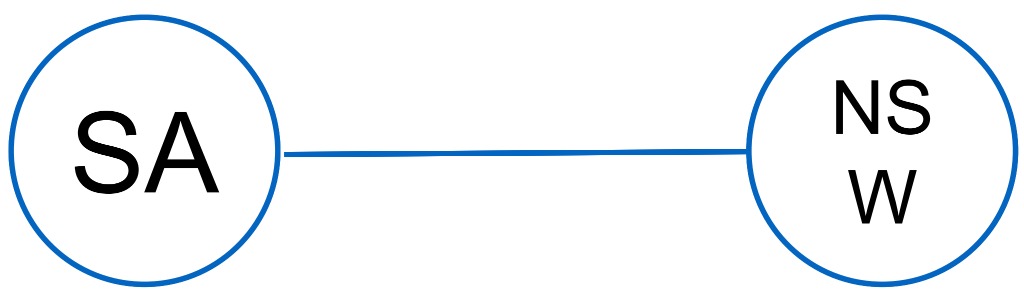
WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB

NT and SA can not both be blue!

Need to reason about constraints

Filtering: Arc Consistency

Given domains D_1 and D_2, an arc is consistent if for all x in D_1 there is a y in D_2 such that x and y are consistent.



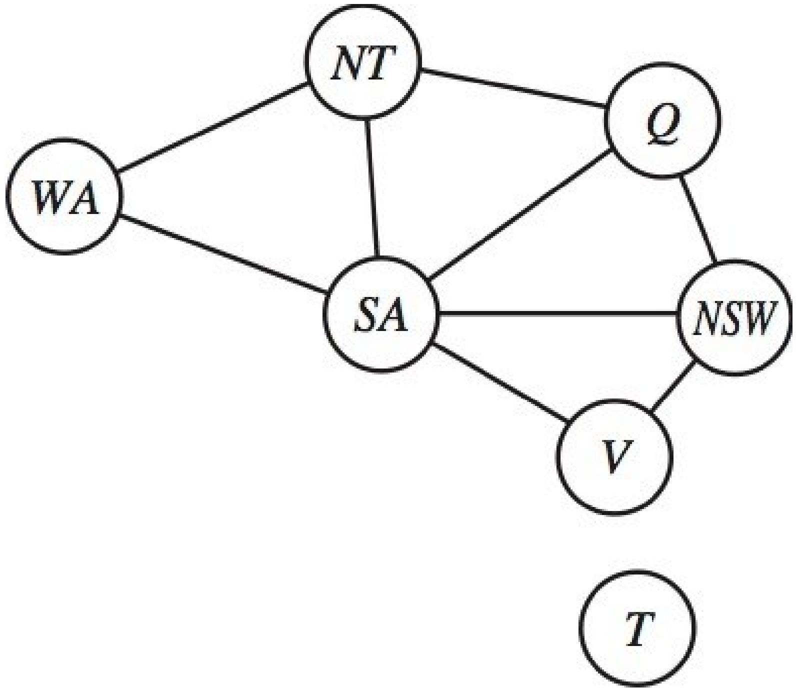
D_{SA}={blue} D_{NSW}={blue,red}

Is the arc from SA to NSW consistent?

Is the arc from NSW to SA consistent?

Structure: Independent Subproblems

Tasmania does not interact with the rest of the problem



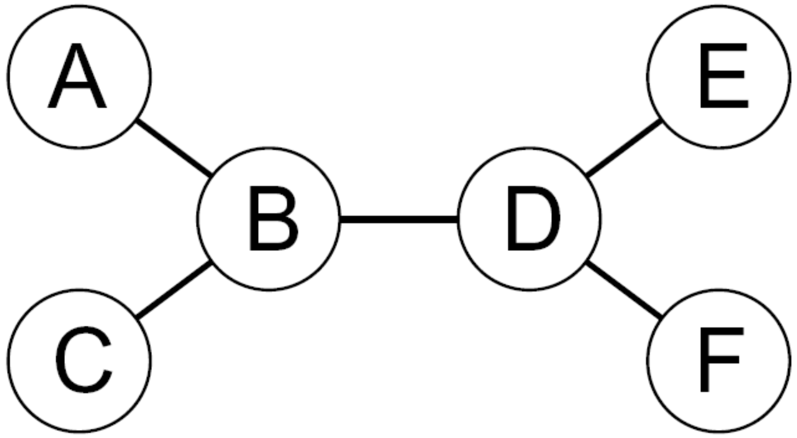
Idea: Break down the graph into its connected components. Solve each component separately.

