**Theorem 1 (Arrow's Theorem)** *If there are three or more alternatives and a finite number of agents, then there is no protocol which satisfies universality, transitivity, Pareto efficiency, independence of irrelevant alternatives, and has no dictator.*

**Proof:**

Let $A$ denote the set of agents, and let $O$ denote the set of alternatives or outcomes. Let $x, y, z \in O$. Let $x >_i y$ denote the preference that agent $i$ prefers outcome $x$ to outcome $y$. Let $x > y$ denote the outcome of the SWF where $x$ is more preferred to $y$.

**Definition 1** *A set $S \subseteq A$ is* decisive *for $x$ over $y$ whenever*

- $x >_i y$ *for all* $i \in S$
- $x <_i y$ *for all* $i \in A \setminus S$

*then $x > y$.*

**Lemma 1** *If $S$ is decisive for $x$ over $y$, then for any other candidate $z$, $S$ is decisive for $x$ over $z$ and for $z$ over $y$.*

**Proof:** Let $S$ be decisive for $x$ over $y$. Consider $x >_i y >_i z$ for all $i \in S$ and $y >_i >_i z >_i x$ for all $i \in A \setminus S$. Since $S$ is decisive for $v$ over $y$ we have $x > y$. Because $y >_i z$ for every $i \in A$, by the Pareto property we have $y > z$. Then, by transitivity, $x > z$. By IIA $(y)$, $x > z$ whenever every agent in $S$ prefers $x$ to $z$ and every agent not in $S$ prefers $z$ to $x$. That is, $S$ is decisive for $x$ over $z$.

To show that $S$ is decisive for $z$ over $y$, consider $z >_i x >_i y$ for all $i \in S$ and $y >_i z >_i x$ for all agents in $A \setminus S$. Then $x > y$ since $S$ is decisive for $x$ over $y$. Also, $z > x$ by the Pareto property, and so $z > y$ from transitivity. Therefore, $S$ is decisive for $z$ over $y$. □

Given that $S$ is decisive for $x$ over $y$, we deduced that $S$ is decisive for $x$ over $z$ and $z$ over $y$. Now reply Lemma 1 with outcome $z$ over $y$ as the hypothesis and conclude that $S$ is decisive for $z$ over $x$ which implies (by Lemma 1) that $S$ is decisive for $y$ over $x$. This implies (by Lemma 1) that $S$ is decisive for $y$ over $z$. Therefore,

**Lemma 2** *If $S$ is decisive for $x$ over $y$ then for any candidates $u$ and $v \in O$, $S$ is decisive for $u$ over $v$. (i.e. $S$ is* decisive*).*

**Lemma 3** *For every $S \subseteq A$, either $S$ is decisive or $A \setminus S$ is decisive (not both).*

**Proof:** Suppose $x >_i y$ for all $i \in S$ and $y >_i x$ for all $i \in A \setminus S$ (note that only such cases need to be addressed, because otherwise the left side of the implication in the definition of decisiveness between candidates does not hold). Because either $x > y$ or $y > x$, $S$ is decisive or $A \setminus S$ is decisive. □

**Lemma 4** *If $S$ is decisive and $T$ is decisive then $S \cup T$ is decisive.*

**Proof:** Let

$$S = \{i : z >_i y >_i x\} \cup \{i : x >_i z >_i y\}$$

Let

$$T = \{i : y >_i x >_i z\} \cup \{i : x >_i z >_i y\}$$

For $i \notin S \cup T$ let $y >_i z >_i x$.

Now, since $S$ is decisive, $z > y$. Since $T$ is decisive $x > z$. Then, by transitivity, $x > y$. So, by IIA $(z)$, $S \cap T$ is decisive for $x$ over $y$. (Note that if $x >_i y$ then $i \in S \cap T$.) Thus, but Lemma 2, $S \cap T$ is decisive. □.

**Lemma 5** *If $S = S_1 \cup S_2$ (where $S_1$ and $S_2$ are disjoint and exhaustive) is decisive, then $S_i$ is decisive and $S_2$ is decisive.*

**Proof:** Suppose neither $S_1$ nor $S_2$ is decisive. Then $\sim S_1$ and $\sim S_2$ are decisive. By Lemma 4, $\sim S_1 \cap \sim S_2 = \ \sim S$ is decisive. But this is a contradiction since $S$ is decisive. $\Box$.

Now we are ready to prove Arrow's theorem.

Clearly the set of all agents is decisive. By Lemma 5 we can keep splitting a decisive set into two subsets, at least one of which is decisive. Keep on splitting the decisive set(s) further until only one agent remains in any decisive set. That agent is a dictator. $\Box$.