CS 886:
Agents and Game Theory

Sept 15

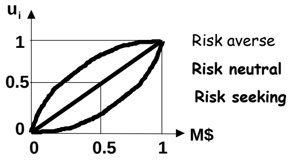
Agenthood

- We use economic definition of *agent* as locus of self-interest
  - Could be implemented e.g. as *several* mobile "agents" …
- Agent attempts to **maximize its expected utility**
- Utility function u_i of agent i is a mapping from outcomes to reals
  - Can be over a multi-dimensional outcome space
  - Incorporates agent's risk attitude (allows quantitative tradeoffs) E.g. outcomes over money

Lottery 1: \$0.5M w.p. 1

Lottery 2: \$1M w.p. 0.5
\$0 w.p. 0.5

Agent's strategy is the choice of lottery



Risk aversion => insurance companies

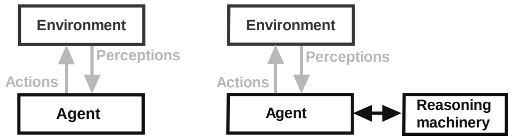
Utility functions are scale-invariant

- *Agent i chooses a strategy that maximizes expected utility*
- $\max_{strategy} \Sigma_{outcome}$ p(outcome | strategy) u_i(outcome)
- If $u_i'() = a\, u_i() + b$ for $a > 0$ then the agent will choose the same strategy under utility function u_i' as it would under u_i
- Note that u_i has to be finite for each possible outcome
  - Otherwise expected utility could be infinite for several strategies, so the strategies could not be compared.

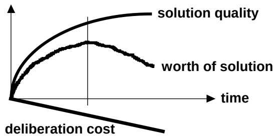
Full vs bounded rationality

Full rationality

Bounded rationality



Descriptive vs. prescriptive theories of bounded rationality



Criteria for evaluating multiagent systems

- Computational efficiency
- Distribution of computation
- Communication efficiency
- Social welfare: $\max_{outcome} \Sigma_i u_i$(outcome)
- Surplus: social welfare of outcome – social welfare of status quo
  - Constant sum games have 0 surplus. Markets are not constant sum
- Pareto efficiency: An outcome o is Pareto efficient if there exists no other outcome o' s.t. some agent has higher utility in o' than in o and no agent has lower
  - Implied by social welfare maximization
- Individual rationality: Participating in the negotiation (or individual deal) is no worse than not participating
- Stability: No agents can increase their utility by changing their strategies
- Symmetry: No agent should be inherently preferred, e.g. dictator

---

Game Theory

Kate Larson

CS 886: Electronic Market Design

Sept 15, 2004

The Basics

- **A game:** Formal representation of a situation of strategic interdependence
  - Set of agents, I (|I|=n)
    - AKA players
  - Each agent, j, has a set of actions, Aj
    - AKA moves
  - Actions define outcomes
    - For each possible set of actions there is an outcome.
  - Outcomes define payoffs
    - Agents' derive utility from different outcomes

Normal form game*

(matching pennies)

		Agent 2	
		H	T
Agent 1	H	-1, 1	1, -1
	T	1, -1	-1, 1

Action

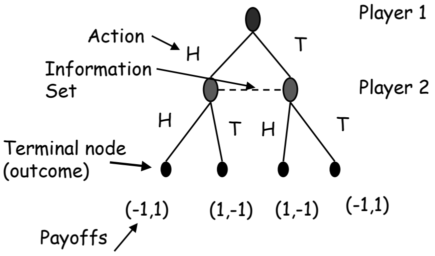
Outcome

Payoffs

*aka strategic form, matrix form

Extensive form game

(matching pennies)



Strategies

- Strategy:
  - A strategy, sj, is a **complete contingency plan**; defines actions agent j should take for all possible states of the world
- Strategy profile: s=(s1,…,sn)
  - s_{-i} = **(s1,…,s_{i-1},s_{i+1},…,sn)**
- Utility function: ui(s)
  - Note that the utility of an agent depends on the strategy profile, not just its own strategy
  - We assume agents are **expected utility maximizers**

Normal form game*

(matching pennies)

		Agent 2	
		H	T
Agent 1	H	-1, 1	1, -1
	T	1, -1	-1, 1

Strategy for agent 1: H

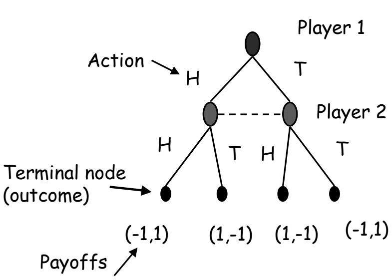
Strategy profile (H,T)

U1((H,T))=1
U2((H,T))=-1

*aka strategic form, matrix form

Extensive form game

(matching pennies)



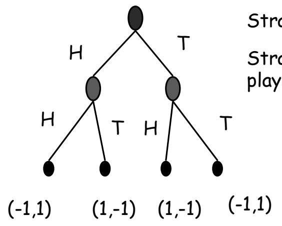
Strategy for agent 1: T

Strategy profile: (T,T)