Significant potential savings:

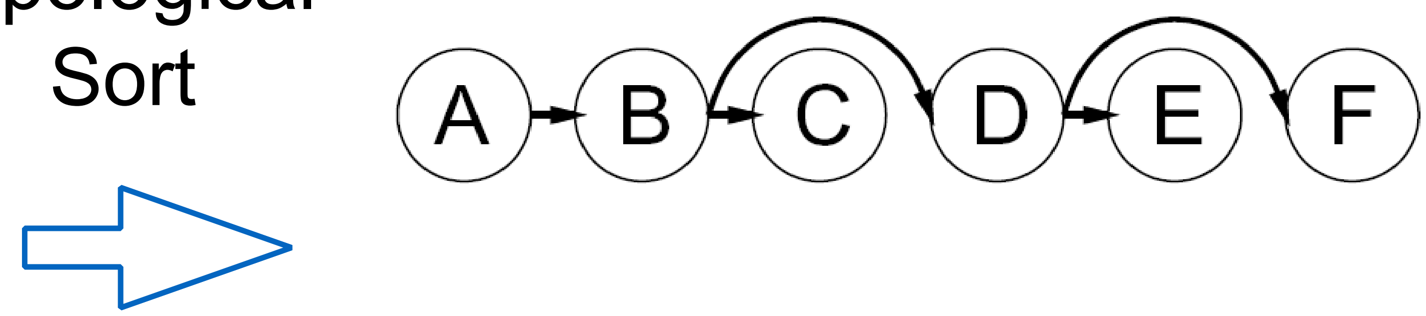
- Assume n variables with domain size d: $O(d^n)$
- Assume each component involves c variables (n/c components) for some constant c: $O(d^c \, n/c)$

