CS 886: Multiagent Systems

Introduction to Mechanism Design

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Outline

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Introduction

Game Theory

- Given a game we are able to analyse the strategies agents will follow

Social Choice

- Given a set of agents' preferences we can choose some outcome

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Today **Mechanism Design**

- Game Theory + Social Choice
- Goal of Mechanism Design is to
  - Obtain some outcome (function of agents' preferences)
  - But agents are rational
    - They may lie about their preferences

Goal

Define the rules of a game so that in equilibrium the agents do what we want.

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1. Introduction
   - Introduction
   - Fundamentals

2. Mechanisms
   - Mechanism Design Problem
   - Direct Mechanisms
   - Revelation Principle
   - Gibbard-Satterthwaite
   - Quasi-Linear Preferences
   - Groves Mechanisms

Fundamentals

- Set of possible outcomes O
- Set of agents N, $|N| = n$
  - Each agent i has type $\theta_i \in \Theta_i$
  - Type captures all private information that is relevent to the agent's decision making
- Utility $u_i(o, \theta_i)$ over outcome $o \in O$
- Recall: goal is to implement some system wide solution
  - Captured by a social choice function

$$f : \Theta_1 \times \ldots \times \Theta_n \to O$$

where $f(\theta_1, \ldots, \theta_n) = o$ is a collective choice

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Examples of Social Choice Functions

- **Voting:**
  - Choose a candidate among a group
- **Public project:**
  - Decide whether to build a swimming pool whose cost must be funded by the agents themselves
- **Allocation:**
  - Allocate a single, indivisible item to one agent in a group

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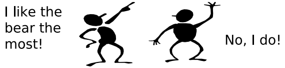
Mechanisms

Recall that we want to implement a social choice function

- Need to know agents' preferences
- They may not reveal them to us truthfully

Example:



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Mechanism Design Problem

- By having agents interact through an institution we might be able to solve the problem
- Mechanism:
$$M = (S_1, \ldots, S_n, g(\cdot))$$
where
  - S_i is the strategy space of agent i
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Implementation

Definition

A mechanism $M = (S_1, \ldots, S_n, g(\cdot))$ **implements** *social choice function* $f(\Theta)$ *if there is an equilibrium strategy profile*

$$s^* = (s_1^*(\theta_1, \ldots, s_n^*(\theta_n))$$

of the game induced by M such that

$$g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = f(\theta_1, \ldots, \theta_n)$$

for all

$$(\theta_1, \ldots, \theta_n) \in \Theta_1 \times \ldots \times \Theta_n$$

Implementation

We did not specify the type of equilibrium in the definition

- Nash

$$u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i) \geq u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)$$

$\forall i, \forall \theta_i, \forall s_i' \neq s_i^*$

- Bayes-Nash

$$E[u_i(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)] \geq E[u_i(g(s_i'(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i)]$$

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Properties for Mechanisms

- Efficiency
  - Select the outcome that maximizes total utility
- Fairness
  - Select outcome that minimizes the variance in utility
- Revenue maximization
  - Select outcome that maximizes revenue to a seller (or, utility to one of the agents)
- Budget-balanced
  - Implement outcomes that have balanced transfers across agents
- Pareto Optimal
  - Only implement outcomes o^* for which for all $o' \neq o^*$ either $u_i(o', \theta_i) = u_i(o^*, \theta_i) \forall i$ or $\exists i \in N$ with $u_i(o', \theta_i) < u_i(o^*, \theta_i)$

Participation Constraints

We can not force agents to participate in the mechanism. Let $\hat{u}_i(\theta_i)$ denote the (expected) utility to agent i with type θ_i of its outside option.

- **ex ante individual-rationality**: agents choose to participate before they know their own type

$$E_{\theta\in\Theta}[u_i(f(\theta),\theta_i)] \geq E_{\theta_i\in\Theta_i}\hat{u}_i(\theta_i)$$

- **interim individual-rationality**: agents can withdraw once they know their own type

$$E_{\theta_{-i}\in\Theta_{-i}}[u_i(f(\theta_i,\theta_{-i}),\theta_i)] \geq \hat{u}_i(\theta_i)$$

- **ex-post individual-rationality**: agents can withdraw from the mechanism at the end

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Direct Mechanisms

Definition

A **direct mechanism** *is a mechanism where*

$$S_i = \Theta_i \text{ for all } i$$

and

$$g(\theta) = f(\theta) \text{ for all } \theta \in \Theta_1 \times \ldots \times \Theta_n$$

Incentive Compatibility

Definition

A direct mechanism is **incentive compatible** *if it has an equilibrium* s^* *where*

$$s_i^*(\theta_i) = \theta_i$$

for all $\theta_i \in \Theta_i$ *and for all* i. *That is, truth-telling by all agents is an equilibrium.*

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A direct mechanism is **strategy-proof** *if it is incentive compatible and the equilibrium is a dominant strategy equilibrium.*

