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Analysis and numerics for a parabolic equation with impulsive forcing

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Abstract

The paper considers a one-dimensional particle-continuum model, with impulsive interaction between the fluid and a number of pointwise particles. A simplification results in a system of ODEs coupled with a parabolic PDE forced by a nonlinear term involving a sum of Dirac delta functions. The existence of a mild solution is proved using a combination of energy estimates and semigroup theory. However, the regularity of these solutions is shown to be limited to $C^{0,1}$ by the impulsive terms. The convergence of a Galerkin method is established simultaneously with a proof of continuous dependence, and thus uniqueness, of solutions for the underlying system. The peculiarities of the system imply this analysis must be performed in L_∞. The $C^{0,1}$ regularity of the solution determines a suboptimal rate of convergence for the Galerkin method. The theoretical results are verified by MATLAB computations.

Keywords: Parabolic equations; Nonlinear systems; Impulsive forcing

1. Introduction

Particle-continuum models arise in a variety of applications. Amsden et al. [2], for example, model spray combustors in gas turbine engines as a collection of particles exchanging mass, momentum and energy with the surrounding gas. The resulting equations raise a number of issues in analysis and numerical approximation, some of which were explored by Hill et al. [5].

A typical model, based on the Navier–Stokes equations, is as follows:

$$\frac{\partial u}{\partial t} + (u\,.\,\nabla)u = \frac{1}{R}\Delta u + \nabla p + f - \gamma \sum_{j=1}^{n} \delta\big(x - y_j(t)\big) F_j,$$

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$$\nabla . u = 0, \qquad \tau \frac{\mathrm{d}^2 y_j}{\mathrm{d}t^2} = F_j, \quad j \in \{1, \ldots, n\}. \tag{1.1}$$

Here, u is the fluid velocity, p is the pressure, y_j is the particle position, f is a non-dimensionalised external force, R is the Reynolds number, τ and γ are non-dimensional constants, and

$$F_j = u\bigl(y_j(t), t\bigr) - \frac{\mathrm{d}y_j}{\mathrm{d}t}, \quad j \in \{1, \ldots, n\},$$

is a non-dimensionalised force due to Stokes' Law. Related equations have been considered by Verwer and Sommeijer [11] and Krottje [6] in models of the outgrowth of axons in a developing nervous system.

The validity of such models, in terms of the derivation from the physics, and comparison with experimental data, is considered in the papers already mentioned. The purpose of this paper, on the other hand, is to investigate some of the mathematical difficulties posed by the impulsive forces, as a baseline and guide for more practical investigations. In particular, we are interested in the well-posedness of such equations, as well as the convergence behaviour to be expected of a standard numerical method. In order to address these questions, we reduce to a simplified situation, in which we consider a one-dimensional problem. The qualities we seek in the simplified model are

(i) A sufficiently close relationship with the main model (1.1);
(ii) A sufficiently simple system to be mathematically tractable.

The qualities we seek in the numerical method are

(i) Basic feasibility;
(ii) Sufficient sophistication to deal with known difficulties, such as stiffness, but no special features, such as particle tracking by the mesh, to anticipate the new problems raised by impulsive forces.

With these mathematical objectives in mind, we do not claim that the simplified problem models any particular physical system, nor do we recommend our numerical method, even for this simplified case.

In Section 2, we rigorously state the simplified model problem, and derive a sequence of boot-strap bounds on a mild solution, leading to an existence theorem and a proof of Lipschitz continuity.

In Section 3, we simultaneously establish the well-posedness of the simplified model problem and the convergence of a piecewise linear Galerkin approximation.

In Section 4 the results of a fully discrete implementation of the Galerkin method are presented, computationally verifying the feature of suboptimal convergence proved in Section 3. In particular, it is observed that the accuracy of the method in $L_2(\Omega)$ is the same as that in $L_\infty(\Omega)$, due to the appearance of pointwise values of u on the right side of the equation. This property of the Galerkin approximation may be compared with other cases of nonlinearly induced suboptimality: see the list of Wahlbin [12].

2. Existence and regularity of solutions

2.1. Simplified model problem

Let $\Omega = S_1$, the periodic domain $(0, 1]$, with 0 identified with 1. For $n \in \mathbb{N}$ and $t \geqslant 0$, let $y_j(t) \in \Omega$, $1 \leqslant j \leqslant n$, denote the positions of n particles. The corresponding velocities of these particles are

$z_j = \mathrm{d}y_j/\mathrm{d}t$, $1 \leqslant j \leqslant n$. The underlying velocity field is $u : \Omega \to \mathbb{R}$. The simplified model equations are then given by

$$\frac{\partial u}{\partial t} = \frac{1}{R}\frac{\partial^2 u}{\partial x^2} + f - \gamma \sum_{j=1}^{n} \delta\big(x - y_j(t)\big)\big[u\big(y_j(t), t\big) - z_j(t)\big], \tag{2.1}$$

$$\tau \frac{\mathrm{d}z_j}{\mathrm{d}t} = u\big(y_j(t), t\big) - z_j(t), \quad j \in \{1, \ldots, n\}, \tag{2.2}$$

$$\frac{\mathrm{d}y_j}{\mathrm{d}t} = z_j(t), \quad j \in \{1, \ldots, n\}, \tag{2.3}$$

for $(x, t) \in \Omega \times (0, T]$, $T > 0$, where derivatives are intended in a weak sense. Note that $f \in L_2(\Omega)$; R, γ and τ are positive constants. The initial data is

$$u_0 \in L_2(\Omega), \qquad \mathbf{z}_0 \in \mathbb{R}^n, \qquad \mathbf{y}_0 \in \Omega^n. \tag{2.4}$$

As the system is defined on $\Omega = S_1$, $y_j(t)$ is identified with $y_j(t) - \lfloor y_j(t) \rfloor$, $1 \leqslant j \leqslant n$, where we use $\lfloor x \rfloor$ to denote the greatest integer strictly less x, for $x \in \mathbb{R}$. Below, we comment on the reasons for the simplifications made in going from (1.1) to (2.1)–(2.3).

The pressure term in (1.1) is essentially coupled with the incompressibility condition. As the latter makes little sense in one dimension, pressure is left out of the simplified model. The convective term uu_x is also omitted in (2.1), this time merely for ease of presentation, since this term is more regular than u_{xx} or $\delta(x - y_j)$ and may be analysed even in several space dimensions using standard semigroup techniques; see Henry [4]. In a final simplification, the domain is chosen to be periodic. If (2.1) were instead defined with Dirichlet conditions, particles would collide with the domain boundary, and would thus have discontinuous velocities, creating additional problems.

2.2. Mild solution

Initially, we seek to prove the existence of a type of weak solution for problem (2.1)–(2.4) called a mild solution. To define a mild solution, we introduce following operator indicated by Eq. (2.1).

Definition 2.1. For $D(A) \equiv H^2(\Omega)$, we define operator $A : D(A) \to L_2(\Omega)$ by

$$Au = -\frac{1}{R}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u, \quad \text{for } u \in D(A).$$

Definition 2.2. Let $D(A)$ be a dense subspace of a complex Banach space X. Then $A : D(A) \to X$ is a sectorial operator, if it is linear and there exist constants $(M, a, \theta) \in [1, \infty) \times \mathbb{R} \times (0, \pi/2)$ such that $(zI - A)^{-1} \in \mathcal{L}(X)$ and

$$\big\|(zI - A)^{-1}\big\|_X \leqslant \frac{M}{|z - a|}, \quad \text{for all } z \neq a \text{ such that } \big|\arg[z - a]\big| > \theta.$$

Sectorial operators are introduced in the books of Henry [4], Lunardi [8] and Miklavčič [9]. There, it is shown that A is indeed sectorial, and that the operator $e^{-At} : L_2(\Omega) \to L_2(\Omega)$ is defined for $t \geqslant 0$. However, the contraction mapping arguments used in those books to prove existence and uniqueness are inapplicable to our simplified model equations, since the nonlinearity fails to satisfy a Lipschitz condition between suitable Sobolev spaces. The following is an adaptation of a definition stated in Lunardi [8] which includes the auxiliary ODEs (2.2) and (2.3):

Definition 2.3. A **mild** solution of (2.1)–(2.4) satisfies $u \in C[0,T;\ L_2(\Omega)] \cap L_\infty[(0,T];\ D(A^\alpha)]$, for some $\alpha \in (1/4, 3/4)$, and for all $t \in (0,T]$,

