CS 886: Multiagent Systems

Extensive Form Games

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Outline

1. Perfect Information Games

2. Imperfect Information Games
   - Bayesian Games
Extensive Form Games
aka Dynamic Games, aka Tree-Form Games

- Extensive form games allows us to model situations where agents take actions over time
- Simplest type is the perfect information game
Perfect Information Game

Perfect Information Game: \( G = (N, A, H, Z, \alpha, \rho, \sigma, u) \)

- \( N \) is the player set \( |N| = n \)
- \( A = A_1 \times \ldots \times A_n \) is the action space
- \( H \) is the set of non-terminal choice nodes
- \( Z \) is the set of terminal nodes
- \( \alpha : H \rightarrow 2^A \) action function, assigns to a choice node a set of possible actions
- \( \rho : H \rightarrow N \) player function, assigns a player to each non-terminal node (player who gets to take an action)
- \( \sigma : H \times A \rightarrow H \cup Z \), successor function that maps choice nodes and an action to a new choice node or terminal node where
  \[ \forall h_1, h_2 \in H \text{ and } a_1, a_2 \in A \text{ if } h_1 \neq h_2 \text{ then } \sigma(h_1, a_1) \neq \sigma(h_2, a_2) \]
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Tree Representation

- The definition is really a tree description
- Each node is defined by its history (sequence of nodes leading from root to it)
- The descendents of a node are all choice and terminal nodes in the subtree rooted at the node.
Example

Sharing two items

Diagram:

1

(2,0) (1,1) (0,2)

2

y n y n y n

2,0 0,0 1,1 0,0 0,2 0,0
Strategies

- A strategy, $s_i$ of player $i$ is a function that assigns an action to each non-terminal history, at which the agent can move.
- **Outcome:** $o(s)$ of strategy profile $s$ is the terminal history that results when agents play $s$
- **Important:** The strategy definition requires a decision at each choice node, regardless of whether or not it is possible to reach that node given earlier moves.
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Example

Strategy sets for the agents

\[ S_1 = \{(A, G), (A, H), (B, G), (B, H)\} \]

\[ S_2 = \{(C, E), (C, F), (D, E), (D, F)\} \]
Example

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Example

We can transform an extensive form game into a normal form game.

<table>
<thead>
<tr>
<th></th>
<th>(C,E)</th>
<th>(C,F)</th>
<th>(D,E)</th>
<th>(D,F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A,G)</td>
<td>3,8</td>
<td>3,8</td>
<td>8,3</td>
<td>8,3</td>
</tr>
<tr>
<td>(A,H)</td>
<td>3,8</td>
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</tr>
<tr>
<td>(B,G)</td>
<td>5,5</td>
<td>2,10</td>
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Nash Equilibria

**Definition (Nash Equilibrium)**

*Strategy profile \( s^* \) is a Nash Equilibrium in a perfect information, extensive form game if for all \( i \)*

\[
u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \forall s_i'
\]

**Theorem**

Any perfect information game in extensive form has a pure strategy Nash equilibrium.

Intuition: Since players take turns, and everyone sees each move there is no reason to randomize.
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Example: Bay of Pigs

What are the NE?
Subgame Perfect Equilibrium

Nash Equilibrium can sometimes be too weak a solution concept.

**Definition (Subgame)**

Given a game $G$, the subgame of $G$ rooted at node $j$ is the restriction of $G$ to its descendents of $h$.

**Definition (Subgame perfect equilibrium)**

A strategy profile $s^*$ is a subgame perfect equilibrium if for all $i \in N$, and for all subgames of $G$, the restriction of $s^*$ to $G'$ ($G'$ is a subgame of $G$) is a Nash equilibrium in $G'$. That is

$$\forall i, \forall G', u_i(s_i^* | G', s_{-i}^* | G') \geq u_i(s_i' | G', s_{-i}^* | G') \forall s_i'$$
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$$\forall i, \forall G', u_i(s^*_i|_{G'}, s^*_{-i}|_{G'}) \geq u_i(s'_i|_{G'}, s^*_{-i}|_{G'}) \forall s'_i$$
Example: Bay of Pigs

What are the SPE?
Existence of SPE

**Theorem (Kuhn’s Thm)**

*Every finite extensive form game with perfect information has a SPE.*

You can find the SPE by backward induction.
- Identify equilibria in the bottom-most trees
- Work upwards
Existence of SPE

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You can find the SPE by backward induction.

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Centipede Game

1

D 1,0
A

2

D 0,2
A

1

D 3,1
A

2

D 2,4
A

1

D 5,3
A

4,6
Imperfect Information Games

Sometimes agents have not observed everything, or else can not remember what they have observed.

**Imperfect information games**: Choice nodes $H$ are partitioned into *information sets*.

- If two choice nodes are in the same information set, then the agent cannot distinguish between them.
- Actions available to an agent must be the same for all nodes in the same information set.
Example

Information sets for agent 1

\[ I_1 = \{\emptyset\}, \{(L, A), (L, B)\} \]

\[ I_2 = \{\{L\}\} \]
Example

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\[ I_1 = \{\{\emptyset\}, \{(L, A), (L, B)\}\} \]

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More Examples

Simultaneous Moves

- Perfect Information Games
- Imperfect Information Games
- Bayesian Games

Imperfect Recall
Strategies

- **Pure strategy**: a function that assigns an action in $A_i(l_i)$ to each information set $l_i \in \mathcal{I}_i$
- **Mixed strategy**: probability distribution over pure strategies
- **Behavioral strategy**: probability distribution over actions available to agent $i$ at each of its information sets (independent distributions)
Strategies

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Behavorial Strategies

Definition

Given extensive game $G$, a behavorial strategy for player $i$ specifies, for every $l_i \in \mathcal{I}_i$ and action $a_i \in A_i(l_i)$, a probability $\lambda_i(a_i, l_i) \geq 0$ with

$$\sum_{a_i \in A_i(l_i)} \lambda(a_i, l_i) = 1$$
Example

Mixed Strategy:
(0.4(A,G), 0.6(B,H))

Behavioral Strategy:
- Play A with probability 0.5
- Play G with probability 0.3
Mixed and Behavioral Strategies

In general you can not compare the two types of strategies.

