Utility functions are scale-invariant

- Agent $i$ chooses a strategy that maximizes expected utility
- $\max_{\text{strategy}} \sum_{\text{outcome}} P(\text{outcome} \mid \text{strategy}) u(\text{outcome})$
- If $u'(z) = a u(z) + b$ for $a > 0$ then the agent will choose the same strategy under utility function $u'$ as it would under $u$.
- Note that $u_i$ has to be finite for each possible outcome.
  - Otherwise expected utility could be infinite for several strategies, so the strategies could not be compared.

Criteria for evaluating multiagent systems

- Computational efficiency
- Distribution of computation
- Communication efficiency
- Social welfare: $\max_{\text{strategy}} \sum_{\text{outcome}} u(\text{outcome})$
- Surplus: social welfare of outcome - social welfare of status quo
  - Constant sum games have 0 surplus. Markets are not constant sum.
- Pareto efficiency: An outcome $a$ is Pareto efficient if there exists no other outcome $b$ s.t. some agent has higher utility in $a$ than in $b$ and no agent has lower.
  - Implied by social welfare maximization
- Individual rationality: Participating in the negotiation (or individual deal) is no worse than not participating
- Stability: No agents can increase their utility by changing their strategies
- Symmetry: No agent should be inherently preferred, e.g. dictator

Agenthood

- We use economic definition of agent as locus of self-interest
  - Could be implemented e.g. as several mobile “agents”.
- Agent attempts to maximize its expected utility
- Utility function $u_i$ of agent $i$ is a mapping from outcomes to reals
  - Can be over a multi-dimensional outcome space
  - Incorporates agent's risk attitude (allows quantitative tradeoffs) E.g. outcomes over money

$R$ is a version $\Rightarrow$ insurance companies

Game Theory

Kate Larson
CS 886: Electronic Market Design
Sept 15, 2004
The Basics

- **A game**: Formal representation of a situation of strategic interdependence
  - Set of agents, I (|I|= n)
    - AKA players
  - Each agent, j, has a set of actions, Aj
    - AKA moves
  - Actions define outcomes
    - For each possible set of actions there is an outcome.
  - Outcomes define payoffs
    - Agents' derive utility from different outcomes

Normal form game*

(matching pennies)

<table>
<thead>
<tr>
<th>Action</th>
<th>Agent 1</th>
<th>Agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
<tr>
<td>T</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

Outcome

Payoffs

*aka strategic form, matrix form

Strategies

- Strategy:
  - A strategy, sj, is a complete contingency plan: defines actions agent j should take for all possible states of the world
  - Strategy profile: s=(s1,...,sn)
    - s_j = (s_1,...,s_j,...,s_n)
  - Utility function: u(s)
    - Note that the utility of an agent depends on the strategy profile, not just its own strategy
    - We assume agents are expected utility maximizers

Extensive form game

(matching pennies)

<table>
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<tr>
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</tr>
</thead>
<tbody>
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<td>H</td>
<td>H</td>
</tr>
<tr>
<td>T</td>
<td>H</td>
<td>T</td>
</tr>
<tr>
<td>Terminal node (outcome)</td>
<td>(-1,1)</td>
<td>(1,-1)</td>
</tr>
<tr>
<td>Payoffs</td>
<td>(1,-1)</td>
<td>(1,-1)</td>
</tr>
</tbody>
</table>

*aka strategic form, matrix form

Extensive form game

(matching pennies)

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<td>H</td>
<td>H</td>
</tr>
<tr>
<td>T</td>
<td>H</td>
<td>T</td>
</tr>
<tr>
<td>Strategy profile: (H,H)</td>
<td>U1((H,H))=1</td>
<td></td>
</tr>
<tr>
<td>Strategy profile: (H,T)</td>
<td>U2((H,T))=-1</td>
<td></td>
</tr>
<tr>
<td>Terminal node (outcome)</td>
<td>(-1,1)</td>
<td></td>
</tr>
<tr>
<td>Payoffs</td>
<td>(1,-1)</td>
<td>(1,-1)</td>
</tr>
</tbody>
</table>
### Extensive form game
(matching pennies, seq moves)

Recall: A strategy is a contingency plan for all states of the game.