Structure: Tree Structures

CSPs can be solved in $O(nd^2)$ if there are no loops in the constraint graph



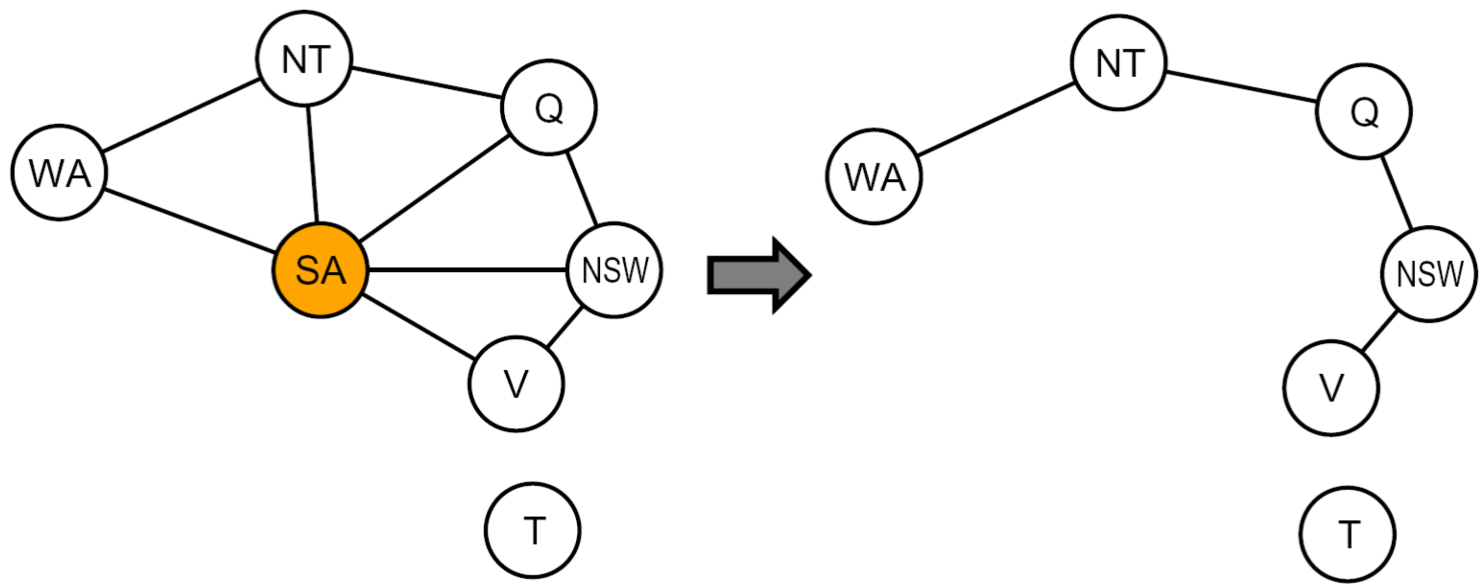
Topological Sort



Step 1: For i=n to 1, make-consistent(X_i,parent(X_i))

Step 2: For i=1 to n, assign value to X_i consistent with parent(X_i) [Note: No backtracking!]

Structure: Non-Trees?



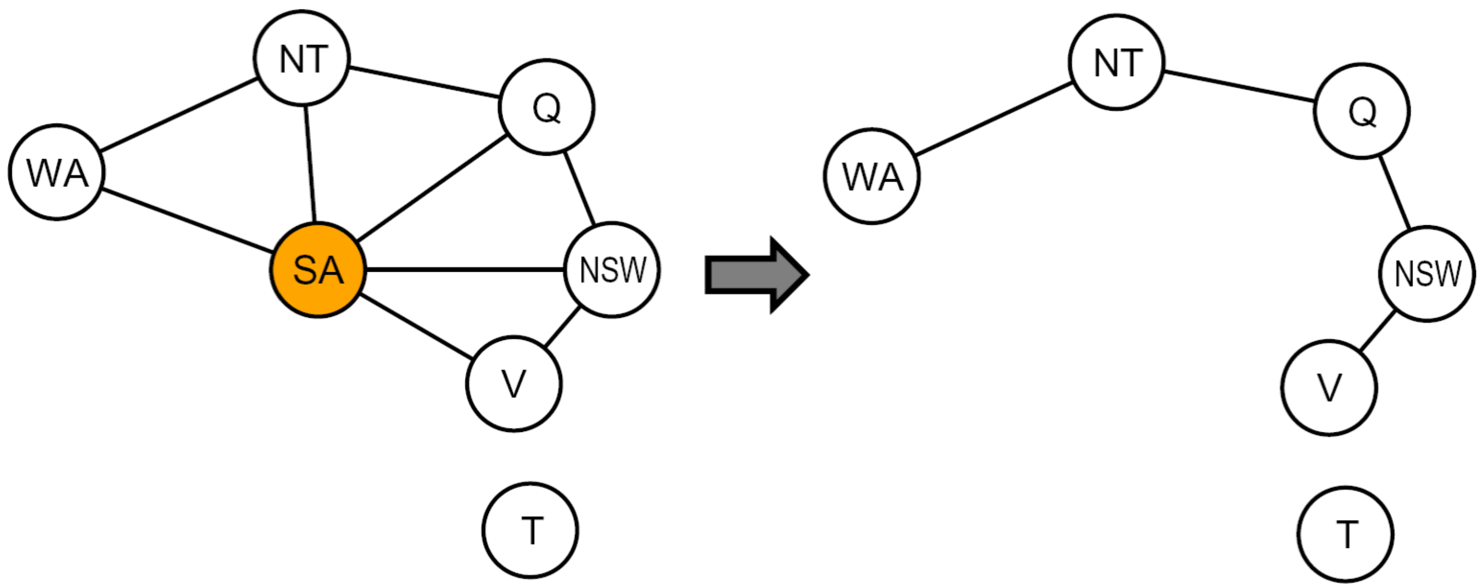
If we assign SA a colour and then remove that colour from the domains all other variables, then we have a tree