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}\left[f + u(s) - \gamma \sum_{j=1}^n S_j(s)\right] \mathrm{d}s, \tag{2.5}$$

where $S_j(s)(x) = \delta(x - y_j(s))[u(y_j(s), s) - z_j(s)]$, $j \in \{1, \ldots, n\}$. Furthermore, for $j \in \{1, \ldots, n\}$, $z_j, y_j \in C[0,T]$ satisfy

$$z_j(t) = e^{-t/\tau} z_j(0) + \frac{1}{\tau}\int_0^t e^{-(t-s)/\tau} u\big(y_j(s), s\big)\, \mathrm{d}s, \quad t \in [0,T], \tag{2.6}$$

$$y_j(t) = y_j(0) + \int_0^t z_j(s)\, \mathrm{d}s, \quad t \in [0,T]. \tag{2.7}$$

2.3. *Spectral Galerkin spatial semidiscretisation*

This discretisation is introduced purely as a theoretical tool. It is well known that the set of eigenpairs of A is $(\lambda_k, \psi_k(x))_{k=-\infty}^{\infty} = (1 + k^2\pi^2/R, e^{ikx})_{k=-\infty}^{\infty}$. Let $N \in \mathbb{N}$ be fixed, and let $V^N = \mathrm{Span}\{\psi_k\}_{k=-N}^{N}$ be a $(2N+1)$-dimensional Galerkin subspace of $D(A)$. Let $P^N : H^{-1}(\Omega) \to V_N$ denote the projection defined by

$$\big\langle P^N u, \phi^N \big\rangle = \big\langle u, \phi^N \big\rangle, \quad \text{for all } \phi^N \in V^N.$$

In the subspace V^N, functions of A take on a particularly simple form. For analytic $g : G \to \mathbb{C}$, $G \subseteq \mathbb{C}$, and $v^N \in V^N$,

$$g(A)v^N = \sum_{k=-N}^{N} g(\lambda_k)\big\langle v^N, \psi_k \big\rangle \psi_k,$$

where $\langle \cdot\,, \cdot \rangle$ is the $L_2(\Omega)$ inner product.

Given f, R, γ, τ, u_0, $\mathbf{y}_0$ and $\mathbf{z}_0$ from (2.1)–(2.4), consider $(u^N(t), \mathbf{z}^N(t), \mathbf{y}^N(t)) \in V^N \times \mathbb{R}^n \times \Omega^n$, satisfying for $t \in [0,T]$,

$$u_t^N + Au^N = P^N f + u^N - \gamma \sum_{j=1}^n S_j^N, \tag{2.8}$$

$$\tau \frac{\mathrm{d}z_j^N}{\mathrm{d}t} = u^N\big(y_j^N(t), t\big) - z_j^N(t), \quad j \in \{1, \ldots, n\}, \tag{2.9}$$

$$\frac{\mathrm{d}y_j^N}{\mathrm{d}t} = z_j^N(t), \quad j \in \{1, \ldots, n\}, \tag{2.10}$$

where $S_j^N(t) = [u^N(y_j^N(t),\, t) - z^N(t)]P^N\delta(\cdot - y_j^N(t)),\ j \in \{1,\dots,n\}$,

$$u^N(0) = P^N u_0, \qquad \mathbf{z}^N(0) = \mathbf{z}_0, \qquad \mathbf{y}^N(0) = \mathbf{y}_0. \tag{2.11}$$

As it is stated, (2.8) represents a sectorial evolution equation on $L_2(\Omega)$. However, by the above remark on functions of A on V^N, it is also a system of $2N+1$ ODEs. This system is coupled with the further $2n$ ODEs for $\mathbf{z}^N$ and $\mathbf{y}^N$. Since the nonlinearities of the system are locally Lipschitz continuous, classical Cauchy–Lipschitz ODE theory implies that, for some $T^* > 0$, there exists a unique solution to (2.8)–(2.11),

$$(u^N, \mathbf{z}^N, \mathbf{y}^N) \in C[0, T^*;\ V^N \times \mathbb{R}^n \times \Omega^n],$$

which may be continued in time as long as the variables remain bounded.

2.4. Proof of existence of a mild solution

Before we proceed further, some technical results are required. We shall use the notation $\|\cdot\|_{H^1}$ and $\|\cdot\|_{C^{0,1}}$ to respectively denote the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_{C^{0,1}(\Omega)}$ etc. This should cause no confusion, as we shall not use Sobolev norms defined on any domain apart from Ω.

Lemma 2.1. *For $\delta > 0$, there is a constant $C > 0$ such that*

$$\begin{aligned}
&\|u\|_\infty \leqslant C\|A^{1/2}u\|_2^{1/2}\|u\|_2^{1/2}, && \text{for } u \in D(A^{1/2}),\\
&\|u\|_\infty \leqslant C\|A^{1/4+\delta}u\|_2, && \text{for } u \in D(A^{1/4+\delta}),\\
&\|A^{-(1/4+\delta)}u\|_2 \leqslant C\|u\|_1, && \text{for } u \in L_1(\Omega),\\
&\|u\|_{C^{0,1}} \leqslant C\|Au\|_2^{1/2}\|A^{1/2}u\|_2^{1/2}, && \text{for } u \in D(A),\\
&\|u\|_{C^{0,1}} \leqslant C\|A^{3/4+\delta}u\|_2, && \text{for } u \in D(A^{3/4+\delta}),\\
&\|u\|_{C^2} \leqslant C\|A^{5/4+\delta}u\|_2, && \text{for } u \in D(A^{5/4+\delta}).
\end{aligned}$$

Proof. The equivalence of the norms $\|A^\alpha(\cdot)\|_2$ and $\|\cdot\|_{H^{2\alpha}}$, $\alpha \geqslant 0$ is shown, for example, in Henry [4]. The first, second, fourth and sixth results are then standard results; see Adams [1]. For the third inequality, the density of $D(A^{1/4+\delta})$ in $L_\infty(\Omega)$ implies that for $u \in L_1(\Omega)$,

$$\begin{aligned}
\|u\|_1 &= \max_{\phi\in D(A^{1/4+\delta})} \frac{\langle u,\phi\rangle}{\|\phi\|_\infty}\\
&\geqslant \max_{\phi\in D(A^{1/4+\delta})} \frac{\langle u,\phi\rangle}{C\|A^{1/4+\delta}\phi\|_2} \quad \text{[by the second inequality]}\\
&= \|A^{-(1/4+\delta)}u\|_2/C.
\end{aligned}$$

For the fifth inequality, let $u \in D(A) \subset C^1(\Omega)$. Then, by the first inequality,

$$\begin{aligned}
\|u\|_{C^{0,1}} &= \|u'\|_\infty + \|u\|_\infty \leqslant C(\|u\|_{H^2}^{1/2} + \|u\|_2^{1/2})\|u\|_{H^1}^{1/2}\\
&\leqslant C\|Au\|_2^{1/2}\|A^{1/2}u\|_2^{1/2}. \qquad \Box
\end{aligned}$$

Over the next three lemmas, $(u^N, \mathbf{z}^N, \mathbf{y}^N)(t)$ are shown to be bounded and Hölder continuous, uniformly in N. The chosen spaces reflect those in the definition of the mild solution, in preparation for a weak convergence existence argument.

Lemma 2.2. *Let the conditions hold which were stated in defining the spectral Galerkin problem* (2.8)–(2.11)*. Then, for all $\alpha \in (1/4, 3/4)$, there exists a constant C, such that for all $N \in \mathbb{N}$,*

$$\begin{aligned} &\left\|A^\alpha u^N(t)\right\|_2 \leqslant Ct^{-\alpha}, && \text{for } t \in (0, T], \\ &\left\|u^N(t)\right\|_\infty \leqslant Ct^{-1/4}, && \text{for } t \in (0, T], \\ &\left|z_j(t)\right|, \left|y_j(t)\right| \leqslant C, && \text{for } t \in (0, T],\ j \in \{1, \dots, n\}. \end{aligned}$$

Proof. Below, C will, as in the statement of the lemma, denote a generic constant, possibly dependent on initial data and T, but independent of N and t. Initially, assume $t \in (0, T^*]$. Integrating (2.9) and applying Lemma 2.1,

$$\begin{aligned} \left|z_j^N(t)\right| &\leqslant \left|z_j^N(0)\right| + \frac{1}{\tau}\int_0^t e^{-(t-s)/\tau}\left|u^N\left(y_j^N(s), s\right)\right| \mathrm{d}s \\ &\leqslant \left|z_j(0)\right| + C\int_0^t \left\|A^\alpha u^N(s)\right\|_2 \mathrm{d}s, \quad j \in \{1, \dots, n\}. \end{aligned} \tag{2.12}$$