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Revelation Principle

Theorem

Suppose there exists a mechanism $M = (S_1, \ldots, S_n, g(\cdot))$ that implements social choice function f in dominant strategies. Then there is a direct strategy-proof mechanism M' which also implements f.
[Gibbard 73; Green & Laffont 77; Myerson 79]

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Revelation Principle: Proof

1. Construct mechanism $M = (S, g)$ that implements $f(\theta)$ in dominant strategies. Then $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$ where s^* is a dominant strategy equilibrium.
2. Construct direct mechanism $M' = (\Theta, f(\Theta))$.
3. By contradiction suppose

$$\exists \theta_i' \neq \theta_i \text{ s.t. } u_i(f(\theta_i', \theta_{-i}), \theta_i) > u_i(f(\theta_i, \theta_{-i}), \theta_i)$$

   for some $\theta_i' \neq \theta_i$, some θ_{-i}.
4. But, because $f(\theta) = g(s^*(\theta))$ this implies that

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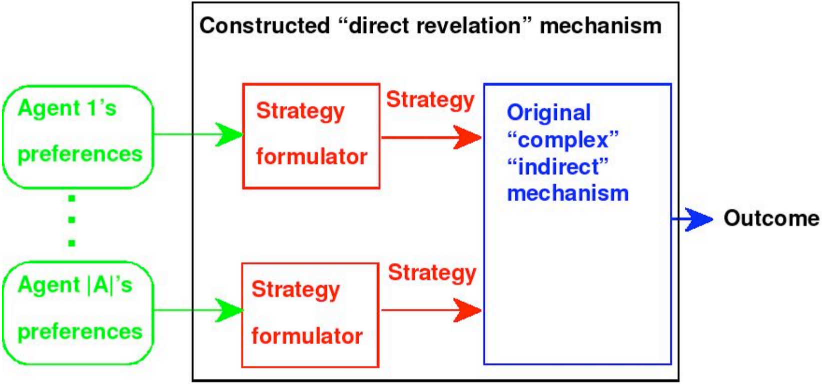
Revelation Principle: Intuition



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- **Literal interpretation:** Need only study direct mechanisms
  - A modeler can limit the search for an optimal mechanism to the class of direct IC mechanisms
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  - Useful because the space of possible mechanisms is huge

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Practical Implications

- Incentive-compatibility is "free"
  - Any outcome implemented by mechanism M can be implemented by incentive-compatible mechanism M'
- "Fancy" mechanisms are unneccessary
  - Any outcome implemented by a mechanism with complex strategy space S can be implemented by a direct mechanism

BUT Lots of mechanisms used in practice are not direct and incentive-compatible!

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Quick Review

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- What a mechanism is
- What it means for a SCF to be dominant-strategy implementable
- Revelation Principle

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- What types of SCF are dominant-strategy implementable

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Gibbard-Satterthwaite Impossibility

Theorem

Assume that

- *O is finite and $|O| \geq 3$,*
- *each $o \in O$ can be achieved by SCF f for some θ, and*
- *Θ includes all possible strict orderings over O.*

Then f is implementable in dominant strategies (strategy-proof) if and only if it is dictatorial.

Definition

SCF f is **dictatorial** *if there is an agent i such that for all θ*

$$f(\theta) \in \{o \in O | u_i(o, \theta_i) \geq u_i(o', \theta_i) \forall o' \in O\}$$

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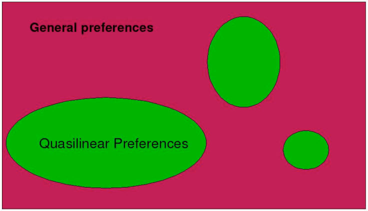
Circumventing Gibbard-Satterthwaite

- Use a weaker equilibrium concept
- Design mechanisms where computing a beneficial manipulation is hard
- Randomization
- Restrict the structure of agents' preferences



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Quasi-linear preferences

- Outcome $o = (x, t_1, \ldots, t_n)$
  - x is a "project choice"
  - $t_i \in \mathbb{R}$ are transfers (money)
- Utility function of agent i

$$u_i(o, \theta_i) = v_i(x, \theta_i) - t_i$$

- Quasi-linear mechanism

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where

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Social Choice Functions and Quasi-linearity

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$$\sum_{i=1}^{n} v_i(x(\theta), \theta_i) \geq \sum_{i=1}^{n} v_i(x'(\theta), \theta_i) \forall x'(\theta)$$

  This is also known as **social welfare maximizing**

- SCF is **budget-balanced** if

$$\sum_{i=1}^{n} t_i(\theta) = 0$$

  Weakly budget-balanced if

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A **Groves mechanism** $M = (S_1, \ldots, S_n, (x, t_1, \ldots, t_n))$ is defined by

- Choice rule

$$x^*(\theta) = \arg\max_x \sum_i v_i(x, \theta_i)$$

- Transfer rules

$$t_i(\theta) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(x^*(\theta), \theta_j)$$

where $h_i(\cdot)$ is an (arbitrary) function that does not depend on the reported type θ_i' of agent i.

