

CS 886: Electronic Market Design
Social Choice
(Preference Aggregation)

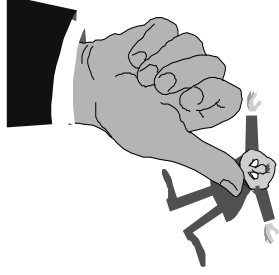
September 20

Social choice theory

- Study of decision problems in which a group has to make the decision
- The decision affects all members of the group
 - Their opinions! should count
- Applications:
 - Political elections
 - Other elections
 - Note that outcomes can be vectors
 - Allocation of money among agents, allocation of goods, tasks, resources...
- CS applications:
 - Multiagent planning [Ephrati&Rosenschein]
 - Computerized elections [Cranor&Cytron]
 - Note: this is not the same as electronic voting
 - Accepting a joint project, rating Web articles [Avery,Resnick&Zeckhauser]
 - Rating CDs...

Assumptions

1. Agents have preferences over alternatives
 - Agents can rank order the outcomes
 - $a > b > c = d$ is read as "a is preferred to b which is preferred to c which is equivalent to d"
2. Voters are sincere
 - They truthfully tell the center their preferences
3. Outcome is enforced on all agents



The problem

- Majority decision:
 - If more agents prefer a to b, then a should be chosen
- Two outcome setting is easy
 - Choose outcome with more votes!
- What happens if you have 3 or more possible outcomes?

Case 1: Agents specify their top preference

Ballot



Canadian Election System

- Plurality Voting
 - One name is ticked on a ballot
 - One round of voting
 - One candidate is chosen

Is this a "good"
system?

What do we mean by good?

Example: Plurality

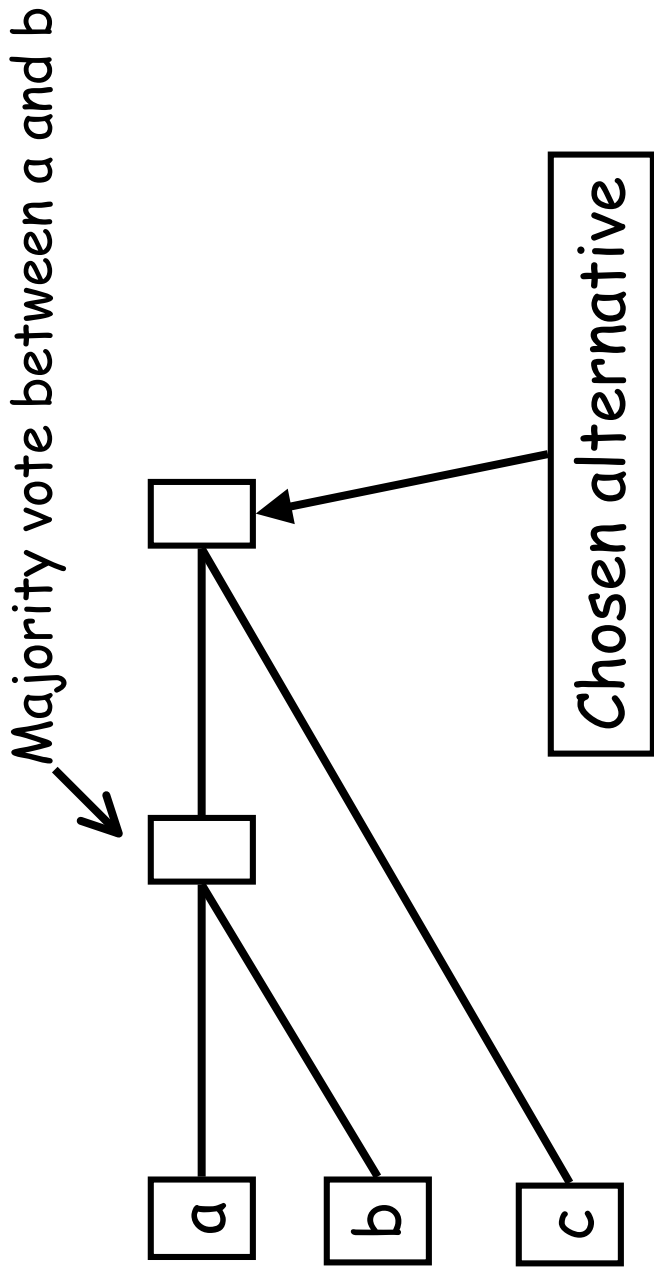
- 3 candidates
 - Lib, NDP, C
- 21 voters with the preferences
 - 10 Lib>NDP>C
 - 6 NDP>C>Lib
 - 5 C>NDP>Lib
- Result: Lib 10, NDP 6, C 5
 - But a majority of voters (11) prefer all other parties more than the Libs!