Step 1: Choose a subset S of variables such that the constraint graph becomes a tree when S is removed (S is the cycle cutset)

Step 2: For each possible valid assignment to the variables in S

1. Remove from the domains of remaining variables, all values that are inconsistent with S
2. If the remaining CSP has a solution, return it

Structure: Cutsets

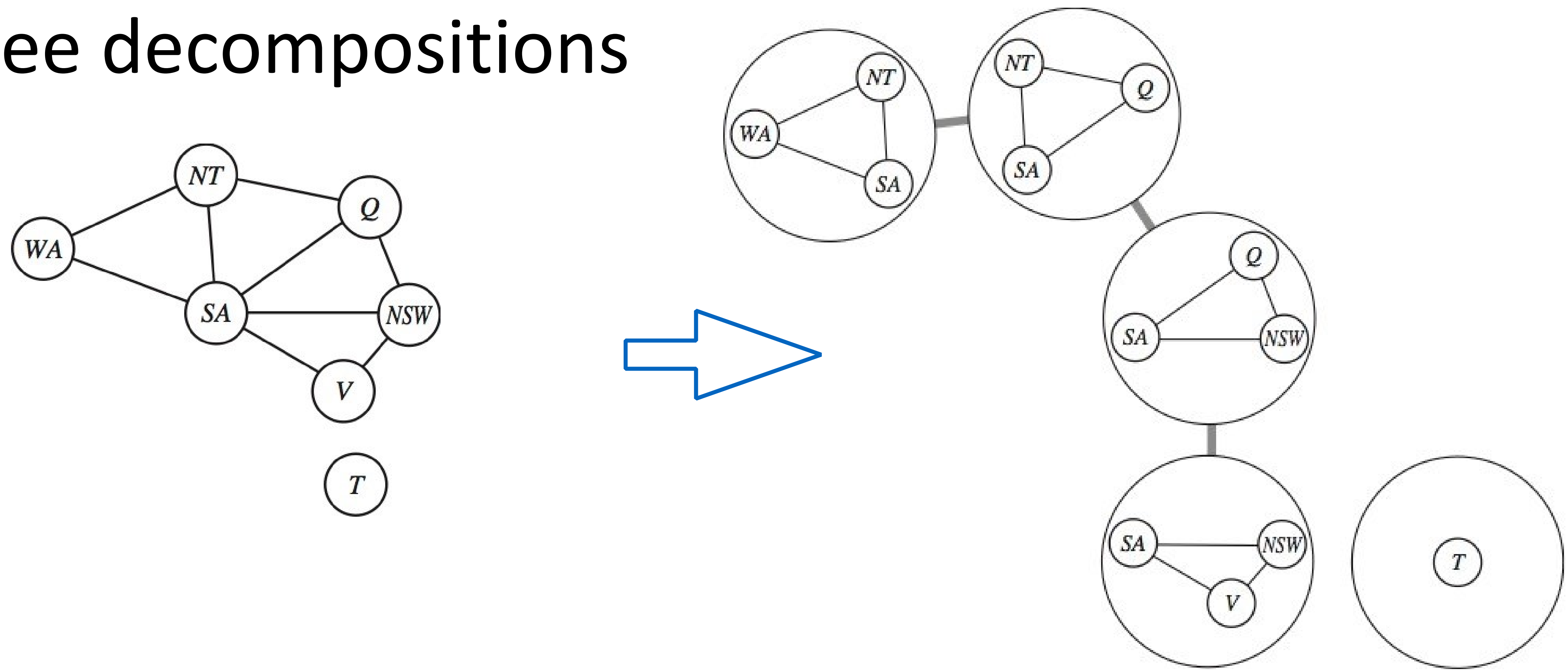


Running time:

- Let c be the size of the cutset then
  - d^c combinations of variables in S
  - For each combination must solve a tree problem of size n-c ($O(n-c)d^2$)
  - Therefore, running time is $O(d^c(n-c)d^2)$
- Finding smallest cutset is NP-hard but efficient approximations exist

Structure: Non-Trees?