Operating on (2.8) by e^{At}, (which is permitted on V^N), integrating over $(0, t)$, and then operating by e^{-At}, we obtain

$$u^N(t) = e^{-At}u^N(0) + \int_0^t e^{-A(t-s)}\left[P^N f + u^N - \gamma \sum_{j=1}^n S_j^N(s)\right]\mathrm{d}s. \tag{2.13}$$

Fixing $\delta = (3/4 - \alpha)/2$, operating on (2.13) by A^α and taking norms,

$$\begin{aligned} \left\|A^\alpha u^N(t)\right\|_2 \leqslant{}& Ct^{-\alpha}\left\|u^N(0)\right\|_2 + C\int_0^t \left\|A^{\alpha+1/4+\delta}e^{-A(t-s)}\right\|_2 \left\|A^{-(1/4+\delta)}\left(f + u^N(s)\right)\right\|_2 \mathrm{d}s \\ &+ \gamma\int_0^t \left\|A^{\alpha+1/4+\delta}e^{-A(t-s)}\right\|_2 \sum_{j=1}^n \left\|A^{-(1/4+\delta)}\delta\left(\cdot - y_j^N(s)\right)\right\|_2 C\left(\left\|A^\alpha u^N(s)\right\|_2 + \left|z_j^N(s)\right|\right) \mathrm{d}s \\ \leqslant{}& Ct^{-\alpha}\|u_0\|_2 + C\int_0^t (t-s)^{-(\alpha+\delta+1/4)}\left(\|f\|_2 + \left\|A^\alpha u^N(s)\right\|_2 + \sum_{j=1}^n \left|z_j^N(s)\right|\right)\mathrm{d}s. \end{aligned}$$

We note that the way in which we have defined e^{-At} and other functions of A on V^N, extends naturally to $L_2(\Omega)$ and other spaces for which $\{\psi_k\}_{k=-\infty}^{\infty}$ are a basis, see Henry [4, Chapter 1]. Hence, are able to use standard N-independent estimates on terms such as $\|A^\alpha e^{-At}\|_2$, which may be also found in Henry [4].

Recalling that $\|u_0\|_2$, $\|f\|_2$ and $\{|z_j(0)|\}_{j=1}^n$ are bounded independently of N, and applying (2.12), we obtain

$$\left\|A^\alpha u^N(t)\right\|_2 \leqslant Ct^{-\alpha} + C\int_0^t (t-s)^{-(\alpha+\delta+1/4)}\left\|A^\alpha u^N(s)\right\|_2 \,\mathrm{d}s, \quad t \in \left(0, T^*\right].$$

Thanks to a variant of the Gronwall lemma; see Henry [4, Lemma 7.1.1],

$$\left\|A^\alpha u^N(t)\right\|_2 \leqslant Ct^{-\alpha}, \quad t \in \left(0, T^*\right].$$

The stated bounds on $\mathbf{z}(t)$ and $\mathbf{y}(t)$ may be obtained for $t \in [0, T^*]$, by substituting this estimate back into (2.12) and (2.7). These bounds imply that the solution of the spectral Galerkin problem may be continued up to time T.

Since $u^N(t) \in H^1(\Omega)$, for $t \in (0, T]$, Lemma 2.1 implies

$$\left\|u^N(t)\right\|_\infty \leqslant C\left\|A^{1/2}u^N(t)\right\|_2^{1/2}\left\|u^N(t)\right\|_2^{1/2} \leqslant Ct^{-1/4}. \qquad \square$$

Lemma 2.3. *Under the assumptions of Lemma* 2.2*, suppose* $\alpha \in [0, 3/4)$ *and* $\theta \in (0, 3/4 - \alpha)$*. Then there is a constant* $C > 0$ *such that*

$$\left\|A^\alpha\left(u^N(t+h) - u^N(t)\right)\right\|_2 \leqslant Ch^\theta t^{-(\alpha+\theta)},$$

for all $t \in (0, T)$*,* $h \in (0, T - t]$ *and* $N \in \mathbb{N}$*.*

Proof. Let $\delta = (3/4 - \alpha - \theta)/2 > 0$. Taking the difference of (2.13) at $t + h$ and t, operating with A^α and taking norms,

$$\begin{aligned}
&\left\|A^\alpha\left(u^N(t+h) - u^N(t)\right)\right\|_2 \\
&\quad\leqslant \left\|A^\alpha e^{-At/2}\right\|_2\left\|e^{-A(t/2+h)} - e^{-At/2}\right\|_2\left\|u^N(0)\right\|_2 \\
&\qquad + \int_0^t \left\|A^{\alpha+\delta+1/4}e^{-A(t-s)/2}\right\|_2\left\|e^{-A((t-s)/2+h)} - e^{-A(t-s)/2}\right\|_2 \\
&\qquad \times \left\|A^{-(1/4+\delta)}\left[f + u^N(s) - \gamma\sum_{j=1}^n S_j^N(s)\right]\right\|_2 \mathrm{d}s \\
&\qquad + \int_0^h \left\|A^{\alpha+\delta+1/4}e^{-A(h-s)}\right\|_2\left\|A^{-(1/4+\delta)}\left[f + u^N(s+t) - \gamma\sum_{j=1}^n S_j^N(s+t)\right]\right\|_2 \mathrm{d}s.
\end{aligned}$$

In addition to the estimates used in the proof of Lemma 2.2, we also use (see Henry [4, Section 1.4]) that there is a C such that for all $\varepsilon \in [0, 1]$, $h, t \in (0, T)$,

$$\left\|e^{-At} - e^{-A(t+h)}\right\|_2 \leqslant C\left(\frac{h}{t}\right)^\varepsilon.$$

We obtain,

$$\begin{aligned}
&\left\| A^\alpha\left(u^N(t+h) - u^N(t)\right)\right\|_2 \\
&\quad \leqslant C\|u_0\|_2 t^{-\alpha}\left(\frac{h}{t}\right)^\theta \\
&\qquad + \int_0^t C(t-s)^{-(\alpha+\delta+1/4)}\left(\frac{h}{t-s}\right)^\theta\left[\|f\|_2 + \left\|u^N(s)\right\|_2 + \sum_{j=1}^n\left(\left\|A^\beta u^N(s)\right\|_2 + \left|z_j^N(s)\right|\right)\right]\mathrm{d}s \\
&\qquad + \int_0^h C(h-s)^{-(\alpha+\delta+1/4)}\left[\|f\|_2 + \left\|u^N(s+t)\right\|_2 + \left\|A^\beta u^N(s+t)\right\|_2 + \sum_{j=1}^n\left|z_j^N(s+t)\right|\right]\mathrm{d}s,
\end{aligned}$$

for some $\beta \in (1/4, 3/4)$. Applying Lemma 2.2,

$$\left\| A^\alpha\left(u^N(t+h) - u^N(t)\right)\right\|_2 \leqslant Ch^\theta t^{-(\alpha+\theta)} + Ch^\theta + Ch^{3/4-\alpha-\delta} \leqslant Ch^\theta t^{-(\alpha+\theta)},$$

for C a constant independent of N, h and t. □

Lemma 2.4. *Under the assumptions of Lemma* 2.2*, suppose* $\alpha \in (1/4, 3/4)$. *Then there exists* $C > 0$ *such that, for all* $t \in (0, T)$, $h \in (0, T-t]$, $j \in \{1, \ldots, n\}$ *and* $N \in \mathbb{N}$,

$$\left|z_j^N(t+h) - z_j^N(t)\right| \leqslant Cht^{-\alpha}, \qquad \left|y_j^N(t+h) - y_j^N(t)\right| \leqslant Ch.$$

Proof. By Lemma 2.2, z_j^N is bounded in $C[0,T]$, uniformly in N, for each $j \in \{1, \ldots, n\}$. Integrating (2.9), one obtains the Galerkin form of (2.7). Taking the difference of solutions at times $t+h$ and t (for $t \in (0,T)$ and $h \in (0, T-t)$),

$$\begin{aligned}
&\left|z_j^N(t+h) - z_j^N(t)\right| \\
&\quad \leqslant \left|e^{-(t+h)/\tau} - e^{-t/\tau}\right|\left|z_j(0)\right| + \frac{1}{\tau}\int_0^h Ce^{-(h-s)/\tau}(t+s)^{-\alpha}\,\mathrm{d}s \\
&\qquad + \frac{1}{\tau}\int_0^t Cs^{-\alpha}\left|e^{-(t-s+h)/\tau} - e^{-(t-s)/\tau}\right|\mathrm{d}s \\
&\quad \leqslant Cht^{-\alpha}.
\end{aligned}$$

For y_j, the required result follows from (2.7) and Lemma 2.2. □

The following lemma establishes existence of a mild solution for the simplified model problem, using a weak convergence argument.

