Multiagent Graph Coloring: Pareto Efficiency, Fairness and Individual Rationality

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Outline

- Introduction
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- Coloring from an Economics Perspective
- Pareto Efficient Colorings
- Socially Fair Colorings
- Partitions of Cycles
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Introduction

- ***Liliput Civil Administration***

Introduction

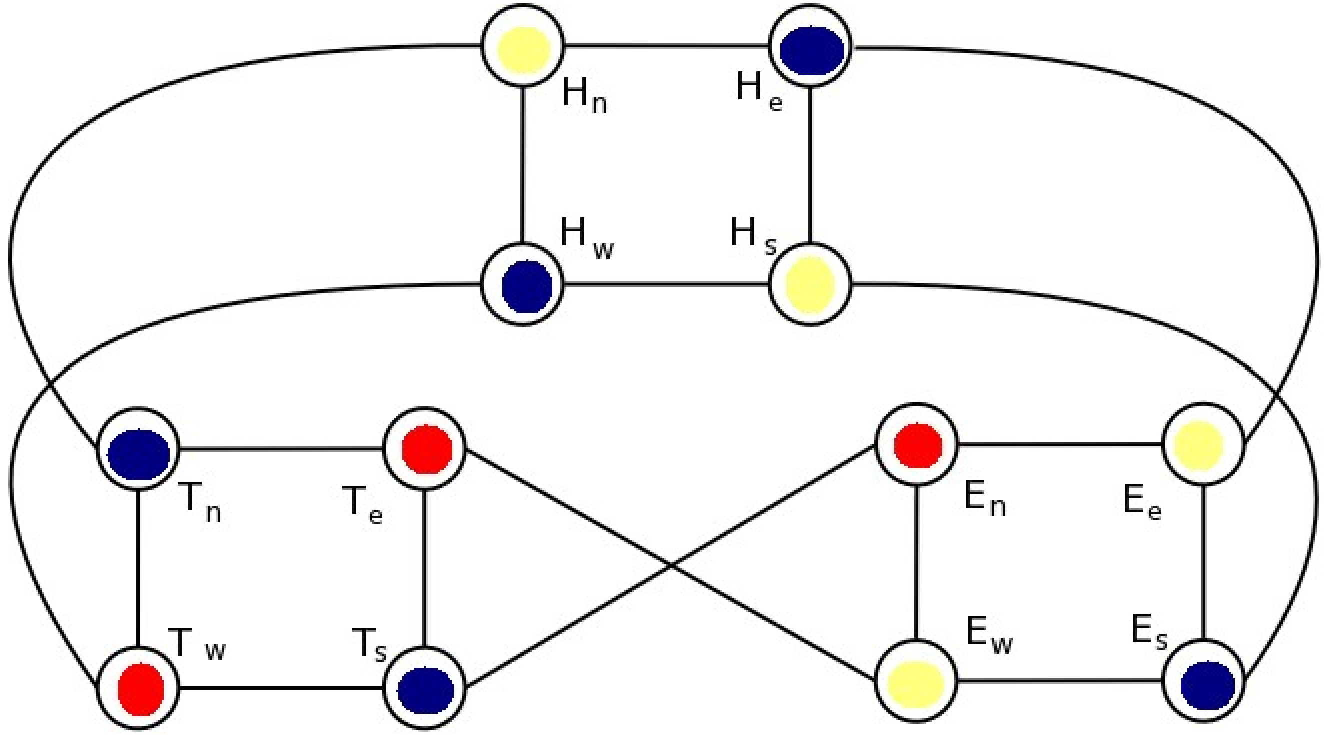
- Weekly committee meetings of offices: Housing, Treasury, Environment
- Offices broken into (cardinal) districts: North, West, South, East
- At times topics lie on the border → need members from each district

Introduction

- Liliput Administration scheduling:
  - Draw vertex for every weekly meeting
  - Two vertices adjacent ⇔ Two meetings have common member (Conflict)
  - Meetings colored with the same color can occur concurrently

Introduction

- Example:



Background: Graph Coloring

- ***χ(G)***: ***Chromatic number*** of a graph ***G***
  - Example: $\chi(K_n) = n$
- Finding the chromatic number is NP-Hard
  - Decision version is NP-Complete: Is there a coloring with at most ***k*** colors?
  - $k = 2$: Equivalent to bipartite, Polytime
  - $k \geq 3$: NP-Complete
  - No approximation within a factor of 4/3 unless

P = NP

Background: Graph Coloring

- ***Precoloring Extension*** problem:
  - Subset of vertices already colored
  - Extend to the entire graph
- ***List Coloring*** problem:
  - Coloring assigned to each vertex chosen from prescribed list
  - Extension: may allow some vertices to choose colors not on original list

Background: Game Theory

- **Individual Rationality**
- **Pareto Optimality**
  - Cannot improve one agent without making other agents worse off
- **Fairness**
  - Minimize some function of loss (due to coordination costs)

Coloring from an Economics Perspective

- Set of agents $A = \{A_1, \ldots, A_m\}$
- Partition of global graph $G = (V,E)$ into disjoint subgraphs $G_i = (V_i, E_i)$
  - Cross-agent edges allowed
  - $\chi_i = \chi(G_i)$
  - $\alpha(G_i)$ = size of maximal independent set on G_i
  - For set S, $c(S) = \{c(x) \mid x \text{ in } S\}$

Coloring from an Economics Perspective

- ***Globally optimal coloring*** c is ***individually rational*** if $|c(V_i)| = \chi_i$, for all A_i in A
- $\chi(G) = \chi_1 + \ldots + \chi_m$?
  - K_n : Yes
  - In general: Combining individual assignments ***not*** globally optimal!

Coloring from an Economics Perspective

- Partition $\boldsymbol{P = \{G_1,\ldots,G_r\}}$ is ***individually rational*** if it has individually rational coloring $\boldsymbol{c}$ with $|\boldsymbol{c(V)}| = \boldsymbol{\chi(G)}$
- Otherwise $\boldsymbol{P}$ is ***irrational***
  - ***Degree of irrational partition***: $\boldsymbol{d} = \min_c \max_{A_i} \{|\boldsymbol{c(V_i)}| - \chi_i\}$
- Agents have ***linear orderings*** on color set $\boldsymbol{L}$
  - Bijection $<_i : \{1,\ldots|L|\} \rightarrow L$
  - $<_i(j)$ is the j^{th} favorite color of player A_i

Coloring from an Economics Perspective

- ***Loss function*** L_i for each agent A_i
  - Cost of using color *s*: $\mathbf{L_i(s) := <_i^{-1}(s)}$
  - Total cost of coloring *c*: $L_i^c := \sum_{s \text{ in } c(V_i)} L_i(s)$
- ***Private/Public*** models
  - Public: Identical color preferences
- ***Pareto Optimal*** coloring
  - No other coloring can improve one agent without degrading others

Coloring from an Economics Perspective

- ***L_i^* : minimum loss*** for A_i (over all colorings)
- ***Coordination cost***: $D_i^c = L_i^c - L_i^*$
- Average coordination cost:
  - $\mu_c = 1/|A| * (\sum_{Ai} D_i^c)$
- ***Socially Fair*** coloring f minimizes variance:
  - $f = \arg\min_{c:c(V) = \chi} \{ 1/|A| * \sum_{Ai} (D_i^c - \mu_c)^2 \}$

Pareto Efficient Colorings

- K_n graph
  - Public preference model: Every proper coloring is Pareto optimal
  - Private: No! Consider K_2 with {Red, Blue}
- Two players (each may have multiple colors)
- ***Partial order*** $\leq^*$ with respect to set of colors L
  - Label R $\leq^*$ B if both agents prefer color R to B
  - R $\parallel$ B if incomparable