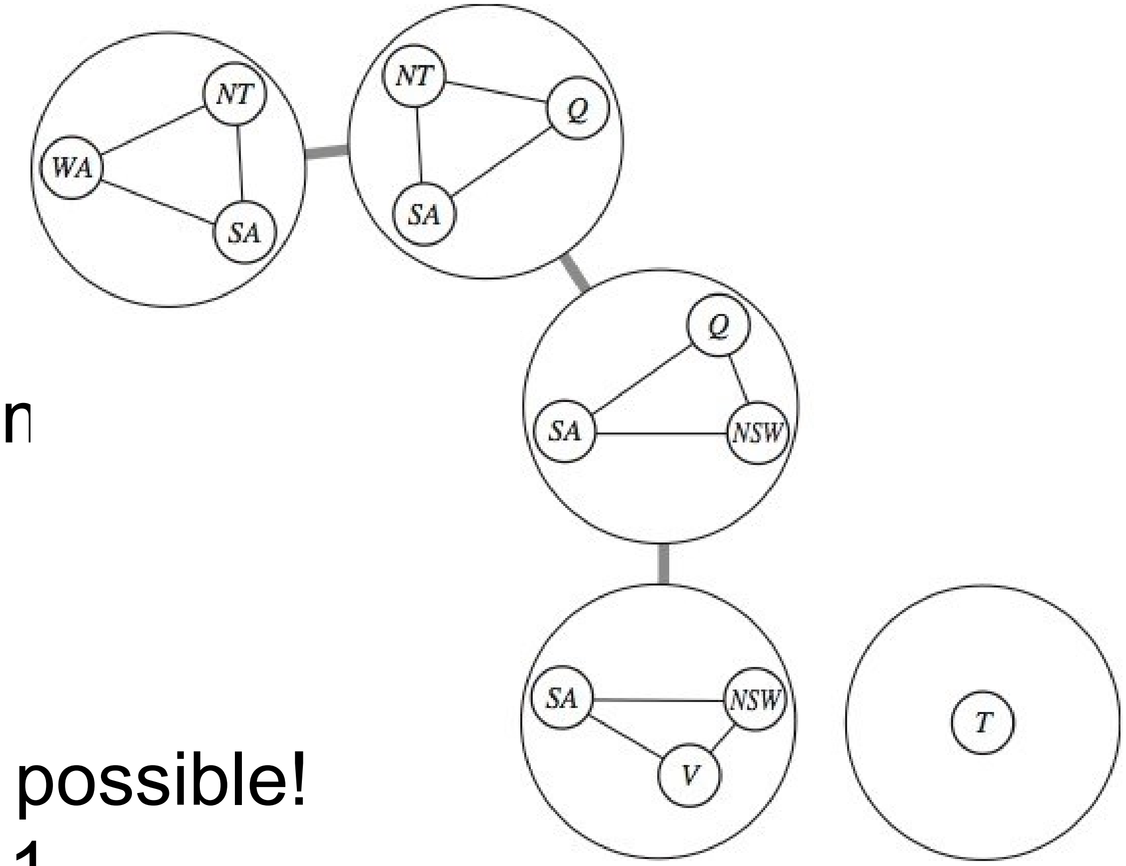
Tree decompositions



1. Each variable appears in at least one subproblem
2. If two variables are connected by a constraint, then they (and the constraint) must appear together in at least one subproblem
3. If a variable appears in two subproblems in the tree, it must appear in every subproblem along the path connecting those subproblems

Structure: Tree Decompositions

- Solve each subproblem independently
  - e.g {(WA=r,NT=g,SA=b),(WA=b, NT=g,SA=r),…}
- Solve constraints connecting the subproblems using tree-based algorithn (to make sure that subproblems with shared variables agree)



Want to make the subproblems as small as possible!
Tree width: w= Size of largest subproblem-1
Running time $O(nd^{w+1})$

Finding tree decomposition with min tree-width is NP-hard, but good heuristics exist

Summary

- How to formalize problems as CSPs
- Backtracking search
- Improvements using
  - Ordering
  - Filtering
  - Structure