Lemma 2.5. *Assume that the conditions hold, which were stated in defining the simplified model problem* (2.1)–(2.4). *Then that problem possesses a mild solution* $(u, \mathbf{z}, \mathbf{y})$. *Furthermore, any mild solution of* (2.1)–(2.4) *satisfies the estimates of Lemmas* 2.2–2.4*, with* $(u^N, \mathbf{z}^N, \mathbf{y}^N)$ *replaced by* $(u, \mathbf{z}, \mathbf{y})$.

Proof. See Appendix A. □

2.5. Optimal regularity

Here, it is shown that $u(t) \in C^{0,1}(\Omega)$, $t \in (0, T]$.

Lemma 2.6. *Given a mild solution u of* (2.1)–(2.4)*, there exist a solution $v(t) \in C^{0,1}(\Omega)$, $t \in (0, T]$, of the elliptic problem*

$$L[v] \equiv -\frac{1}{R}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + v = f - \gamma \sum_{j=1}^{n} \delta\bigl(\cdot - y_j(t)\bigr)\bigl[u\bigl(y_j(t), t\bigr) - z_j(t)\bigr], \quad x \in \Omega,$$

and a constant C, independent of t, such that $\|v(t)\|_{C^{0,1}} \leqslant Ct^{-1/4}$.

Proof. It is well known, from the method of Fourier series, that $L[v_1] = f$ has a solution $v_1 \in W^{2,2}(\Omega) \subset C^{0,1}(\Omega)$. On the other hand, for $t \in (0, T]$,

$$L\bigl[v_2(t)\bigr] = -\gamma \sum_{j=1}^{n} \delta\bigl(\cdot - y_j(t)\bigr)\bigl[u\bigl(y_j(t), t\bigr) - z_j(t)\bigr]$$

has solution

$$v_2(x, t) = \frac{\gamma\sqrt{R}}{2} \sum_{j=1}^{n} \exp\bigl(-\sqrt{R}\,\bigl|x - y_j(t)\bigr|\bigr)\bigl(u\bigl(y_j(t), t\bigr) - z_j(t)\bigr) + C_1(t)e^{-\sqrt{R}x} + C_2(t)e^{\sqrt{R}x},$$
$$x \in \Omega,$$

where $C_1(t)$ and $C_2(t)$ are chosen so that

$$\lim_{x\to 0+} v_2^{(k)}(x, t) = \lim_{x\to 1-} v_2^{(k)}(x, t), \quad k = 0, 1.$$

For each $t \in (0, T]$, we observe that $v_2(t) \in C^{0,1}(\Omega) \setminus C^1(\Omega)$; i.e., $C^{0,1}$ is essentially the optimal regularity of $v_2(t)$.

The time dependence of $v_2(t)$ in $C^{0,1}(\Omega)$ comes through the spatially uniform quantities $u_j(y_j(t), t)$ and $z_j(t)$, $y_j(t)$, $j \in \{1, \ldots, n\}$. By Lemmas 2.2 and 2.5, the y_j's and z_j's are uniformly bounded for $t \in [0, T]$ and $|u(y_j(t), t)| \leqslant Ct^{-1/4}$, for $t \in (0, T]$, $j \in \{1, \ldots, n\}$, for C independent of t. The dependence of $C_1(t)$, $C_2(t)$ and hence $\|v_2(t)\|_{C^{0,1}}$ upon $u(y_j(t), t)$, $j \in \{1, \ldots, n\}$, is linear, which implies that $\|v_2(t)\|_{C^{0,1}} \leqslant Ct^{-1/4}$. Thus, as v_1 is independent of t,

$$\bigl\|v(t)\bigr\|_{C^{0,1}} \leqslant \bigl\|v_1(t)\bigr\|_{C^{0,1}} + \bigl\|v_2(t)\bigr\|_{C^{0,1}} \leqslant Ct^{-1/4}. \qquad \square$$

Lemma 2.7. *Given a mild solution of* (2.1)–(2.4)*, there is a $C > 0$, independent of t, such that*

$$\bigl\|u(t)\bigr\|_{C^{0,1}} \leqslant Ct^{-3/4}, \quad t \in (0, T].$$

Proof. For $t \in (0, T]$,

$$\int_0^t e^{-A(t-s)}\Biggl[f - \gamma \sum_{j=1}^{n} S_j(t)\Biggr]\mathrm{d}s = \bigl[I - e^{-At}\bigr]A^{-1}\Biggl[f - \gamma \sum_{j=1}^{n} S_j(t)\Biggr] = \bigl[I - e^{-At}\bigr]v(t),$$

where $v(t)$ is as in Lemma 2.6. Hence,

$$u(t) = v(t) + e^{-At}u_0 + \int_0^t e^{-A(t-s)}u(s)\,\mathrm{d}s - e^{-At}v(t) - \gamma\int_0^t e^{-A(t-s)}\sum_{j=1}^n \bigl(S_j(s) - S_j(t)\bigr)\,\mathrm{d}s$$
$$= T_1 + T_2 + T_3 + T_4 + T_5. \tag{2.14}$$

By Lemma 2.6, $\|T_1\|_{C^{0,1}} = \|v(t)\|_{C^{0,1}} \leqslant Ct^{-1/4}$.

For T_2, we note that $e^{-At}u_0 \in D(A)$, $t > 0$. Hence, Lemmas 2.1, 2.2 and 2.5 imply that

$$\|T_2\|_{C^{0,1}} = \|e^{-At}u_0\|_{C^{0,1}} \leqslant C\bigl\|Ae^{-At}u_0\bigr\|_2^{1/2}\bigl\|A^{1/2}e^{-At}u_0\bigr\|_2^{1/2} \leqslant Ct^{-3/4}.$$

Considering T_3 and T_4 together, for some fixed $\delta \in (0, 1/8)$, Lemma 2.1 implies

$$\|T_3 + T_4\|_{C^{0,1}} \leqslant C\bigl\|A^{3/4+\delta}(T_3 + T_4)\bigr\|_2 = C\left\|A^{3/4+\delta}\left(\int_0^t e^{-A(t-s)}u(s)\,\mathrm{d}s - e^{-At}v(t)\right)\right\|_2$$
$$\leqslant \int_0^t C(t-s)^{-(3/4+\delta)}\bigl\|u(s)\bigr\|_2\,\mathrm{d}s + C\bigl\|A^{2\delta}e^{-At}\bigr\|_2\left\|A^{-(1/4+\delta)}\left(f - \gamma\sum_{j=1}^n S_j(t)\right)\right\|_2$$
$$\leqslant C + Ct^{-2\delta}\left(\|f\|_1 + \sum_{j=1}^n\bigl(\bigl\|u(t)\bigr\|_\infty + \bigl|z_j(t)\bigr|\bigr)\right)$$
$$\leqslant Ct^{-(1/4+3\delta)} \leqslant Ct^{-5/8}.$$

$$\bigl\|A^{3/4+\delta}T_5\bigr\|_2 = \left\|A^{3/4+\delta}\int_0^t e^{-A(t-s)}\gamma\sum_{j=1}^n\bigl(S_j(s) - S_j(t)\bigr)\,\mathrm{d}s\right\|_2$$
$$\leqslant \gamma\int_0^t \bigl\|A^{1+2\delta}e^{-A(t-s)}\bigr\|_2$$
$$\times \sum_{j=1}^n\bigl\|A^{-(1/4+\delta)}\delta\bigl(\cdot - y_j(s)\bigr)\bigl[\bigl(u\bigl(y_j(s), s\bigr) - u\bigl(y_j(t), t\bigr)\bigr) - \bigl(z_j(s) - z_j(t)\bigr)\bigr]\bigr\|_2\,\mathrm{d}s$$
$$+ \gamma\int_0^t \bigl\|A^{5/4+\delta}e^{-A(t-s)}\bigr\|_2\sum_{j=1}^n\bigl\|A^{-1/2}\bigl[\delta\bigl(\cdot - y_j(s)\bigr) - \delta\bigl(\cdot - y_j(t)\bigr)\bigr]\bigr\|_2$$
$$\times\bigl|u\bigl(y_j(t), t\bigr) - z_j(t)\bigr|\,\mathrm{d}s$$
$$= I_1 + I_2.$$

