CS 886: Multiagent Systems
Normal Form Games

Kate Larson
Cheriton School of Computer Science
University of Waterloo

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Outline

1. Self-Interested Agents
2. What is Game Theory?
3. Quick Utility Theory Review
4. Normal Form Games
   - Nash Equilibria
Self-Interested Agents

We are interested in **self-interested** agents.

It does not mean that

- they want to harm other agents
- they only care about things that benefit them

It means that

- the agent has its *own* description of states of the world that it likes, and that its actions are motivated by this description
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What is game theory?

The study of games!

- Bluffing in poker
- What move to make in chess
- How to play Rock-Scissors-Paper

Also study of auction design, strategic deterrence, election laws, coaching decisions, routing protocols,...
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What is game theory?

Game theory is a formal way to analyze **interactions** among a **group** of **rational** agents who behave **strategically**.

**Group:** Must have more than one decision maker
- Otherwise you have a decision problem, not a game

Solitaire is not a game.
What is game theory?

Game theory is a formal way to analyze interactions among a group of rational agents who behave strategically.

**Interaction:** What one agent does directly affects at least one other agent

**Strategic:** Agents take into account that their actions influence the game

**Rational:** An agent chooses its best action (maximizes its expected utility)
Example

Pretend that the entire class is going to go for lunch:

1. Everyone pays their own bill
2. Before ordering, everyone agrees to split the bill equally

Which situation is a game?
Impact

Influential in a variety of fields, including

- economics
- political science
- linguistics
- psychology
- biology
- computer science
- ...

2 branches

- Non-cooperative: basic unit is the individual
- Cooperative: basic unit is the group
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Preferences and Utility

Agents have preferences over outcomes ($A \succ B$, $B \succ A$, $A \sim B$).
Agents can also have preferences over lotteries with possible outcomes $C_1, \ldots, C_n$

$$L = [p_1 : C_1, \ldots, p_n : C_n]$$
Properties (Axioms)

- Orderability
- Transitivity
- Continuity

\[ A \succ B \succ C \Rightarrow \exists p [p : A, (1 - p) : C] \sim B \]

- Substitutability

\[ A \sim B \Rightarrow [p : A, (1 - p) : C] \sim [p : B, (1 - p) : C] \]

- Monotonicity

\[ A \succ B \Rightarrow (p \geq q \iff [p : A, (1 - p) : B] \succeq [q : A, (1 - q) : B]) \]

- Decomposability

Utility Principle

**Theorem (Utility Principle)**

*If the axioms are followed then there exists a function* $U : O \rightarrow \mathbb{R}$ *such that* $\forall A, B \in O$

$$U(A) > U(B) \iff A \succ B$$

$$U(A) = U(B) \iff A \sim B.$$  

**Maximum Expected Utility**: Rational choice – select lottery $L^*$ such that

$$L^* = \arg \max_L \sum_i p_i U_i(C_i)$$
Utility

- The “units” do not matter
- Affine transformations do not really change anything;

\[ U'(o) = aU(o) + b \]

will result in the same decision.

Note: Risk attitudes are important.
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**Note:** Risk attitudes are important.
An Example
A normal form game is defined by

- Finite set of agents (or players) $N$, $|N| = n$
- Each agent $i$ has an action space $A_i$
  - $A_i$ is non-empty and finite
- Outcomes are defined by action profiles $(a = (a_1, \ldots, a_n))$ where $a_i$ is the action taken by agent $i$
- Each agent has a utility function $u_i : A_1 \times \ldots \times A_n \mapsto \mathbb{R}$
Normal Form

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Examples

Prisoners’ Dilemma

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\[ c > a > d > b \]

Pure coordination game

\[
\forall \text{ action profiles } \\
a \in A_1 \times \ldots \times A_n \text{ and } \forall i, j, \\
u_i(a) = u_j(a).
\]

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Agents do not have conflicting interests. Their sole challenge is to coordinate on an action which is good for all.
Zero-sum games

$$\forall a \in A_1 \times A_2, \ u_1(a) + u_2(a) = 0.$$ That is, one player gains at the other player's expense.

Matching Pennies

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Given the utility of one agent, the other's utility is known.
More Examples

Most games have elements of both cooperation and competition.