U1((T,T))=-1
U2((T,T))=1

Extensive form game

(matching pennies, seq moves)



Recall: A strategy is a contingency plan for all states of the game

Strategy for agent 1: T

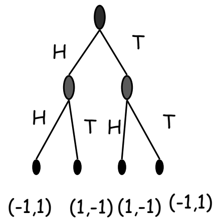
Strategy for agent 2: H if 1 plays H, T if 1 plays T (H,T)

Strategy profile: (T,(H,T))

U1((T,(H,T)))=-1

U2((T,(H,T)))=1

Game Representation



	H,H	H,T	T,H	T,T
H	-1,1	-1,1	1,-1	1,-1
T	1,-1	-1,1	1,-1	-1,1

Potential combinatorial explosion →

Example: Ascending Auction

- State of the world is defined by (x,p)
  - x∈{0,1} indicates if the agent has the object
  - p is the current price
- Strategy si((x,p))

$$si((x,p)) = \begin{cases} p, \text{ if } vi<p \text{ and } x=0 \\ \text{No bid otherwise} \end{cases}$$

Dominant Strategies

- Recall that
  - Agents' utilities depend on what strategies other agents are playing
  - Agents' are expected utility maximizers
- Agents' will play best-response strategies

si* is a best response if ui(si*,s_{-i})≥ui(si',s_{-i}) for all si'

- A dominant strategy is a best-response for all s_{-i}
  - They do not always exist
  - Inferior strategies are called dominated

Dominant Strategy Equilibrium

- A dominant strategy equilibrium is a strategy profile where the strategy for each player is dominant
  - s*=(s*1,…,s*n)
  - ui(s*i,s_{-i})≥ui(s'i,s_{-i}) for all i, for all s'i, for all s_{-i}
- **GOOD**: Agents do not need to counterspeculate!

Example: Prisoner's Dilemma

- Two people are arrested for a crime. If neither suspect confesses both are released. If both confess then they get sent to jail. If one confesses and the other does not, then the confessor gets a light sentence and the other gets a heavy sentence.

	Confess	Don't Confess
Confess	-5,-5	-1,-10
Don't Confess	-10,-1	-2,-2

Dom. Str. Eq

Pareto Optimal Outcome

Example: Vickrey Auction

- Each agent i has value vi
- Strategy bi(vi)∈[0,∞)

$$ui(bi,b_{-i}) = \begin{cases} vi\text{-}max\{bj\} \text{ where } j\neq i \text{ if } bi>bj \text{ for all } j \\ 0 \text{ otherwise} \end{cases}$$

Given value vi, bi(vi)=vi is (weakly) dominant.

Let b'=$max_{j\neq i}$bj. If b'<vi then any bid bi(vi)>b' is optimal. If b'≥vi, then any bid bi(vi)≤ vi is optimal. Bid bi(vi)=vi satisfies both constraints.

Example: Bach or Stravinsky

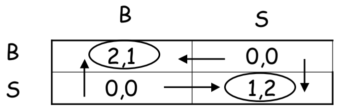
- *A couple likes going to concerts together. One loves Bach but not Stravinsky. The other loves Stravinsky but not Bach. However, they prefer being together than being apart.*

	B	S
B	2,1	0,0
S	0,0	1,2

No dom. str. equil.

Nash Equilibrium

- Sometimes an agent's best-response depends on the strategies other agents are playing
  - No dominant strategy equilibria
- A strategy profile, *s**, is a **Nash equilibrium** if no player has incentive to deviate from his strategy given that others do not deviate:
  - for every agent i, $u_i(s_i^*,s_{-i}) \geq u_i(s_i',s_{-i})$ for all s_i'



	B	S
B	2,1	0,0
S	0,0	1,2

Iterated Elimination of Dominated Strategies

- Let Ri∈Si be the set of removed strategies for agent i
- Initially Ri=Ø
- Choose agent i, and strategy si such that si∈Si\Ri and there exists si' ∈Si\Ri such that

  $ui(si',s_{-i})>ui(si,s_{-i})$ for all $s_{-i} \in S_{-i}\backslash R_{-i}$
- Add si to Ri, continue
- **Thm:** If a unique strategy profile, *s**, survives then it is a Nash Eq.
- **Thm:** If a profile, *s**, is a Nash Eq then it must survive iterated elimination.

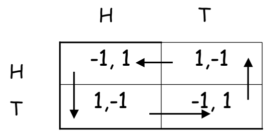
Example: Iterated Dominance

	r	l	c
U	~~3,-3~~	~~7,-7~~	~~15,-15~~
D	9,-9	8,-8	10,-10

Nash Equilibrium

- Interpretations:
  - Focal points, self-enforcing agreements, stable social convention, consequence of rational inference..
- Criticisms
  - They may not be unique (Bach or Stravinsky)
    - Ways of overcoming this
      - Refinements of equilibrium concept, Mediation, Learning
  - Do not exist in all games (in form defined)
  - They may be hard to find
  - People don't always behave based on what equilibria would predict (ultimatum games and notions of fairness,…)

Example: Matching Pennies



	H	T
H	-1, 1	1,-1
T	1,-1	-1, 1

So far we have talked only about **pure** strategy equilibria.

