Competitive Auction

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Introduction

- We study a class of single-round, sealed-bid auctions for an item in unlimited supply.
- Set optimal single price profit as a benchmark
- The approach is motivated by competitive analysis of on-line algorithms.

Preliminaries and notation

- *Single-round sealed-bid auction*, A
  - **b** the vector of all submitted bids(maximum amount)

    The *i*th component of b is b_i , the bid submitted by bidder *i*
  - *allocation,* x= ($x_1 , \dots , x_n$)

    price, p=($p_1 , \dots , p_n$)
  - *profit* of the auction A(b)= $\sum_i p_i$

We make the following assumptions about the bidders:

- Each bidder has a private *valuation*,
- Each bidder bids so as to maximize their *utility*,
- Bidders bid with full knowledge of the auctioneer's mechanism.
- Bidders do not collude.
- The bidders in the auction are indistinguishable from the perspective of the auctioneer.

Bid-independent Auction, $A_f(\mathrm{b})$

- $t_i \leftarrow f(b_{-i})$
- If $t_i \leq b_i$, set $x_i \leftarrow 1$ and $p_i \leftarrow t_i$
- Otherwise, set $x_i = p_i = 0$

A *deterministic auction* is *truthful* if and only if it is equivalent to a *deterministic bid-independent auction*.

*Same applied to *randomized auction*

Optimal omniscient auctions

- *Optimal single price omniscient auction*, F

$$F(b) = \max_{1 \leq i \leq n} i z_i$$

- *Optimal Multiple price omniscient auction,* T

$$T(b) = \sum_{1 \leq i \leq n} b_i$$

Competitive analysis

- Let **b** be a vector of bids. Denote by opt(**b**) the sale price for b that gives the optimal profit

$$\text{opt(b)} = \underset{z_i}{argmax}\, iz_i$$

- The *Deterministic Optimal Price* (DOP) auction is defined by the bid independent function f

$$f(b_{-i}) = opt(b_{-i})$$

Competitive analysis

- DOP is a profit maximizing auction for a large class of distributions over bid vectors.
- Easy to exhibit classes of bid vectors where DOP's profit is very far from optimal

Competitive auction framework

- *There exist bid vectors* **b** *for which*

  $$\mathcal{F}(\mathbf{b}) = \Theta\big(\mathcal{T}(\mathbf{b}) / \ln n\big).$$

  Moreover, for all bid vectors **b**

  $$\mathcal{F}(\mathbf{b}) \geqslant \mathcal{T}(\mathbf{b}) / \ln n.$$

- *For any truthful auction* $\mathcal{A}_f$ *and any* $\beta \geqslant 1$*, there is a bid vector b such that the expected profit of* $\mathcal{A}_f$ *on b is less than* $\mathcal{F}(\mathbf{b})/\beta$.

- *Optimal single price omniscient auction that sells at least two units,* $\mathcal{F}^{(2)}$

$$\mathcal{F}^{(2)}(\mathbf{b}) = \max_{2 \leqslant k \leqslant n} k z_k .$$

- *m-optimal single price omniscient auction,* $\mathcal{F}^{(m)}$

$$\mathcal{F}^{(m)}(\mathbf{b}) = \max_{m \leqslant k \leqslant n} k z_k .$$

- We say that auction $\mathcal{A}$ is *β-competitive against* $\mathcal{F}^{(m)}$ if for all bid vectors **b**, the expected profit of $\mathcal{A}$ on **b** satisfies

$$\mathbf{E}\left[\mathcal{A}(\mathbf{b})\right] \geqslant \frac{\mathcal{F}^{(m)}(\mathbf{b})}{\beta}.$$

- *β as the competitive ratio of* $\mathcal{A}$.
- $\mathcal{F}^{(2)}$ is the strongest omniscient auction that we will be able to feasibly compete with.

Deterministic auctions are not competitive

Let $\mathcal{A}_f$ *be any symmetric deterministic auction defined by bid-independent function f. Then* $\mathcal{A}_f$ *is not competitive: for any* $1 \leqslant m \leqslant n$ *there exists a bid vector b of length n such that the profit on b is at most* $\mathcal{F}^{(m)}(\mathbf{b})\frac{m}{n}$.

A lower bound on the competitive ratio

We show that for any randomized truthful auction $\mathcal{A}$, there exists an input bid vector b on which

$$\mathbf{E}[\mathcal{A}(\mathbf{b})] \leqslant \frac{\mathcal{F}^{(2)}(\mathbf{b})}{2.42}.$$

Competitive auctions via random sampling

- The Random Sampling Optimal Price auction(RSOP) is:
  (1)Partition bids b uniformly at random into two sets: b', b''

  (2)Let p'=opt(b') and p''=opt(b'')

  (3)Use p' as a take-it-or-leave-it offer for all bids in b''

  (4)Use p'' as a take-it-or-leave-it offer for all bids in b'

- The RSOP auction is truthful.
- RSOP *is constant competitive against* $\mathcal{F}^{(2)}$.
- *Let* b *be any bounded-range bid vector, i.e., any bid vector of n bids with* $b_i \in [1, h]$ *for all i. Then*

$$\lim_{n \to \infty} \max_{\mathbf{b}} \frac{\mathcal{F}(\mathbf{b})}{\text{RSOP}(\mathbf{b})} = 1.$$

- Random Sampling Profit Extraction auction (RSPE) is:
  - (1)Partition bids b uniformly at random into two sets: b' and b''.
  - (2)Compute F'=$\mathcal{F}$ (b') and F''= $\mathcal{F}$(b'').
  - (3)Compute the auction results by running $ProfitExtract_{F''}$ on b' and $ProfitExtract_{F'}$ on b''
  - * $ProfitExtract_R$, given target profit R, is defined:
    - (1)Find the largest k such that highest k bidders' bids are at least R/k
    - (2)Charge these k bidders R/k and reject all others.

- RSPE is truthful.
- RSPE is 4-competitive

Limited supply

- *The limited supply version of* RSOP *is constant competitive against* $\mathcal{F}^{(2,k)}$.
- *The limited supply version of* RSPE *is 4-competitive against* $\mathcal{F}^{(2,k)}$.

Better than F?

- An auction is monotone if for any pair of bidders i and j with $b_i \leqslant b_j$,we have:

$$\forall x \leqslant b_i,\ \mathbf{Pr}[\text{bidder } i \text{ wins at price} \leqslant x] \leqslant \mathbf{Pr}[\text{bidder } j \text{ wins at price} \leqslant x].$$

- Let $\mathcal{A}$ be any monotone(truthful) randomized auction. For all bid vectors b, the revenue $R = \mathcal{A}(\mathbf{b})$ of $\mathcal{A}$ on input b satisfies $\mathbf{E}[R] \leqslant \mathcal{F}(\mathbf{b})$.
- No monotone auction can achieve an expected profit higher than F on any input.

Conclusions and future works

- Lower bound 13/6-competitive three bidder auction.
- Best possible bound on RSOP's competitive ratio?

- Some of my thoughts about this paper:
  - All the proofs are beautiful
  - Are RSOP and RSPE good examples?

Discussion

Questions and Comments