What can we do?

- Majority system
 - Works well when there are 2 alternatives
 - Not great when there are more than 2 choices
- Proposal:
 - Organize a series of votes between 2 alternatives at a time
 - How this is organized is called an agenda
 - Or a cup (often in sports)

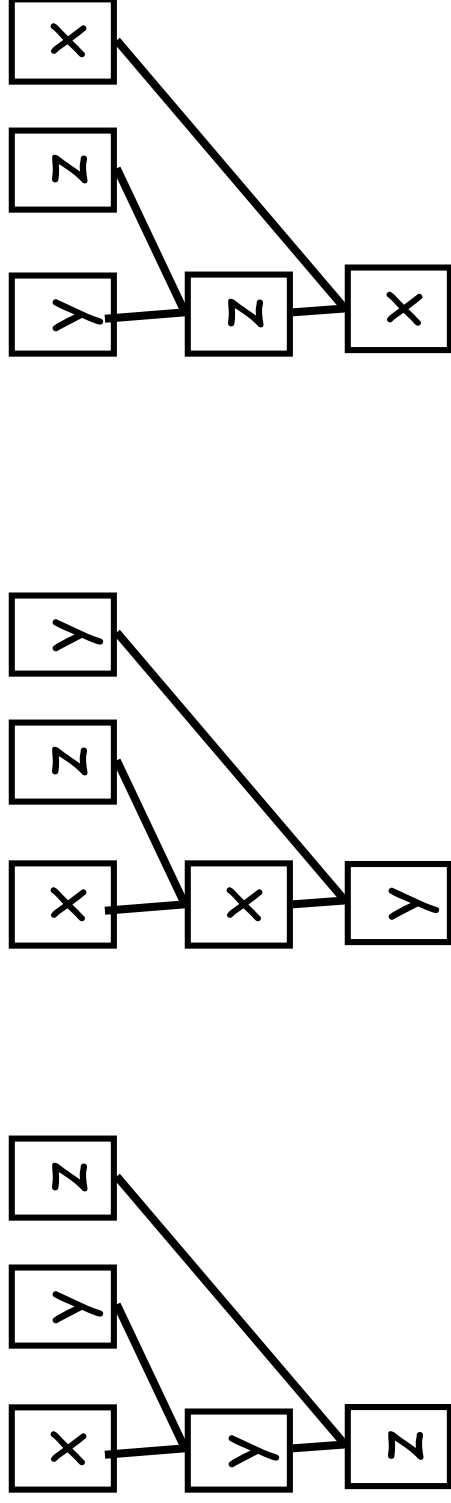
Agendas

- 3 alternatives {a,b,c}
- Agenda a,b,c



Agenda paradox

- Binary protocol (majority rule) = cup
- Three types of agents:
 1. $x > z > y$ (35%)
 2. $y > x > z$ (33%)
 3. $z > y > x$ (32%)