Pareto Efficient Colorings

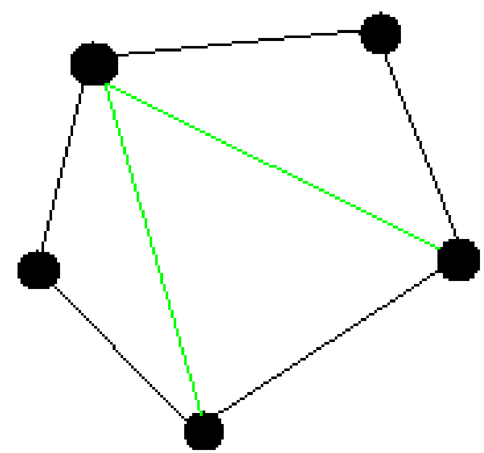
- Subset $\boldsymbol{S}$ is a ***minimum*** with respect to set of colors $\boldsymbol{L}$ if:
  - For all labels R, B, where R in $\boldsymbol{L} \backslash \boldsymbol{S}$, B in $\boldsymbol{S} \rightarrow$ B $\leq^*$ R or B || R
- *[Lemma 1] Two agent, individually rational coloring c is Pareto optimal ⇔ (1) minimality (preserves optimality) and (2) condition to prevent opposite allocations*

Socially Fair Colorings

- *Observation*: If P is an individually rational partition, then $L_i^* = \frac{1}{2} * \chi_i(\chi_i+1)$
- *[Lemma 2] Socially fair colorings of* K_n*, for public preference models, can be constructed*

Partitions of Cycles: Individual Rationality

- ***Chordal (triangulated) graph***: Each cycle of length greater than 3 has a chord



- If C_n is a chordless cycle
  - $X(C_n) = 2$ if n even
  - $X(C_n) = 3$ otherwise
- ***Floating Agent***: vertices form an independent set

Partitions of Cycles

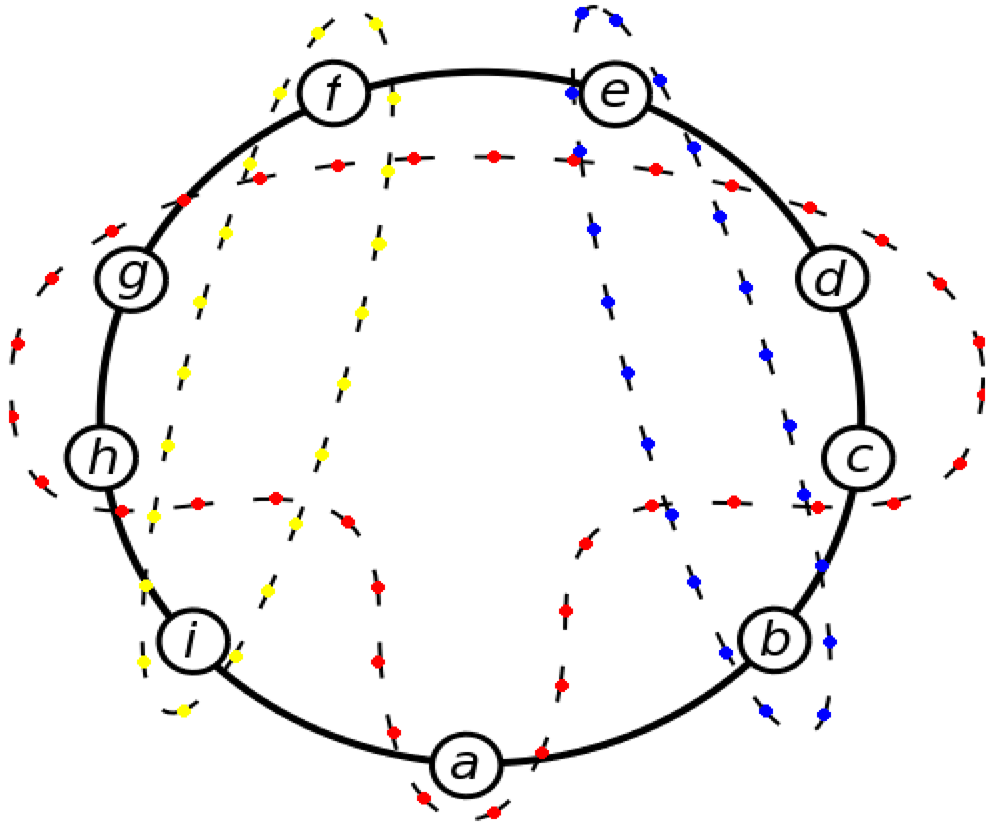
- *[Lemma 3] A partition **P** of an even cycle is individually rational ⇔ **P** does not include floating agents with vertices of different parity*
  - Note: Even cycles have ***2*** optimal colorings
  - Similarly on trees: ***P*** is individually rational ⇔ no floating agent has vertices at both even and odd levels of the tree
  - Generally: Uniquely colorable graphs