Note that the following inequality holds for arbitrary $w \in D(A^{1/2})$:

$$\bigl|w\bigl(y_j(s)\bigr) - w\bigl(y_j(t)\bigr)\bigr| = \left|\int_{y_j(t)}^{y_j(s)} w'(\theta)\,\mathrm{d}\theta\right| \leqslant \bigl|y_j(t) - y_j(s)\bigr|^{1/2}\|w'\|_2$$
$$\leqslant C(t-s)^{1/2}\bigl\|A^{1/2}w\bigr\|_2, \tag{2.15}$$

by the Hölder continuity of y_j, $j \in \{1, \dots, n\}$; see Lemmas 2.4, 2.5. Applying (2.15) and taking $\theta = 1/4$ in the statement of Lemma 2.3, (for u, rather than u^N),

$$I_1 \leqslant \int_0^t C(t-s)^{-(1+2\delta)}\Bigl(\bigl|u\bigl(y_j(s), s\bigr) - u\bigl(y_j(t), s\bigr)\bigr| + \bigl\|u(s) - u(t)\bigr\|_\infty + \max_{j \in [1,n]}\bigl|z_j(s) - z_j(t)\bigr|\Bigr)\,\mathrm{d}s$$

$$\leqslant \int_0^t C(t-s)^{-(1+2\delta)}\bigl((t-s)^{1/2}s^{-1/2} + (t-s)^{1/4}s^{-(1/2+\delta)} + (t-s)s^{-(1/4+\delta)}\bigr)\,\mathrm{d}s \leqslant C.$$

Considering I_2, let $B = \{\phi \in D(A^{1/2}) \mid \|A^{1/2}\phi\|_2 = 1\}$. Then, for $s \in (0, t)$

$$\begin{aligned}
&\bigl\|A^{-1/2}\bigl[\delta\bigl(\cdot - y_j(s)\bigr) - \delta\bigl(\cdot - y_j(t)\bigr)\bigr]\bigr\|_2 \\
&\quad = \sup_{\phi \in B}\bigl\langle \delta\bigl(\cdot - y_j(s)\bigr) - \delta\bigl(\cdot - y_j(t)\bigr), \phi\bigr\rangle \\
&\quad = \sup_{\phi \in B}\bigl[\phi\bigl(y_j(s)\bigr) - \phi\bigl(y_j(t)\bigr)\bigr] \leqslant C(t-s)^{1/2}, \quad j \in \{1, \dots, n\},
\end{aligned} \tag{2.16}$$

where we have applied (2.15) in the last line. Hence,

$$I_2 \leqslant \int_0^t C(t-s)^{-(5/4+\delta)} n(t-s)^{1/2} t^{-1/2}\,\mathrm{d}s \leqslant Ct^{-(1/4+\delta)}.$$

In combination with the bound for I_1, we deduce that

$$\|T_5\|_{C^{0,1}} \leqslant I_1 + I_2 \leqslant Ct^{-3/8}.$$

The proof is now completed by taking $C^{0,1}$ norms in (2.14) and summing the bounds for $\|T_k\|_{C^{0,1}}$, $k = 1, \dots, 5$. $\square$

3. Well–posedness and finite element spatial semidiscretisation

In this section we analyse a standard Galerkin finite element discretisation in space of the simplified model problem (2.1)–(2.4). We simultaneously establish an optimal order error bound in L_∞ and a proof of the well-posedness of the system (2.1)–(2.4).

3.1. Properties of the finite element space

Let $V^h \subset H^1(S_1)$, be an N-dimensional space of piecewise linear polynomials.

Projection: Let $\{\psi_i^h\}_{i=1}^N$ be a basis for V^h, and let $\mathcal{M}_{ij}^h = \langle \psi_i^h, \psi_j^h \rangle$. Then $P^h : H^{-1} \to V^h$ is defined by

$$P^h u = \sum_{i,j=1}^N \bigl(\mathcal{M}^h\bigr)_{ij}^{-1}\bigl\langle u, \psi_j^h\bigr\rangle \psi_i, \quad \text{for } u \in H^{-1}(\Omega).$$

(i) *Approximation inequality*: There exists C independent of h such that for all $s \in [1,2]$ and all $v \in H^s(\Omega) = D(A^{s/2})$,

$$\|v - P^h v\|_2 + h\|(v - P^h v)'\|_2 \leqslant Ch^s \|A^{s/2} v\|_2.$$

(ii) *Inverse inequalities*: There is a $C > 0$ independent of h such that

$$\|(\phi^h)'\|_2 \leqslant Ch^{-1}\|\phi^h\|_2, \qquad \|\phi^h\|_\infty \leqslant Ch^{-1/2}\|\phi^h\|_2, \quad \text{for all } \phi^h \in V^h.$$

(iii) *Selfadjointness* Let $A^h : V^h \to V^h$ be defined by

$$\langle A^h u^h, \phi^h\rangle \equiv \frac{1}{R}\langle (u^h)', (\phi^h)'\rangle + \langle u^h, \phi^h\rangle, \quad \text{for all } \phi^h \in V^h.$$

It is assumed that A^h is selfadjoint; i.e.,

$$\langle A^h u^h, v^h\rangle = \langle u^h, A^h v^h\rangle, \quad \text{for all } u^h, v^h \in V^h.$$

3.2. Finite element discretisation

Consider initial data, $(\hat{u}_0, \hat{\mathbf{z}}_0, \hat{\mathbf{y}}_0) \in L_2(\Omega) \times \mathbb{R}^n \times \Omega^n$. Given f, R, γ, τ from (2.1)–(2.4), for $t \in [0,T]$ we seek a solution $(u^h(t), \mathbf{z}^h(t), \mathbf{y}(t)) \in V^h \times \mathbb{R}^n \times \Omega^n$ of the finite element problem

$$u_t^h + A^h u^h = P^h f + u^h - \gamma \sum_{j=1}^n \left[u^h\left(y_j^h(t), t\right) - z_j^h(t)\right] P^h \delta\left(\cdot - y_j^h(t)\right), \tag{3.1}$$

$$\tau \frac{\mathrm{d}z_j^h}{\mathrm{d}t} = u^h\left(y_j^h(t),\ t\right) - z_j^h(t), \quad j \in \{1, \ldots, n\}, \tag{3.2}$$

$$\frac{\mathrm{d}y_j^h}{\mathrm{d}t} = z_j^h(t), \quad j \in \{1, \ldots, n\}, \tag{3.3}$$

$$u^h(0) = P^h \hat{u}_0, \qquad \mathbf{z}^h(0) = \hat{\mathbf{z}}_0, \qquad \mathbf{y}^h(0) = \hat{\mathbf{y}}_0. \tag{3.4}$$

Lemma 3.1. *Under the assumptions for the finite element method* (3.1)–(3.4)*, there exists a solution* $(u^h, \mathbf{z}^h, \mathbf{y}^h) \in C[0,T; V^h \times \mathbb{R}^n \times \Omega^n]$. *For* $t \in [0,T]$,

$$\|u^h(t)\|_2^2 + \tau\|\mathbf{z}^h(t)\|_{l_2}^2 \leqslant e^t\left(\|\hat{u}_0\|_2^2 + \|f\|_2^2 + \tau\|\hat{\mathbf{z}}_0\|_{l_2}^2\right). \tag{3.5}$$

Here, we use the notation $\|\mathbf{z}\|_{l_p} \equiv (\sum_{i=1}^n |z_i|^p)^{1/p}$, $p \in [1,\infty)$, $\mathbf{z} \in \mathbb{R}^n$.

Proof. Since (3.1)–(3.4) are a finite-dimensional system of ODEs, with locally Lipschitz nonlinearities, there exists $T^* > 0$ and a unique solution, $(u^h, \mathbf{z}^h, \mathbf{y}^h) \in C[0, T^*; V^h \times \mathbb{R}^n \times \Omega^n]$.