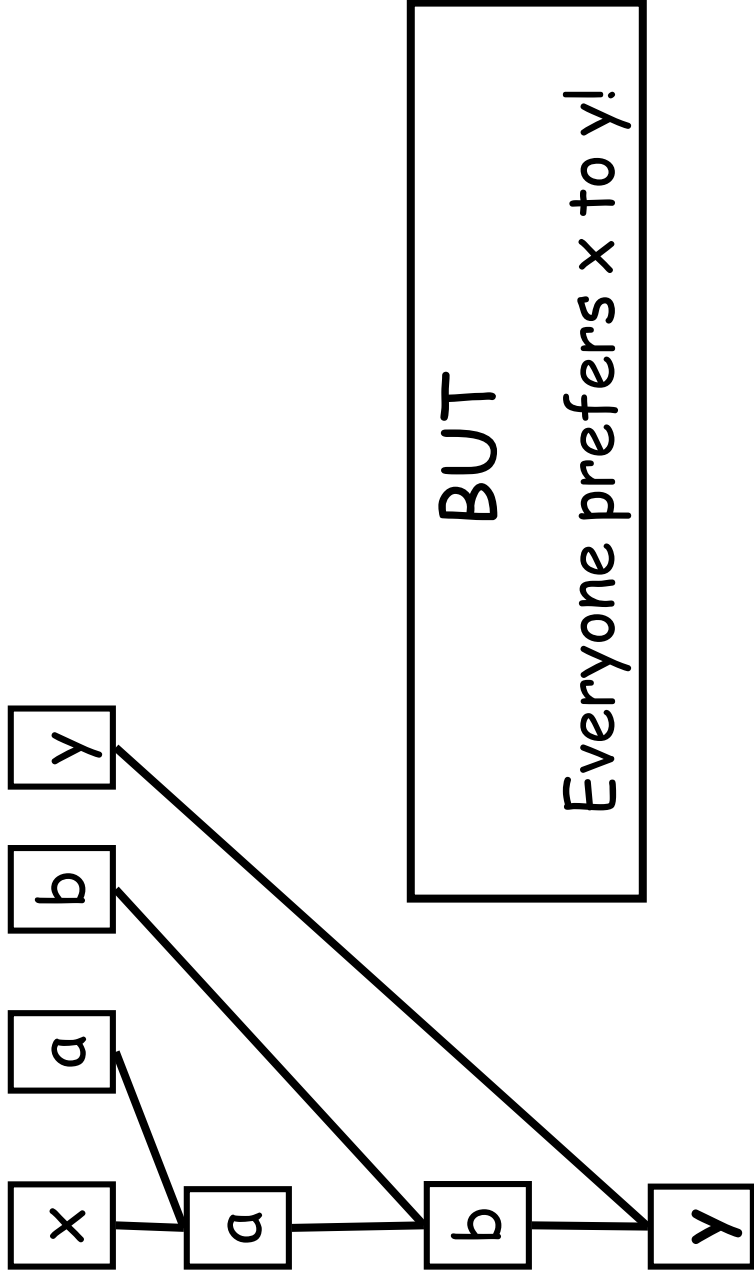


- Power of agenda setter (e.g. chairman)
- Vulnerable to irrelevant alternatives (z)

Another problem: Pareto dominated winner paradox

Agents:

1. $x > y > b > a$
2. $a > x > y > b$
3. $b > a > x > y$



Case 2: Agents specify their complete preferences

Maybe the problem was with the ballots!

Ballot

$X > Y > Z$



Now have more information

Condorcet

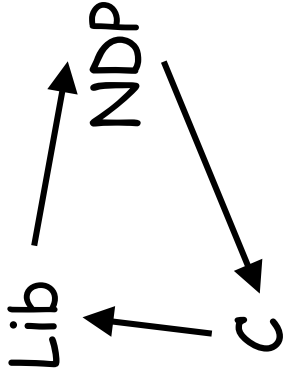
- Proposed the following
 - Compare each pair of alternatives
 - Declare "a" is socially preferred to "b" if more voters strictly prefer a to b
- Condorcet Principle: If one alternative is preferred to all other candidates then it should be selected

