CS 886: Multiagent Sysytems
Assignment 1

October 20, 2008

Due Wednesday, October 8 in class. In this assignment you are expected to work individually. You may use any sources that you want, but you must cite them. You can email me or drop by my office if you have questions.

1. (10 points) Throughout this question, you may restrict your analysis to pure strategies.

   (a) Draw the normal form game of the game tree in Figure 1.
   I acccepted several answers to this. Let (x, y) denote that agent 1 will use action x if agent 1 takes action L and y if agent 2 takes action R.

	(U, U)	(U, D)	(D, U)	(D, D)
L	3, 3	3, 3	3, 3	3, 3
R	2, 7	4, 4	2, 7	4, 4

   (b) Name the dominant strategy equilibria, if there are any.
   There are no dominant strategy equilibria, since agent 1 does not have a dominant strategy.

   (c) Name the Nash equilibria of this game, if there are any.
   There are two Nash equilibria: (L, (U, U) and (L, (D, U)). Note that in marking this question, I used your normal form representation

   (d) Name the subgame perfect Nash equilibria in the game, if there are any.
   The subgame-perfect equilibrium is (L, (U, U))

   (e) Name the Pareto efficient outcomes of this game, if there are any.
   There are two Pareto optimal outcomes. Both (4, 4) and (2, 7) are Pareto optimal.

   (f) Name the social welfare maximizing outcomes of this game, if there are any.
   The social-welfare maximising outcome is (2,7). (I also accepted other answers iif people explicitly noted what social-welfare function they were using. In particular I accepted (4,4) if people noted that they were using the egalitarian social welfare function.)

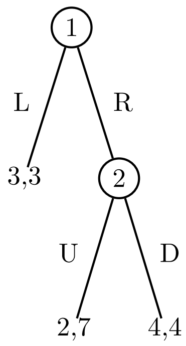

Figure 1:

2. *Game of Chicken* Two teenagers play the following risky game. They drive towards each other at stop speed in separate cars. Just before collision each one has the choice of continuing straight or avoiding collision by turning right. If both continue straight then they both die. If one continues straight while the other turns they both live, but the one who went straight gets boasting rights and the is humiliated. If both turn, then both survive and both are moderately humiliated. The game is represented in the table in Figure 2.

	straight	turn
straight	-3,-3	2,0
turn	0,2	1,1

Figure 2:

(a) (5 pts) Does this game have pure strategy Nash equilibria? If so, what are they?

There are two pure strategy Nash equilibria: (straight, turn) and (turn, straight)

(b) (13 pts) What are the mixed strategy Nash equilibria of this game?

Let p be the probability with which agent 1 goes straight, and let q be the probability that agent 2 goes straight. The mixed strategy equilibria is at $p = \frac{1}{4}$ and $q = \frac{1}{4}$.

(c) (2 pts) In each equilibrium, what is the probability that the teenagers will die?

The only time the teenagers will die is when they both go straight. In both pure strategy Nash equilbria this outcome never happens. Thus the probability is zero. In the mixed strategy Nash equilibria the probability is $pq = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

3. (10 pts) Agents 1 and 2 play Split-the-Dollar. Each agent simultaneously name shares that they want, where s_1 is agent 1's requested share and s_2 is agent 2's requested share. If $s_1 + s_2 \leq 1$ then both agent gets their requested share. If $s_1 + s_2 > 1$ then both agents get zero. What are the pure strategy Nash equilibria in this game?

   The pure strategy Nash equilibria take the form $s_1, s_2 \in [0, 1]$ such that $s_1 + s_2 = 1$. Note that if agent 1 announces s_1 then agent 2 maximises its utility when it announces $s_2 = 1 - s_1$. Similarly, if agent 2 announces $s_2 = 1 - s_1$ then agent 1 is best off announcing s_1. This holds for any value of $s_1, s_2 \in [0, 1]$.

4. An agent's strategy is *strictly dominated* if that agent has another strategy that gives strictly better payoff to the agent no matter what strategies other agents do. An agent's strategy is *weakly dominated* if that agent has another strategy that gives at least equally high payoff to the agent no matter what other agents do, and strictly higher payoff to the agent for at least one choice of strategies of by the others. To solve a game, we can iteratively eliminate dominated strategies until all remaining strategies are undominated.

   (a) (10 pts) Prove that if strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a Nash equilibrium in a normal form game, then they survive iterated elimination of strictly dominated strategies. (Hint: By contradiction, assume that one of the strategies in the Nash equilibrium is eliminated by iterated elimination of dominated strategies).
   Assume that s_i^* was eliminted by iterated elimination of dominated strategies. Therefore, there exists s_i' such that

$$u_i(s_i', s_{-i}) > u_i(s_i^*, s_{-i}), \forall s_{-i}. \tag{1}$$

   In particular, since constraint 1 holds for all s_{-i} it must also hold for s_{-i}^*. That is

$$u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*). \tag{2}$$

   Since $s^* = (s_i^*, s_{-i}^*)$ is a Nash equilibrium, then

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*). \tag{3}$$

   Contradiction.

   (b) (10 pts) Prove that if the process of iterated elimination of strictly dominated strategies results in a unique strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ then this is a Nash equilibrium of the game. (Hint: By contradiction, assume there exists some agent i for which $s_i \neq s_i^*$ is preferred over s_i^*, and show a contradiction with the fact that s_i was eliminated.)
   Assume that s^* is not a Nash equilibrium. Then for some agent i, there exists strategy $s_i \neq s_i^*$ such that

$$u_i(s_i, s_{-i}^*) \geq u_i(s_i^*, s_{-i}^*). \tag{4}$$

However, s_i was eliminated. Therefore, there exists strategy s'_i such that

$$u_i(s'_i, s^*_{-i}) > u_i(s_i, s^*_{-i}). \tag{5}$$

If $s'_i = s^*_i$ then we have

$$u_i(s^*_i, s^*_{-i}) > u_i(s_i, s^*_{-i}) > u_i(s^*_i, s^*_{-i}), \tag{6}$$

a constradiction, and so we are done.

If $s'_i \neq s^*_i$ then, since s'_i was elimintated, there must exist s''_i such that

$$u_i(s''_i, s^*_{-i}) > u_i(s'_i, s^*_{-i}). \tag{7}$$

If $s''_i = s^*_i$ then we have a contradiction.

If $s''_i \neq s^*_i$ then continue on as in the last step, finding the strategy that causes the elimination of s''_i. Since we are working with pure strategies, there is only a finite number of strategies, and so eventually we will reach a sitation where the dominating strategy is s^*_i which leads to the contradiction that

$$u_i(s^*_i, s^*_{-i}) > u_i(s^*_i, s^*_{-i}). \tag{8}$$

Thus, the assumption that s^* is not a Nash equilibrium is wrong.