Taking the inner product of (3.1) with $u^h(t)$, for $t \in (0, T^*)$, and adding to the sum over j of (3.3) times $z_j(t)$, one obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u^h(t)\|_2^2 + \tau\|\mathbf{z}^h(t)\|_{l_2}^2\right) + \frac{1}{R}\langle (u^h)', (u^h)'\rangle = \langle f, u^h(t)\rangle - \gamma \sum_{j=1}^n \left[u\left(y_j^h(t), t\right) - z_j^h(t)\right]^2.$$

After applying the Cauchy–Schwarz and Young inequalities,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|u^h(t)\right\|_2^2+\tau\left\|\mathbf{z}^h(t)\right\|^2\right)\leqslant\|f\|_2^2+\left\|u^h(t)\right\|_2^2.$$

Inequality (3.5) follows, for $T=T^*$, after time integration. In turn, this bound, together with an obvious bound for $\mathbf{y}(t)$, implies that the solution of the ODE system remains bounded for finite times, and may therefore be continued until $t=T$. We finally deduce (3.5) by a repetition of the above energy inequality argument. □

3.3. Linear approximation properties

For $1\leqslant p\leqslant\infty$, let V_p^h denote the complexification of V^h with norm $\|\cdot\|_p$.

Lemma 3.2. *The operator A^h, defined under the finite element space assumptions is sectorial for $D(A^h)=X=V_2^h$, with norm $\|\cdot\|_2$, for constants M,a,θ independent of h. Furthermore, there exists $C>0$, independent of h, such that*

$$\left\|\left(A^h\right)^{1/2}\phi^h\right\|_2^2=\frac{1}{R}\left\langle\left(\phi^h\right)',\left(\phi^h\right)'\right\rangle+\left\|\phi^h\right\|_2^2\geqslant C\left\|\phi^h\right\|_{H^1}^2,\quad \textit{for all } \phi^h\in V_2^h. \tag{3.6}$$

Proof. We remark that the numerical range, (see Edmunds and Evans [3]), of A^h, $\Theta(A^h)$ satisfies

$$\begin{aligned}\Theta(A^h)&\equiv\left\{\left\langle A^h\phi^h,\phi^h\right\rangle\mid\phi^h\in V_2^h\text{ s.t. }\left\|\phi^h\right\|_2=1\right\}\\&\subseteq\left\{\left\langle A^{1/2}\phi,A^{1/2}\phi\right\rangle\mid\phi\in H^1(\Omega)\text{ s.t. }\|\phi\|_2=1\right\}\\&=\Theta(A)\subset[1,\infty).\end{aligned}$$

A standard argument, see Edmunds and Evans [3, p. 100] or Henry [4, p.18], now shows that A^h is a sectorial operator on V^h in the L_2 norm for constants (M,a,θ) which depend only on the set $[1,\infty)$. The evolution operator e^{-A^ht} and fractional powers of A^h may be defined, as in Henry [4]. These functions of A^h are subject to h-independent bounds identical to those of the corresponding functions of A. Inequality (3.6) may be shown independently. □

Lemma 3.3. *Under the assumptions defining A, in Section 2, and A^h, in Section 3, there exists C independent of h such that, for all $t\in(0,T]$,*

$$\begin{aligned}&\left\|e^{-At}-e^{-A^ht}P^h\right\|_{L_2\to L_\infty}\leqslant Cht^{-3/4},\\&\left\|e^{-At}-e^{-A^ht}P^h\right\|_{H^1\to L_\infty}\leqslant Cht^{-1/4},\\&\left\|\left[e^{-At}-e^{-A^ht}P^h\right]\delta(\cdot-y)\right\|_\infty\leqslant Cht^{-1},\quad\textit{for all } y\in\Omega.\end{aligned}$$

Proof. See Appendix A. □

3.4. Maximum norm error bounds

Let $(u,\mathbf{z},\mathbf{y})$ be a mild solution of (2.1)–(2.4). We define difference variables

$$w(t)=\left(u-u^h\right)(t);\qquad\boldsymbol{\zeta}(t)=\left(\mathbf{z}-\mathbf{z}^h\right)(t),\qquad\boldsymbol{\eta}(t)=\left(\mathbf{y}-\mathbf{y}^h\right)(t),\quad t\in[0,T]. \tag{3.7}$$

The theory of ODEs implies that the solution of (3.1)–(3.4) may be expressed in the following mild form for $t \in (0, T^*]$ and $j \in \{1, \dots, n\}$:

$$u^h(t) = e^{-A_h t} P^h u_0 + \int_0^t e^{-A^h(t-s)} \left[P^h f + u^h(s) - \gamma \sum_{j=1}^n S_j^h(s) \right] \mathrm{d}s, \tag{3.8}$$

$$z_j^h(t) = e^{-t/\tau} z_j^h(0) + \int_0^t e^{-(t-s)/\tau} u\big(y_j^h(s), s\big)\, \mathrm{d}s, \tag{3.9}$$

$$y_j^h(t) = y_j^h(0) + \int_0^t z_j^h(s)\, \mathrm{d}s, \tag{3.10}$$

where $S_j^h(s) = [u^h(y_j^h(s), s) - z_j^h(s)] P^h \delta(\cdot - y_j^h(s))$.

Lemma 3.4. *Let the assumptions hold which were made in defining the simplified model equations,* (2.1)–(2.4) *and its finite elements approximation,* (3.1)–(3.4). *Then, there exists a constant C, independent of t, such that, for all $t \in [0, T]$ and $j \in \{1, \dots, n\}$,*

$$\big|\zeta_j(t)\big| \leqslant C\left\{\big|\eta_j(0)\big| + \big|\zeta_j(0)\big| + \int_0^t \big\|w(s)\big\|_\infty \,\mathrm{d}s\right\},$$

$$\big|\eta_j(t)\big| \leqslant C\left\{\big|\eta_j(0)\big| + \big|\zeta_j(0)\big| t + \int_0^t \big\|w(s)\big\|_\infty \,\mathrm{d}s\right\}.$$

Proof (*Sketch*). The estimates for the $|\eta_j|$, $j \in \{1, \dots, n\}$, may be approached by taking the difference of (2.7) and (3.10), and using Lemma 2.7 to bound $\|u(s)\|_{C^{0,1}}$. This bound may be completed using Gronwall's Lemma. The bounds on $|\zeta_j|$ are obtained by a similar approach, which in addition makes use of the bounds on $|\eta_j|$. □

Lemma 3.5. *Let the assumptions of Lemma* 3.4 *hold. Then, there is a constant C independent of h, such that for all $t \in (0, T]$,*

$$\big\|w(t)\big\|_\infty \leqslant C\big[ht^{-3/4} + \big\|\boldsymbol{\zeta}(0)\big\|_{l_1} + t^{-1/4}\big(\big\|\boldsymbol{\eta}(0)\big\|_{l_1} + \big\|w(0)\big\|_\infty\big)\big],$$
$$\big\|\boldsymbol{\zeta}(t)\big\|_{l_1} \leqslant C\big(h\big\|\boldsymbol{\eta}(0)\big\|_{l_1} + \big\|\boldsymbol{\zeta}(0)\big\|_{l_1} + \big\|w(0)\big\|_\infty\big),$$
$$\big\|\boldsymbol{\eta}(t)\big\|_{l_1} \leqslant C\big(h + \big\|\boldsymbol{\eta}(0)\big\|_{l_1} + t\big\|\boldsymbol{\zeta}(0)\big\|_{l_1} + \big\|w(0)\big\|_\infty\big).$$

Proof. See Appendix A. □

3.5. Well–posedness

Lemma 3.5 implies (2.1)–(2.4) is well-posed:

Theorem 3.6. *Given initial data* (2.4)*, the system* (2.1)–(2.3) *has a unique mild solution over a time interval* $[0, T]$*, satisfying the estimates of Lemmas* 2.5 *and* 2.7*. If* $\hat{u}(t)$*,* $t \in [0, T]$ *is the solution of* (2.1)–(2.3) *for initial data* $(\hat{u}_0, \hat{\mathbf{z}}_0, \hat{\mathbf{y}}_0) \in L_2(\Omega) \times \mathbb{R}^n \times \Omega^n$*, then for all* $t \in (0, T]$*,*

$$\begin{aligned}
&\|(u-\hat{u})(t)\|_\infty \leqslant Ct^{-1/4}\big(\|u_0-\hat{u}_0\|_2 + \|\mathbf{y}_0-\hat{\mathbf{y}}_0\|_{l_1}\big) + C\|\mathbf{z}_0-\hat{\mathbf{z}}_0\|_{l_1},\\
&\big|(z_j-\hat{z}_j)(t)\big| \leqslant C\big(\|u_0-\hat{u}_0\|_2 + \|\mathbf{y}_0-\hat{\mathbf{y}}_0\|_{l_1} + \|\mathbf{z}_0-\hat{\mathbf{z}}_0\|_{l_1}\big),\\
&\big|(y_j-\hat{y}_j)(t)\big| \leqslant C\big(\|u_0-\hat{u}_0\|_2 + \|\mathbf{y}_0-\hat{\mathbf{y}}_0\|_{l_1} + \|\mathbf{z}_0-\hat{\mathbf{z}}_0\|_{l_1}\big).
\end{aligned}$$

Proof. The existence of a mild solution for initial data (2.4) is shown by Lemma 2.5. If the initial data for the Galerkin discretisation (3.1)–(3.4) is $(P^h u_0, \mathbf{z}_0, \mathbf{y}_0)$, then Lemma 3.5 implies

$$\lim_{h\to 0}\big(u^h, \mathbf{z}^h, \mathbf{y}^h\big)(t) = (u, \mathbf{z}, \mathbf{y})(t), \quad \text{for each } t \in (0, T]$$

in $L_\infty(\Omega) \times \mathbb{R}^n \times \Omega^n$, where $(u, \mathbf{z}, \mathbf{y})$ is any mild solution of (2.1)–(2.4). Thus, the mild solution is unique.