Example: Condorcet

- 3 candidates
 - Lib, NDP, C
- 21 voters with the preferences
 - 10 Lib>NDP>C
 - 6 NDP>C>Lib
 - 5 C>NDP>Lib
- Result:
 - NDP win! (11/21 prefer them to Lib, 16/21 prefer them to C)

A Problem

- 3 candidates
 - Lib, NDP, C
- 3 voters with the preferences
 - Lib>NDP>C
 - NDP>C>Lib
 - C>Lib>NDP
- Result:
 - No Condorcet Winner



Borda Count

- Each ballot is a list of ordered alternatives
- On each ballot compute the rank of each alternative
- Rank order alternatives based on decreasing sum of their ranks

A>B>C

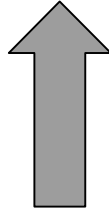
A>C>B

C>A>B

A: 4

B: 8

C: 6



Borda Count

- Simple
- Always a Borda Winner
- BUT does not always choose Condorcet winner!
- 3 voters
 - 2: $b > a > c > d$
 - 1: $a > c > d > b$

Borda scores:

a:5, b:6, c:8, d:11

Therefore a wins

BUT b is the
Condorcet winner

Inverted-order paradox

- Borda rule with 4 alternatives
 - Each agent gives 1 point to best option, 2 to second best...
- Agents:
 1. $x > c > b > a$
 2. $a > x > c > b$
 3. $b > a > x > c$
 4. $x > c > b > a$
 5. $a > x > c > b$
 6. $b > a > x > c$
 7. $x > c > b > a$
- $x=13, a=18, b=19, c=20$
- Remove x : $c=13, b=14, a=15$

Borda rule vulnerable to irrelevant alternatives

- Three types of agents:
 1. $x > z > y$ (35%)
 2. $y > x > z$ (33%)
 3. $z > y > x$ (32%)
- Borda winner is x
- Remove z : Borda winner is y

Desirable properties for a voting protocol

- **Universality**
 - It should work with any set of preferences
- **Transitivity**
 - It should produce an ordered list of alternatives
- **Paretian (or unanimity)**
 - If all agents prefer x to y then in the outcome x should be preferred to y
- **Independence**
 - The comparison of two alternatives should depend only on their standings among agents' preferences, not on the ranking of other alternatives
- **No dictators**

Arrow's Theorem (1951)

- If there are 3 or more alternatives and a finite number of agents then there is no protocol which satisfies the 5 desired properties

Is there anything that can be done?

- Can we relax the properties?
- No dictator
 - Fundamental for a voting protocol
- Paretian
 - Also seems to be pretty desirable
- Transitivity
 - Maybe you only need to know the top ranked alternative
 - Stronger form of Arrow's theorem says that you are still in trouble
- Independence
- Universality
 - Some hope here (1 dimensional preferences, spacial preferences)

Take-home Message

- Despair?
 - No ideal voting method
 - That would be boring!
- A group is more complex than an individual
- Weigh the pro's and con's of each system and understand the setting they will be used in
- Do not believe anyone who says they have the best voting system out there!

Proof of Arrow's theorem (slide 1 of 3)