But for games with perfect recall
- Any mixed strategy can be replaced with a behavioral strategy
- Any behavioral strategy can be replaced with a mixed strategy
Mixed Strategy:
(<0.3(A,L)>, <0.2(A,R)>,
<0.5(B,L)>)

Behavioral Strategy:

- At $I_1$: (0.5, 0.5)
- At $I_2$: (0.6, 0.4)
Outline

1. Perfect Information Games
2. Imperfect Information Games
   - Bayesian Games
Bayesian Games

So far we have assumed that all players know what game they are playing

- Number of players
- Actions available to each player
- Payoffs associated with strategy profiles

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Bayesian Games

There are different possible representations.

**Information Sets**

- $N$ set of agents
- $G$ set of games
  - Same strategy sets for each game and agent
- $\Pi(G)$ is the set of all probability distributions over $G$
  - $P(G) \in \Pi(G)$ common prior
- $I = (I_1, \ldots, I_n)$ are information sets (partitions over games)
Example
Extensive Form With Chance Moves
A special player, Nature, makes probabilistic moves.

Nature

0.6

0.4

1

2

U

D

U

D

2

2

2

2

L

R

L

R

L

R

L

R
Epistemic Types

Epistemic types captures uncertainty directly over a game’s utility functions.

- $N$ set of agents
- $A = (A_1, \ldots, A_n)$ actions for each agent
- $\Theta = \Theta_1 \times \ldots \times \Theta_n$ where $\Theta_i$ is *type space* of each agent
- $p : \Theta \rightarrow [0, 1]$ is common prior over types
- Each agent has utility function $u_i : A \times \Theta \rightarrow \mathbb{R}$
Example

BoS

- 2 agents
- $A_1 = A_2 = \{\text{soccer, hockey}\}$
- $\Theta = (\Theta_1, \Theta_2)$ where
  $\Theta_1 = \{H, S\}$, $\Theta_2 = \{H, S\}$
- Prior: $p_1(H) = 1$, $p_1(S) = \frac{2}{3}$, $p_2(H) = \frac{2}{3}$, $p_2(S) = \frac{1}{3}$

Utilities can be captured by matrix-form

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Strategies and Utility

- A strategy $s_i(\theta_i)$ is a mapping from $\Theta_i$ to $A_i$. It specifies what action (or what distribution of actions) to take for each type.

**Utility**: $u_i(s|\theta_i)$

- **ex-ante EU** (know nothing about types)

  $$EU = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s_i|\theta_i)$$

- **interim EU** (know own type)

  $$EU = EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \Pi_{j \in N} s_j(a_j, \theta_j) u_i(a, \theta_{-i}, \theta_i)$$

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- *ex-post* EU (know everyones type)
Example

- 2 firms, 1 and 2, competing to create some product.
- If one makes the product then it has to share with the other.
- Product development cost is $c \in (0, 1)$
- Benefit of having the product is known only to each firm
  - Type $\theta_i$ drawn uniformly from $[0, 1]$
  - Benefit of having product is $\theta_i^2$
Bayes Nash Equilibrium

**Definition (BNE)**

*Strategy profile $s^*$ is a Bayes Nash equilibrium if $\forall i, \forall \theta_i$

$$EU(s_i^*, s_{-i}^* | \theta_i) \geq EU(s_i^', s_{-i}^* | \theta_i) \forall s_i^' \neq s_i^*$$*
Example Continued

- Let \( s_i(\theta_i) = 1 \) if \( i \) develops product, and 0 otherwise.
- If \( i \) develops product

\[
u_i = \theta_i^2 - c
\]

If it does not then

\[
u_i = \theta_i^2 Pr(s_j(\theta_j) = 1)
\]

Thus, develop product if and only if

\[
\theta_i^2 - c \geq \theta_i^2 Pr(s_j(\theta_j) = 1) \Rightarrow \theta_i \geq \sqrt{\frac{c}{1 - Pr(s_j(\theta_j) = 1)}}
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  $$u_i = \theta_i^2 \Pr(s_j(\theta_j) = 1)$$

- Thus, develop product if and only if

  $$\theta_i^2 - c \geq \theta_i^2 \Pr(s_j(\theta_j) = 1) \Rightarrow \theta_i \geq \sqrt{\frac{c}{1 - \Pr(s_j(\theta_j) = 1)}}$$
Example Continued

Suppose $\hat{\theta}_1, \hat{\theta}_2 \in (0, 1)$ are cutoff values in BNE.

- If so, then $Pr(s_j(\theta_j) = 1) = 1 - \hat{\theta}_j$
- We must have
  \[ \hat{\theta}_i \geq \sqrt{\frac{c}{\hat{\theta}_j}} \Rightarrow \hat{\theta}_i^2 \hat{\theta}_j = c \]

  and

  \[ \hat{\theta}_j^2 \hat{\theta}_i = c \]

- Therefore
  \[ \hat{\theta}_i^2 \hat{\theta}_j = \hat{\theta}_j^2 \hat{\theta}_i \]

  and so

  \[ \hat{\theta}_i = \hat{\theta}_j = \theta^* = c^{\frac{1}{3}} \]
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