Partitions of Cycles

- *[Lemma 4] A two agent partition of an odd cycle is individually rational (3 coloring)*
  - If agent A_1 is floating: Red for A_1, {Blue, Yellow} for A_2
  - Else: remove an edge, tie back cycle, induct
  - Does not hold for three agents ☹

Partitions of Cycles

- Irrational partition of C_9. $A_1=\{a,c,d,g,h\}$, $A_2=\{b,e\}$, $A_3=\{f,i\}$. A_2 and A_3 are floating



Partitions of Cycles

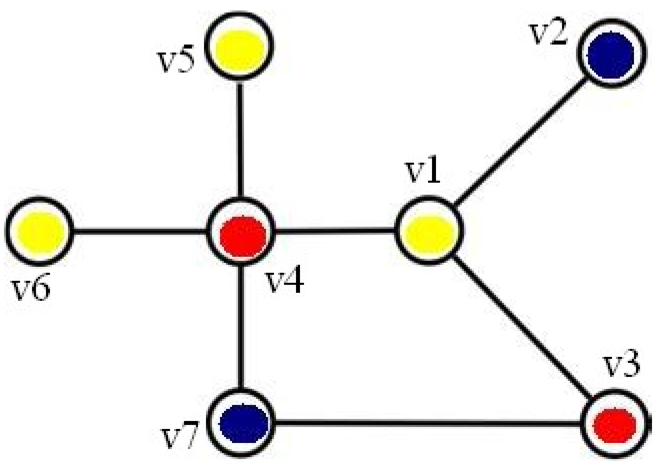
- *[Theorem 6] Determining whether a partition of an odd cycle is individually rational is NP-complete*
  - Reduction from **3-COLORING**
  - Input graph <***G***>: Find ***Odd cycle*** and ***Partition P***
  - Expand every vertex v_i to path P_i of length $2*d(v_i)$
  - Connect P_i and P_{i+1} with intermediate vertex s_i
  - Additional path P_{n+1} to connect P_1 and P_n (1 or 2 vertices)