Suppose now that the initial data (3.4) for (3.1)–(3.4) is $(P^h\hat{u}_0, \hat{\mathbf{z}}(0), \hat{\mathbf{y}}(0))$. This time,

$$\lim_{h\to 0}\big(u^h, \mathbf{z}^h, \mathbf{y}^h\big)(t) = (\hat{u}, \hat{\mathbf{z}}, \hat{\mathbf{y}})(t), \quad \text{for each } t \in (0, T].$$

Taking the limit as $h \to 0+$ in the error bounds of Lemma 3.5 we obtain the continuity estimates in the statement of this lemma. □

3.6. Error bounds

Lemma 3.5 yields various error bounds:

Theorem 3.7. *For* $t \in (0, T]$*, let* $(u, \mathbf{z}, \mathbf{y})$ *be the mild solution of* (2.1)–(2.4) *and* $(u^h, \mathbf{z}^h, \mathbf{y}^h)$ *the solution of the Galerkin system* (3.1)–(3.4) *for initial data* $(P^h u_0, \mathbf{z}_0, \mathbf{y}_0)$*. Then, there is a constant* C *independent of* h *such that for* $t \in (0, T]$*,*

$$\begin{aligned}
&\big\|\big(u-u^h\big)(t)\big\|_\infty \leqslant Cht^{-3/4}, &&\big\|\big(u-u^h\big)(t)\big\|_{H^1} \leqslant Ch^{1/2}t^{-3/4},\\
&\big|\big(z_j-z_j^h\big)(t)\big| \leqslant Ch, &&\big|\big(y_j-y_j^h\big)(t)\big| \leqslant Ch, \quad j \in \{1, \ldots, n\}.
\end{aligned}$$

Proof. After Lemma 3.5, it only remains to prove the H^1 error bound. From the inverse inequalities, we deduce that for $t \in (0, T]$

$$\begin{aligned}
\big\|\big(u-u^h\big)(t)\big\|_{H^1} &\leqslant \big\|\big(I-P^h\big)u(t)\big\|_{H^1} + \big\|\big(P^h u-u^h\big)(t)\big\|_{H^1}\\
&\leqslant \big\|\big(I-P^h\big)u(t)\big\|_{H^1} + h^{-1/2}\big\{\big\|\big(I-P^h\big)u(t)\big\|_\infty + \big\|\big(u-u^h\big)(t)\big\|_\infty\big\}.
\end{aligned} \tag{3.11}$$

As $\|(u-u^h)(t)\|_\infty \leqslant Cht^{-3/4}$, we need only bound $(I-P^h)u(t)$ in $H^1(\Omega)$ and $L_\infty(\Omega)$.

We recall from the proof of Lemmas 2.6 and 2.7 that, for $t > 0$,

$$u(t) = -\gamma\sum_{j=1}^n A^{-1}\delta\big(\cdot - y(t)\big)\big[u\big(y_j(t), t\big) - z_j(t)\big] + u_2(t),$$

where $u_2(t) \in H^{3/2}(\Omega)$. Using the approximation property of V^h, the bound for $|z_j(t)|$ of Lemma 3.1, the bound $|u(y_j(t), t)| \leqslant \|u(t)\|_\infty \leqslant Ct^{-1/4}$, the approximation in mean of the δ-function by a sequence in $L_1(\Omega)$, and the bounds obtained in Lemmas 2.6 and 2.7 and (A.13),

$$
\begin{aligned}
&\big\|\big(I-P^h\big)u(t)\big\|_2+h\big\|\big(I-P^h\big)u(t)\big\|_{H^1}\\
&\quad\leqslant \gamma\sum_{j=1}^{n}\big\|\big(I-P^h\big)A^{-1}\delta\big(\cdot-y_j(t)\big)\big\|_2\big(\|u(t)\|_\infty+|z_j(t)|\big)\\
&\qquad+h\gamma\sum_{j=1}^{n}\big\|\big(I-P^h\big)A^{-1}\delta\big(\cdot-y_j(t)\big)\big\|_{H^1}\big(\|u(t)\|_\infty+|z_j(t)|\big)+Ch^{3/2}\|u_2(t)\|_{H^{3/2}}\\
&\quad\leqslant \big(C\big\|\big(I-P^h\big)A^{-1}\big\|_{L_1\to L_2}+Ch\big\|\big(I-P^h\big)A^{-1}\big\|_{L_1\to H^1}\big)t^{-1/4}+Ch^{3/2}t^{-3/4}\\
&\quad\leqslant C\big\|\big(I-P^h\big)A^{-1}\big\|_{L_1\to L_2}t^{-1/4}+Ch^{3/2}t^{-3/4}.
\end{aligned}\tag{3.12}
$$

Now, duality arguments and the equivalence of norms implies that

$$
\begin{aligned}
&\big\|\big(I-P^h\big)A^{-1}\big\|_{L_1\to L_2}\\
&\quad=\big\|A^{-1}\big(I-P^h\big)\big\|_{L_2\to L_\infty}\leqslant C\big\|A^{-1}\big(I-P^h\big)\big\|_2^{1/2}\big\|A^{-1}\big(I-P^h\big)\big\|_{L_2\to H^1}^{1/2}\\
&\quad\leqslant C\big\|\big(I-P^h\big)A^{-1}\big\|_2^{1/2}\big\|\big(I-P^h\big)A^{-1/2}\big\|_2^{1/2}\leqslant Ch^{3/2}.
\end{aligned}
$$

Substituting this result into (3.12), we obtain

$$
\big\|\big(I-P^h\big)u(t)\big\|_2+h\big\|\big(I-P^h\big)u(t)\big\|_{H^1}\leqslant Ch^{3/2}t^{-3/4}.\tag{3.13}
$$

From this, we deduce

$$
\big\|\big(I-P^h\big)u(t)\big\|_\infty\leqslant C\big\|\big(I-P^h\big)u(t)\big\|_2^{1/2}\big\|\big(I-P^h\big)u(t)\big\|_{H^1}^{1/2}\leqslant Cht^{-3/4}.\tag{3.14}
$$

Substituting the bounds (3.13) and (3.14) into (3.11), we obtain the required $H^1(\Omega)$ bound. □

4. Fully discrete scheme and numerical results

4.1. Semi-implicit Euler finite element full discretisation

Here, for $N\in\mathbb{N}$, V^h is the space of piecewise linears, defined on the nodes ih, $i\in\{1,\dots,N\}$, where $h=1/N$. We consider the nodal basis $\{\psi_j^h\}_{j=1}^N$, satisfying $\psi_j^h(ih)=\delta_{ij}$. Such a method is well known to satisfy the assumptions on the finite element space, made in Section 3.1.

The stiffness matrix $\mathcal{K}^h$ and the delta matrix $\mathcal{D}^h(\mathbf{y})$ are defined by

$$
\mathcal{K}_{ij}^h=\frac{1}{h}\big\langle\psi_i^h,\psi_j^h\big\rangle,\qquad \mathcal{D}_{ij}^h(\mathbf{y})=\frac{1}{h}\sum_{k=1}^{n}\psi_i^h(y_k)\psi_j^h(y_k),\quad 1\leqslant i,j\leqslant N.
$$

The delta vectors are given by $(\mathbf{d}^h(\mathbf{y}))_j=\frac{1}{h}[\psi_1^h(y_j),\dots,\psi_N^h(y_j)]^{\mathrm{T}}$, $1\leqslant j\leqslant n$. The forcing vector $\mathbf{F}=\mathcal{M}^h\mathbf{f}^h$, where $\sum_{i=1}^N(\mathbf{f}^h)_i\psi_i^h=P^hf$.