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- Choice rule

$$x^*(\theta) = \arg\max_x \sum_i v_i(x, \theta_i)$$

- Transfer rules

$$t_i(\theta) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(x^*(\theta), \theta_j)$$

where $h_i(\cdot)$ is an (arbitrary) function that does not depend on the reported type θ_i' of agent i.

Groves Mechanisms

Theorem

Groves mechanisms are strategy-proof and efficient.

We have gotten around Gibbard-Satterthwaite.

Proof

Agent i's utility for strategy $\hat{\theta}_i$, given $\hat{\theta}_{-i}$ from agents $j \neq i$ is

$$\begin{aligned} u_i(\hat{\theta}_i) &= v_i(x^*(\hat{\theta}, \theta_i) - t_i(\hat{\theta}) \\ &= v_i(x^*(\hat{\theta}, \theta_i) + \sum_{j \neq i} v_j(x^*(\hat{\theta}, \hat{\theta}_j) - h_i(\hat{\theta}_{-i}) \end{aligned}$$

Ignore $h_i(\hat{\theta}_{-i})$ and notice $x^*(\hat{\theta}) = \arg\max_x \sum_i v_i(x, \hat{\theta}_i)$ i.e it maximizes the sum of reported values. Therefore, agent i should announce $\hat{\theta}_i = \theta_i$ to maximize its own payoff.

Thm: Groves mechanisms are unique (up to $h_i(\theta_{-i})$).

Vickrey-Clarke-Groves Mechanism

aka Clarke mechansism, aka Pivotal mechanism

- Implement efficient outcome

$$x^* = \arg\max_x \sum_i v_i(x, \theta_i)$$

- Compute transfers

$$t_i(\theta) = \sum_{j \neq i} v_j(x^{-i}, \theta_j) - \sum_{j \neq i} v_j(x^*, \theta_j)$$

where $x^{-i} = \arg\max_x \sum_{j \neq i} v_j(x, \theta_j)$

VCG are efficient and strategy-proof.

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VCG Mechanism

Agent's equilibrium utility is

$$\begin{aligned} u_i((x^*, t), \theta_i) &= v_i(x^*, \theta_i) - \left[\sum_{j \neq i} v_j(x^{-i}, \theta_j) - \sum_{j \neq i} v_j(x^*, \theta_j) \right] \\ &= \sum_{j=1}^{n} v_j(x^*, \theta_j) - \sum_{j \neq i} v_j(x^{-i}, \theta_j) \\ &= \text{marginal contribution to the welfare of the system} \end{aligned}$$

Example: Building a Pool

- Cost of building the pool is \$300
- If together all agents value the pool more than \$300 then it will be built
- Clarke Mechanism
  - Each agent announces v_i and if $\sum_i v_i \geq 300$ then it is built
  - Payments $t_i = \sum_{j \neq i} v_j(x^{-i}, v_j) - \sum_{j \neq i} v_j(x^*, v_j)$

Assume $v_1 = 50$, $v_2 = 50$, $v_3 = 250$. Clearly, the pool should be built.
Transfers: $t_1 = (250 + 50) - (250 + 50) = 0 = t_2$ and $t_3 = (0) - (100) = -100$. Note that it is not budget-balanced.

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Vickrey Auction

- Highest bidder gets the item and pays an amount equal to the second highest bid
- This is also a VCG mechanism
  - Allocation rule: get item if $b_i = \max_j[b_j]$
  - Every agent pays

$$t_i(v) = \sum_{j \neq i} v_j(x^{-i}, v_j) - \sum_{j \neq i} v_j(x^*, v_j)$$

Note that $\sum_{j \neq i} v_j(x^{-i}, v_j) = \max_{j \neq i} b_j$ and

$$\sum_{j \neq i} v_j(x^*, v_j) = \begin{cases} \max_{j \neq i}[b_j] & \text{if } i \text{ is not the higest bidder} \\ 0 & \text{if it is.} \end{cases}$$

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London Bus System[1]

- 5 million passengers daily
- 7500 buses
- 700 routes
- The system has been privatized since 1997 by using competitive tendering
- *Idea*: Run an auction to allocate routes to companies

[1] As of April 2004

Auction Protocol

- Let G be set of all routes, I be the set of bidders
- Agent i submits bid $v_i(S)$ for all bundles $S \subseteq G$
- Compute allocation S^* to maximize sum of reported bids

$$V^*(I) = \max_{(S_1,\ldots,S_n)} \sum_i v_i(S_i)$$

- Compute best allocation without each agent

$$V^*(I \setminus i) = \max_{(S_1,\ldots,S_n)} \sum_{j \neq i} v_j^*(S_j)$$

- Allocate each agent S_i^*, each agent pays

$$P(i) = v_i^*(S_i^*) - [V^*(I) - V^*(I \setminus i)]$$

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For Further Reading I

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