**BoS**

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**Hawk-Dove**

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Strategies

**Notation:** Given set $X$, let $\Delta X$ be the set of all probability distributions over $X$.

**Definition**

*Given a normal form game, the set of mixed strategies for agent $i$ is*

$$S_i = \Delta A_i$$

The set of mixed strategy profiles is $S = S_1 \times \ldots \times S_n$.

**Definition**

*A strategy $s_i$ is a probability distribution over $A_i$. $s_i(a_i)$ is the probability action $a_i$ will be played by mixed strategy $s_i$.*
Strategies

Definition

The support of a mixed strategy $s_i$ is

$$\{a_i | s_i(a_i) > 0\}$$

Definition

A pure strategy $s_i$ is a strategy such that the support has size 1, i.e.

$$|\{a_i | s_i(a_i) > 0\}| = 1$$

A pure strategy plays a single action with probability 1.
Expected Utility

The expected utility of agent $i$ given strategy profile $s$ is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a_j)$$

Example

Given strategy profile

$s = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{10}, \frac{9}{10}))$

$$u_1 = -1\left(\frac{1}{2}\right)\left(\frac{1}{10}\right) - 4\left(\frac{1}{2}\right)\left(\frac{9}{10}\right) - 3\left(\frac{1}{2}\right)\left(\frac{9}{10}\right) = -3.2$$

$$u_2 = -1\left(\frac{1}{2}\right)\left(\frac{1}{10}\right) - 4\left(\frac{1}{2}\right)\left(\frac{1}{10}\right) - 3\left(\frac{1}{2}\right)\left(\frac{9}{10}\right) = -1.6$$
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Best-response

Given a game, what strategy should an agent choose? We first consider only pure strategies.

**Definition**

Given $a_{-i}$, the best-response for agent $i$ is $a_i \in A_i$ such that

$$u_i(a_i^*, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \in A_i$$

Note that the best response may not be unique. A best-response set is

$$B_i(a_{-i}) = \{ a_i \in A_i | u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \forall a_i' \in A_i \}$$
Nash Equilibrium

Definition

A profile $a^*$ is a Nash equilibrium if $\forall i$, $a^*_i$ is a best response to $a^*_{-i}$. That is

$$\forall i u_i(a^*_i, a^*_{-i}) \geq u_i(a'_i, a^*_{-i}) \forall a'_i \in A_i$$

Equivalently, $a^*$ is a Nash equilibrium if $\forall i$

$$a^*_i \in B(a^*_{-i})$$
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Nash Equilibria

We need to extend the definition of a Nash equilibrium. Strategy profile $s^*$ is a Nash equilibrium is for all $i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$$

$$\forall s_i' \in S_i$$

Similarly, a best-response set is

$$B(s_{-i}) = \{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \forall s_i' \in S_i\}$$
Examples
Characterization of Mixed Nash Equilibria

$s^*$ is a (mixed) Nash equilibrium if and only if

- the expected payoff, given $s_{-i}^*$, to every action to which $s_i^*$ assigns positive probability is the same, and
- the expected payoff, given $s_{-i}^*$, to every action to which $s_i^*$ assigns zero probability is at most the expected payoff to any action to which $s_i^*$ assigns positive probability.
Existence

**Theorem (Nash, 1950)**

*Every finite normal form game has a Nash equilibrium.*

**Proof:** Beyond scope of course.

**Basic idea:** Define set $X$ to be all mixed strategy profiles. Show that it has nice properties (compact and convex). Define $f : X \mapsto 2^X$ to be the best-response set function, i.e. given $s$, $f(s)$ is the set all strategy profiles $s' = (s'_1, \ldots, s'_n)$ such that $s'_i$ is $i$’s best response to $s'_{-i}$. Show that $f$ satisfies required properties of a fixed point theorem (Kakutani’s or Brouwer’s). Then, $f$ has a fixed point, i.e. there exists $s$ such that $f(s) = s$. This $s$ is mutual best-response – NE!
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Interpretations of Nash Equilibria

- Consequence of rational inference
- Focal point
- Self-enforcing agreement
- Stable social convention
- ...

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