- **Strategy for agent 1**: $T$
- **Strategy for agent 2**: $H$ if 1 plays $H$, $T$ if 1 plays $T$ ($H,T$)

Strategy profile: $(T,(H,T))$

- $U_1((T,(H,T))) = -1$
- $U_2((T,(H,T))) = 1$

Potential combinatorial explosion

### Example: Ascending Auction

- State of the world is defined by $(x,p)$
  - $x \in \{0,1\}$ indicates if the agent has the object
  - $p$ is the current price
- **Strategy** $s_i(x,p)$

\[
s_i(x,p) = \begin{cases} 
  p, & \text{if } x=p \text{ and } x=0 \\
  \text{No bid otherwise}
\end{cases}
\]

### Dominant Strategies

- Recall that
  - Agents’ utilities depend on what strategies other agents are playing
  - Agents’ are expected utility maximizers
- **Agents’ will play best-response strategies**

$s^*_{i}$ is a best response if $u_i(s^*_{i},s_{-i}) \geq u_i(s'_i,s_{-i})$ for all $s'_i$

- A dominant strategy is a best-response for all $s_{-i}$
  - They do not always exist
  - Inferior strategies are called dominated

### Dominant Strategy Equilibrium

- A dominant strategy equilibrium is a strategy profile where the strategy for each player is dominant
  - $s^*=(s^*_1,\ldots,s^*_n)$
  - $u_i(s^*_1,s_{-i}) \geq u_i(s'_i,s_{-i})$ for all $i$, for all $s'_i$, for all $s_{-i}$

- **GOOD**: Agents do not need to counterspeculate!

### Example: Prisoner’s Dilemma

- Two people are arrested for a crime. If neither suspect confesses both are released. If both confess then they get sent to jail. If one confesses and the other does not, then the confessor gets a light sentence and the other gets a heavy sentence.

<table>
<thead>
<tr>
<th></th>
<th>Confess</th>
<th>Don’t Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confess</td>
<td>(-5,-5)</td>
<td>(-1,-10)</td>
</tr>
<tr>
<td>Don’t Confess</td>
<td>(-10,1)</td>
<td>(-2,-2)</td>
</tr>
</tbody>
</table>

Pareto Optimal Outcome
**Example: Vickrey Auction**

- Each agent $i$ has value $v_i$
- Strategy $b_i(v_i) \in [0, \infty)$
  
  $$u(b_i, b_j) = \begin{cases} v_i - \max(b_j) & \text{if } j \neq i \text{ and } b_i \geq b_j \\ 0 & \text{otherwise} \end{cases}$$

  Given value $v_i$, $b_i(v_i) = v_i$ is (weakly) dominant.
  Let $b'_\max = b_j$. If $b'_i(v_i) \geq v_i$ then any bid $b_i(v_i) \geq b'_i$ is optimal. If $b'_i(v_i) < v_i$, then any bid $b_i(v_i) < v_i$ is optimal. Bid $b_i(v_i) = v_i$ satisfies both constraints.

**Example: Bach or Stravinsky**

- A couple likes going to concerts together. One loves Bach but not Stravinsky. The other loves Stravinsky but not Bach. However, they prefer being together than being apart.

<table>
<thead>
<tr>
<th></th>
<th>$B$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>$S$</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

No dom. str. equil.

**Nash Equilibrium**

- Sometimes an agent’s best-response depends on the strategies other agents are playing
  - No dominant strategy equilibria
  - A strategy profile, $s^*$, is a Nash equilibrium if no player has incentive to deviate from his strategy given that others do not deviate:
    - for every agent $i$, $u(s^*, s_i) \geq u(s_i, s_i)$ for all $s_i$

<table>
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**Iterated Elimination of Dominated Strategies**

- Let $R_i, S_i$ be the set of removed strategies for agent $i$
  - Initially $R_i = \emptyset$
  - Choose agent $i$, and strategy $s_i$ such that $s_i \in S_i \setminus R_i$ and there exists $s'_i \in S_i \setminus R_i$ such that $u(s'_i, s_i) \geq u(s_i, s_i)$ for all $s_i \in S_i \setminus R_i$
  - Add $s_i$ to $R_i$, continue

- Thm: If a unique strategy profile, $s^*$, survives then it is a Nash Eq.
- Thm: If a profile, $s^*$, is a Nash Eq then it must survive iterated elimination.

**Example: Iterated Dominance**

<table>
<thead>
<tr>
<th></th>
<th>$r$</th>
<th>$l$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>3,3</td>
<td>7,7</td>
<td>15,15</td>
</tr>
<tr>
<td>$D$</td>
<td>9,9</td>
<td>8,8</td>
<td>10,10</td>
</tr>
</tbody>
</table>

**Nash Equilibrium**

- Interpretations:
  - Focal points, self-enforcing agreements, stable social convention, consequence of rational inference...
- Criticisms
  - They may not be unique (Bach or Stravinsky)
    - Ways of overcoming this
      - Refinements of equilibrium concept, Mediation, Learning
    - Do not exist in all games (in form defined)
    - They may be hard to find
    - People don’t always behave based on what equilibria would predict (ultimatum games and notions of fairness...).
Example: Matching Pennies

\[
\begin{array}{c|cc}
 & H & T \\
\hline
H & -1,1 & 1,-1 \\
T & 1,-1 & -1,1 \\
\end{array}
\]

So far we have talked only about pure strategy equilibria. Not all games have pure strategy equilibria. Some equilibria are mixed strategy equilibria.