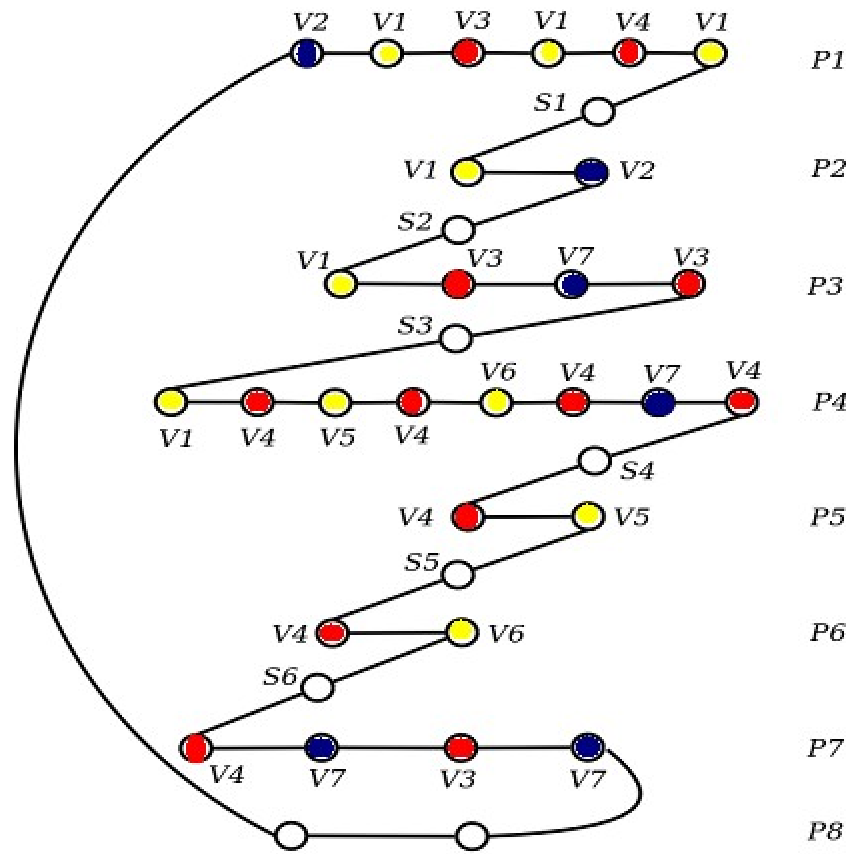
Partitions of Cycles

- Construct partition of the odd cycle: $\boldsymbol{P = \{G_1, \ldots, G_{2n}\}}$
  - If $\boldsymbol{i}$ odd: $\boldsymbol{G_i}$ = {Even nodes of $\boldsymbol{P_{(i+1)/2}}$} + {Odd vertex $\boldsymbol{v_j}$ on every $\boldsymbol{P_j}$ path | $\boldsymbol{(v_i v_j)}$ in $\boldsymbol{E}$}
  - If $\boldsymbol{i}$ even and $\boldsymbol{i \leq 2n-2}$: $\boldsymbol{G_i = \{s_{i/2}\}}$
  - Else $\boldsymbol{G_{2n} = P_{n+1}}$

Partitions of Cycles

- 3-COLORING Instance → Global Cycle

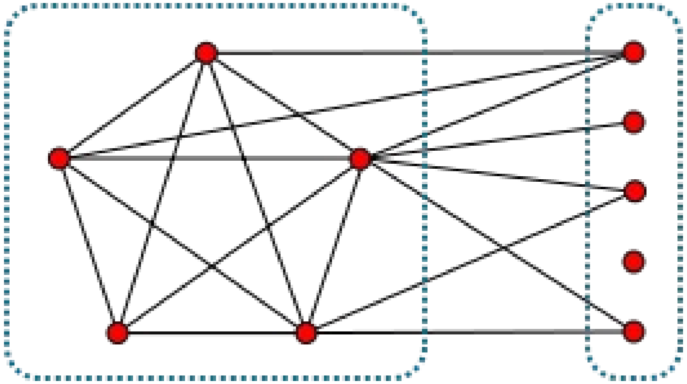




Partitions of Cycles

- *[Lemma 7] If **P** is a partition of the odd cycle, then **P** lacks individuality of degree at most 1*
- Undirected graph ***G = (V,E)*** is a ***split graph*** if

V = {I, K} : ***I*** independent set, ***K*** complete graph

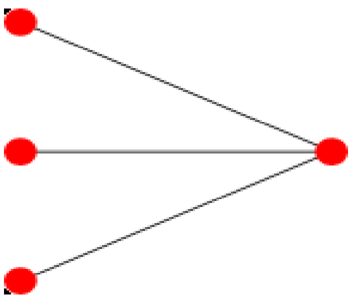


Split Graphs

- Recognizing split graphs: $\boldsymbol{O(|V|)}$
  - Sort vertex degrees $d_1 \geq \ldots \geq d_n$
  - Let q = greatest index with $d_q \geq q-1$
  - G is split $\Leftrightarrow \sum_{i=1,q} d_i = q(q-1) + \sum_{i=q+1,n} d_i$
- Partition ***{I, K}*** not unique: a-b-c
- Canonical partition: $\boldsymbol{O(|V|)}$