Not all games have pure strategy equilibria.
Some equilibria are **mixed** strategy equilibria.

Mixed strategy equilibria

- Mixed strategy:

$\sigma i \in \Sigma i$ defines a probability distribution over Si

- Strategy profile: $\sigma=(\sigma 1,\ldots, \sigma n)$
- Expected utility: $ui(\sigma)=\Sigma_{s\in S}(\Pi_j\ \sigma(sj))ui(s)$
- Nash Equilibrium:
  - σ^* is a (mixed) Nash equilibrium if

$ui(\sigma^*i, \sigma^*_{-i})\geq ui(\sigma i, \sigma^*_{-i})$ for all $\sigma i\in\Sigma i$, for all i

Example: Matching Pennies

	q H	1-q T
p H	-1, 1	1,-1
1-p T	1,-1	-1, 1

Want to play each strategy with a certain probability so that the competitor is indifferent between its own strategies.

$p-(1-p)=-p+(1-p) \Longrightarrow p=1/2$

$q-(1-q)=-q+(1-q) \Longrightarrow q=1/2$

Mixed Nash Equilibrium

- Thm (Nash 50):
  - Every game in which the strategy sets, S1,…,Sn, have a finite number of elements has a mixed strategy equilibrium.
- Finding Nash Equil is another problem
  - "Together with factoring, the complexity of finding a Nash Eq is, in my opinion, the most important concrete open question on the boundary of P today" (Papadimitriou)

Bayesian-Nash Equil

(Harsanyi 68)

- So far we have assumed that agents have complete information about each other (including payoffs)
  - Very strong assumption!
- Assume agent i has type $\theta i\in\Theta i$, defines the payoff $ui(s, \theta i)$
- Agents have common prior over distribution of types $p(\theta)$
  - Conditional probability $p(\theta_{-i}|\ \theta i)$ (obtained by Bayes Rule when possible)

Bayesian-Nash Equil

- Strategy: $\sigma i(\theta i)$ is the (mixed) strategy agent i plays if its type is θi
- Strategy profile: $\sigma=(\sigma 1,\ldots, \sigma n)$
- Expected utility:
  - $Ui(\sigma i(\theta i),\sigma_{-i}(),\theta i)=\Sigma_{\theta\text{-}i}\ p(\theta_{-i}|\theta i)ui(\sigma i(\theta i),\sigma_{-i}(\theta_{-i}),\theta i)$
- Bayesian Nash Eq: Strategy profile σ^* is a Bayesian-Nash Eq if for all i, for all θi,

$Ui(\sigma^*i(\theta i),\sigma^*_{-i}(),\theta i)\geq Ui(\sigma i(\theta i),\sigma^*_{-i}(),\theta i)$

(best responding w.r.t. its beliefs about the types of the other agents, assuming they are also playing a best response)

Example: 1st price sealed-bid auction

2 agents (1 and 2) with values v_1,v_2 drawn uniformly from [0,1].

Utility of agent i if it bids b_i and wins the item is $u_i=v_i-b_i$.

Assume agent 2's bidding strategy is $b_2(v_2)=v_2/2$

How should 1 bid? (i.e. what is $b_1(v_1)=z$?)

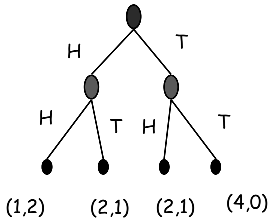
$$\boxed{U_1=\int_{z=0}^{2z}(v_1-z)dz = (v_1-z)2z=2zv_1-2z^2}$$

Note: given z=b2(v2)=v2/2, 1 only wins if v2<2z

Therefore, $\text{Max}_z[2zv_1-2z^2\ ]$ when $z=b_1(v_1)=v_1/2$

Similar argument for agent 2, assuming $b_1(v_1)=v_1/2$.
We have an equilibrium

Extensive Form Games



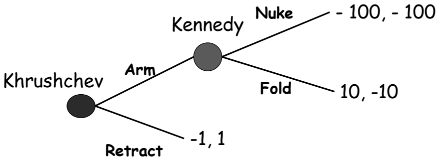
Any finite game of perfect information has a pure strategy Nash equilibrium. It can be found by backward induction.

Chess is a finite game of perfect information. Therefore it is a "trivial" game from a game theoretic point of view.

Subgame perfect equilibrium & credible threats

- Proper subgame = subtree (of the game tree) whose root is alone in its information set
- Subgame perfect equilibrium
  - Strategy profile that is in Nash equilibrium in every proper subgame (including the root), whether or not that subgame is reached along the equilibrium path of play

Example: Cuban Missile Crisis



Pure strategy Nash equilibria: (Arm, Fold) and (Retract, Nuke)

Pure strategy subgame perfect equilibria: (Arm, Fold)

Conclusion: Kennedy's Nuke threat was not credible.

Next week

- Monday: Social Choice (Voting or How to Rig an Election!)
- Wednesday: Mechanism Design I
- *Assignment on Game Theory will be given out on Monday – due in 1 ½ weeks*