Mixed strategy equilibria

- Mixed strategy:
  - \( \sigma_i \in \Sigma_i \) defines a probability distribution over \( S_i \)
  - Strategy profile: \( \sigma = (\sigma_1, \ldots, \sigma_n) \)
  - Expected utility: \( u_i(\sigma) = \sum_{s \in S_i} (\Pi_j \sigma(j)) u(s) \)
- Nash Equilibrium:
  - \( \sigma^* \) is a (mixed) Nash equilibrium if
    \[ u_i(\sigma^*_i, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i}) \text{ for all } \sigma_i \in \Sigma_i \text{ for all } i \]

Example: Matching Pennies

\[
\begin{array}{c|cc}
 & H & 1-q \cdot T \\
\hline
p \cdot H & -1,1 & 1,-1 \\
1-p \cdot T & 1,-1 & -1,1 \\
\end{array}
\]

Want to play each strategy with a certain probability so that the competitor is indifferent between its own strategies.

\[
p \cdot (1-p) = p \cdot (1-p) \quad \Rightarrow \quad p = 1/2 \\
q \cdot (1-q) = q \cdot (1-q) \quad \Rightarrow \quad q = 1/2
\]

Mixed Nash Equilibrium

- Thm (Nash 50):
  - Every game in which the strategy sets, \( S_1, \ldots, S_n \), have a finite number of elements has a mixed strategy equilibrium.
  - Finding Nash Equil is another problem
    - “Together with factoring, the complexity of finding a Nash Eq is, in my opinion, the most important concrete open question on the boundary of P today” (Papadimitriou)

Bayesian-Nash Equil

(Harsanyi 68)

- So far we have assumed that agents have complete information about each other (including payoffs)
  - Very strong assumption!
- Assume agent \( i \) has type \( \theta_i \in \Theta_i \), defines the payoff \( u_i(s, \theta_i) \)
- Agents have common prior over distribution of types \( p(\theta) \)
  - Conditional probability \( p(\theta_i | \theta) \) (obtained by Bayes Rule when possible)

Bayesian-Nash Equil

- Strategy: \( \alpha_i(\theta_i) \) is the (mixed) strategy agent \( i \) plays if its type is \( \theta_i \)
- Strategy profile: \( \sigma = (\sigma_1, \ldots, \sigma_n) \)
- Expected utility:
  - \( U_i(\sigma(\theta_i), \sigma(\cdot), \theta) = \sum_{s \in S_i} p(0, \cdot | \theta) u_i(\sigma_i(\theta_i), \sigma(\cdot), \theta) \)
- Bayesian Nash Eq: Strategy profile \( \sigma^* \) is a Bayesian-Nash Eq if for all \( i, \) for all \( \theta_i \),
  - \( U_i(\sigma^*(\theta_i), \sigma^*(\cdot), \theta) \geq U_i(\sigma(\theta_i), \sigma^*(\cdot), \theta) \)

(best responding w.r.t. its beliefs about the types of the other agents, assuming they are also playing a best response)
Example: 1st price sealed-bid auction

2 agents (1 and 2) with values \( v_1, v_2 \) drawn uniformly from \([0,1]\). Utility of agent \( i \) if it bids \( b_i \) and wins the item is \( u_i = v_i - b_i \). Assume agent 2’s bidding strategy is \( b_2(v_1) = \frac{v_1}{2} \).

How should 1 bid? (i.e. what is \( b_1(v_1) = z^* \))

\[
U_1 = \int_{b_2(v_1)}^{v_1} v^2(1-z)dz = (v_1 - z)2z = 2v_1z - 2z^2
\]

Note: given \( z = b_2(v_2) = v_2/2 \), I only wins if \( v_2 > 2z \)

Therefore, Max\(_z\) \((2v_1z - 2z^2 \) \) when \( z = b_2(v_1) = v_1/2 \)

Similar argument for agent 2, assuming \( b_2(v_1) = v_1/2 \). We have an equilibrium.

Subgame perfect equilibrium & credible threats

- Proper subgame = subtree (of the game tree) whose root is alone in its information set

- Subgame perfect equilibrium
  - Strategy profile that is in Nash equilibrium in every proper subgame (including the root), whether or not that subgame is reached along the equilibrium path of play

Example: Cuban Missile Crisis

Pure strategy Nash equilibria: (Arm, Fold) and (Retract, Nuke)

Pure strategy subgame perfect equilibria: (Arm, Fold)

Conclusion: Kennedy’s Nuke threat was not credible.

Next week

- Monday: Social Choice (Voting or How to Rig an Election!)

- Wednesday: Mechanism Design I

- Assignment on Game Theory will be given out on Monday - due in 1 1/2 weeks