Split Graphs

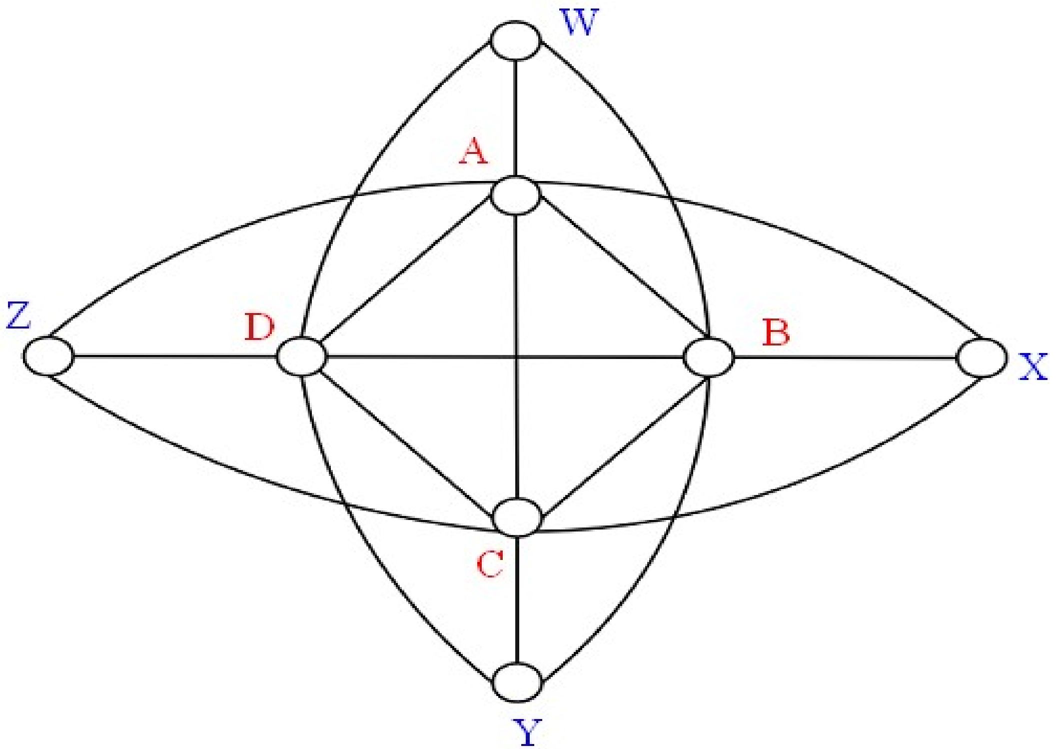
- ***Perfect graph***: For every ***induced subgraph***, the chromatic number equals the size of the largest clique
  - Example: the ***Claw graph***:



- Split graphs are perfect
  - Chromatic number equals size of max clique
  - Non-constant chromatic number → unbounded lack of individuality

Split Graphs

- ***Lilium***: split graph with irrational partition. $A_1 = \{A\text{-}D\}$, $A_2 = \{W\text{-}Z\}$. A_1 needs $|V| - 1$ additional colors to coordinate with A_2



Split Graphs

- Consider:
  - Vertices in the maximal independent set ***and*** in A_j's subgraph: $I^{(j)} := V_j \cap I$
  - Vertices in the clique ***and*** in A_j's subgraph: $K^{(j)} := V_j \cap K$
- Notation:
  - $N(H)$ = neighbors of set H

Split Graphs

- ***Boundary*** of the agent: $\boldsymbol{B_j := \{v \text{ in } I^{(j)} \mid K^{(j)}}$ in $\boldsymbol{\{v\} + N(\{v\})\}}$
  - ***Border vertices***
- ***Cooperative vertex*** (with respect to agent $\boldsymbol{A_j}$)
  - Not included in the neighborhood of the boundary ($\boldsymbol{N(B_j)}$)

Split Graphs

- *[Theorem 8] A partition of a global split graph is individually rational ⇔ every agent has a cooperative vertex*
  - Start with canonical partition $\{I,K\}$
  - Construct sets $\{I^{(j)}, K^{(j)}\}$
  - Pass over every vertex v in $I^{(j)}$: Is $K^{(j)}$ included in $N(v)$?
  - Find boundary sets $B_j \rightarrow$ cooperative vertices
  - Runtime: $O(|V|+|E|)$

Split Graphs

- If every agent has cooperative vertex
  - Can construct individually rational coloring
  - Start from arbitrary coloring of $\boldsymbol{K}$
  - Color non-border vertices with one of the free colors in $\boldsymbol{K}^{(j)}$
  - Match color of boundary with that of cooperative vertex
- Otherwise: irrational partition

Conclusions

- Framework for multiagent graph coloring
  - Conditions for Pareto Optimality
  - Procedure to ensure fairness
- Individually rational colorings hard to find
  - Faster on split graphs
- Future work
  - Individual rationality on other graphs
  - Graph coloring heuristics

Thank you!

- Questions?