- Follows [Mas-Colell, Whinston & Green, 1995]
- Assuming G is Paretian and independent of irrelevant alternatives, we show that G is dictatorial
- **Def.** Set $S \subseteq A$ is decisive for x over y whenever
 - $x \succ_i y$ for all $i \in S$
 - $x \succ_i y$ for all $i \in A-S$
 - $\Rightarrow x \succ y$
- **Lemma 1.** If S is decisive for x over y , then for any other candidate z , S is decisive for x over z and for z over y
- **Proof.** Let S be decisive for x over y . Consider: $x \succ_i y \succ_i z$ for all $i \in S$ and $y \succ_i z \succ_i x$ for all $i \in A-S$
 - Since S is decisive for x over y , we have $x \succ y$
 - Because $y \succ_i z$ for every agent, by the Pareto principle we have $y \succ z$
 - Then, by transitivity, $x \succ z$
 - By independence of irrelevant alternatives (y), $x \succ z$ whenever every agent in S prefers x to z and every agent not in S prefers z to x . I.e., S is decisive for x over z
- To show that S is decisive for z over y , consider: $z \succ_i x \succ_i y$ for all $i \in S$ and $y \succ_i z \succ_i x$ for all $i \in A-S$
 - Then $x \succ y$ since S is decisive for x over y
 - $z \succ x$ from the Pareto principle and $z \succ y$ from transitivity
 - Thus S is decisive for z over y ■

Proof of Arrow's theorem (slide 2 of 3)

- Given that S is decisive for x over y , we deduced that S is decisive for x over z and z over y .
- Now reapply Lemma 1 with decision z over y as the hypothesis and conclude that
 - S is decisive for z over x
 - which implies (by Lemma 1) that S is decisive for y over x
 - which implies (by Lemma 1) that S is decisive for y over z
 - Thus: **Lemma 2.** If S is decisive for x over y , then for any candidates u and v , S is decisive for u over v (i.e., S is *decisive*)
- **Lemma 3.** For every $S \subseteq A$, either S or $A-S$ is decisive (not both)
- **Proof.** Suppose $x \succ_i y$ for all $i \in S$ and $y \succ_i x$ for all $i \in A-S$ (only such cases need to be addressed, because otherwise the left side of the implication in the definition of decisiveness between candidates does not hold). Because either $x \succ y$ or $y \succ x$, S is decisive or $A-S$ is decisive ■

Proof of Arrow's theorem (slide 3 of 3)

- **Lemma 4.** If S is decisive and T is decisive, then $S \cap T$ is decisive
- **Proof.**
 - Let $S = \{i: z \succ_i y \succ_i x\} \cup \{i: x \succ_i z \succ_i y\}$
 - Let $T = \{i: y \succ_i x \succ_i z\} \cup \{i: x \succ_i z \succ_i y\}$
 - For $i \notin S \cup T$, let $y \succ_i z \succ_i x$
 - Now, since S is decisive, $z \succ y$
 - Since T is decisive, $x \succ z$
 - Then by transitivity, $x \succ y$
 - So, by independence of irrelevant alternatives (z), $S \cap T$ is decisive for x over y .
 - (Note that if $x \succ_i y$, then $i \in S \cap T$.)
 - Thus, by Lemma 2, $S \cap T$ is decisive ■
- **Lemma 5.** If $S = S_1 \cup S_2$ (where S_1 and S_2 are disjoint and exhaustive) is decisive, then S_1 is decisive or S_2 is decisive
- **Proof.** Suppose neither S_1 nor S_2 is decisive. Then $\sim S_1$ and $\sim S_2$ are decisive. By Lemma 4, $\sim S_1 \cap \sim S_2 = \sim S$ is decisive. But we assumed S is decisive. Contradiction ■
- **Proof of Arrow's theorem**
 - Clearly the set of all agents is decisive. By Lemma 5 we can keep splitting a decisive set into two subsets, at least one of which is decisive. Keep splitting the decisive set(s) further until only one agent remains in any decisive set. That agent is a dictator. QED

Stronger version of Arrow's theorem

- In Arrow's theorem, social choice functional G outputs a ranking of the outcomes
- The impossibility holds even if only the highest ranked outcome is sought:
- **Thrm.** Let $|O| \geq 3$. If a social choice function $f: R \rightarrow O$ is monotonic and Paretian, then f is dictatorial
- f is monotonic if $[x = f(R) \text{ and } x \text{ maintains its position in } R'] \Rightarrow f(R') = x$
- x maintains its position whenever $x \succ_i y \Rightarrow x \succ_i' y$