For $k=0,1,\dots,u(\cdot,k\Delta t)$ is approximated by $u^{h,k}=\sum_{i=1}^N U_i^k\psi_i^h$ and

$$
z_j(k\Delta t)\approx z_j^{h,k},\qquad y_j(k\Delta t)\approx y_j^{h,k},\quad\text{for } j\in\{1,\dots,n\}.
$$

For $k=0,1,\dots,$ we discretise (3.1)–(3.4) by a semi-implicit method, in which the stiffness component is treated implicitly, while the impulsive components are explicit. The resulting method is

$$\mathcal{M}^h\left(\frac{\mathbf{U}^{k+1}-\mathbf{U}^k}{\Delta t}\right)+\frac{1}{R}\mathcal{K}^h\mathbf{U}^{k+1}+\gamma\mathcal{D}^h\left(\mathbf{y}^{h,k}\right)\mathbf{U}^{k+1}=\mathbf{F}+\gamma\sum_{j=1}^{n}z_j^{h,k}\mathbf{d}_j^h\left(\mathbf{y}^{h,k}\right), \tag{4.1}$$

$$\frac{y_j^{h,k+1}-y_j^{h,k}}{\Delta t}=z_j^{h,k}, \tag{4.2}$$

$$\tau\frac{z_j^{h,k+1}-z_j^{h,k}}{\Delta t}=\left(\sum_{i=1}^{N}U_i^{k+1}\psi_i^h\left(y_j^{h,k+1}\right)\right)-z_j^{h,k+1}, \tag{4.3}$$

$$\sum_{i=1}^{N}\left(\mathbf{U}^0\right)_i\psi_i^h=P^hu_0,\qquad \mathbf{z}^{h,0}=\mathbf{z}_0,\qquad \mathbf{y}^{h,0}=\mathbf{y}_0, \tag{4.4}$$

where u_0, $\mathbf{z}_0$ and $\mathbf{y}_0$ are specified by (2.4).

4.2. Numerical results

The main numerical results were obtained by integrating (4.1)–(4.4) using parameters $R=\tau=1$, $n=8$, $\gamma=10$, $f=0$ and $T=5$, whilst $\Delta t=0.4h^2$. Large T and γ were used to emphasize the low order error due to δ-function terms at the expense of the initially dominant high order error due to approximating $e^{-At}u_0$. The size of the Reynold's number is relatively immaterial in this interaction.

The initial particle positions $\mathbf{y}^{h,0}$ were uniformly distributed, whilst initial velocities $\mathbf{z}^{h,0}$ were randomly assigned and of $O(1)$.

In Table 1, $E^h=u^*-u^h$, where u^* is the solution for $h=1/3200$. Fig. 1 is a log–log plot of the $L_2(\Omega)$ error. Table 2 gives the slopes of the regression line for log–log plots in $L_2(\Omega)$, $L_\infty(\Omega)$ and $H^1(\Omega)$.

4.3. Analysis of numerical results

Preliminary testing showed that the error due to time discretisation was respectively,

$$\mathrm{O}(\Delta t)\quad\text{in } L_2(\Omega),\qquad \mathrm{O}(\Delta t)\quad\text{in } L_\infty(\Omega),\qquad \mathrm{O}\left(\Delta t^{1/2}\right)\quad\text{in } H^1(\Omega).$$

Table 1
The numerical errors measured in the $L_2(\Omega)$, $L_\infty(\Omega)$ and $H_1(\Omega)$ norms

h	$\|E^h\|_2$	$\|E^h\|_\infty$	$\|E^h\|_{H^1}$
1/25	2.32e−02	4.41e−02	3.06e−01
1/50	1.21e−03	2.25e−02	1.71e−01
1/100	6.09e−03	1.20e−02	1.24e−01
1/200	3.04e−03	6.28e−03	9.18e−02
1/400	1.44e−03	3.04e−03	6.41e−02
1/800	6.16e−04	1.28e−03	4.34e−02

Table 2
Linear regression for $\log\|E^h\|=\alpha\log h+\beta$

	$L_2(\Omega)$	$L_\infty(\Omega)$	$H^1(\Omega)$
α	1.04	1.00	0.53

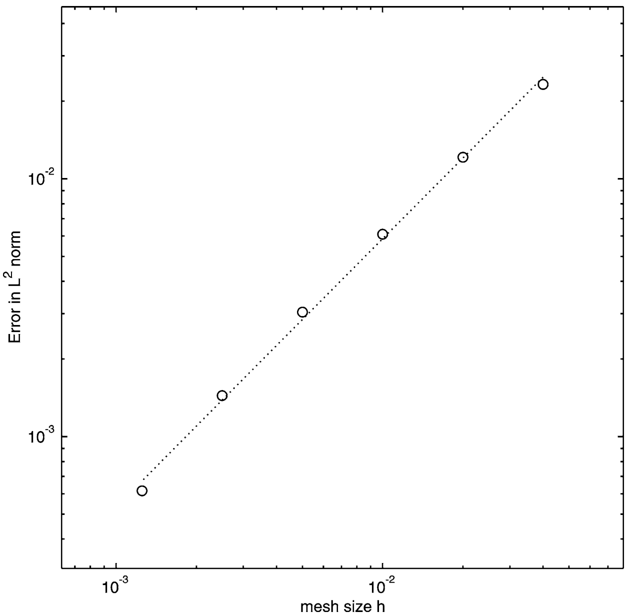

Fig. 1. Log–log plot of the numerical errors measured in the $L_2(\Omega)$ norm.

As $\Delta t = 0.4h^2$, this leads to time discretisation errors of

$$\mathrm{O}(h^2) \quad \text{in } L_2(\Omega), \qquad \mathrm{O}(h^2) \quad \text{in } L_\infty(\Omega), \qquad \mathrm{O}(h) \quad \text{in } H^1(\Omega).$$

On the other hand, from Table 2, we conclude that the overall error is

$$\mathrm{O}(h) \quad \text{in } L_2(\Omega), \qquad \mathrm{O}(h) \quad \text{in } L_\infty(\Omega), \qquad \mathrm{O}(h^{1/2}) \quad \text{in } H^1(\Omega). \tag{4.5}$$

We can therefore be confident that the orders of convergence stated in (4.5) are entirely due to the spatial discretisation and so are directly comparable with the results of Section 3.

We observe that the $L_\infty(\Omega)$ and $H^1(\Omega)$ results are in agreement with Theorem 3.6, for fixed $t = T = 5$. However, more interesting is the implication that

$$\left\|\left(u - u^h\right)(t)\right\|_2 = C(t)h. \tag{4.6}$$

This is suboptimal, since, from the proof of Theorem 3.7, $\|(I - P^h)u(t)\|_2 \leqslant C(t)h^{3/2}$.

To see why this suboptimality occurs, turn ahead to (A.9). Here, the term T_9 is bounded directly in terms of $w(y_j^h(s), s)$ and therefore the u-error in $L_\infty(\Omega)$, which is $\mathrm{O}(h)$. The terms T_6, T_7, T_{10} and T_{11} are bounded in terms of $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$, which are in turn bounded by the u-error in $L_\infty(\Omega)$, as is shown by Lemma 3.4. Thus, provided that these terms do not cancel one another out, the error in any norm is limited by the $\mathrm{O}(h)$ error in $L_\infty(\Omega)$.

5. Conclusions

The immediate conclusions from the analysis are:

(i) The simplified model equations are well-posed;
(ii) The delta function forcing limits the regularity of the solution, so that $u(t) \in C^{0,1}(\Omega)$ for $t > 0$;
(iii) The delta function forcing term depends on a pointwise evaluation of $u(y_j(t), t)$. In a spatial finite element discretisation, the error in $u(y_j(t), t)$ depends upon the error in $L_\infty(\Omega)$. This feeds through to limit the order of accuracy obtainable in $L_p(\Omega)$, for $p \in [1, \infty)$, to be the same as that in $L_\infty(\Omega)$;
(iv) Conclusions (ii) and (iii) indicate that if a standard finite element is to be used, then one of low order is appropriate.

From a broader perspective, it is to be expected that, whilst higher dimensional problems will raise many new difficulties, aspects of such problems will retain some features similar to the one-dimensional case.

Appendix A

Proof of Lemma 2.5. Following Ladyženskaja et al. [7], consider $a_k^N : [0, T] \to \mathbb{R}$, where $a_k^N(t) = \langle u^N(t), \psi_k \rangle$. Applying Lemma 2.3, with $\alpha = 0$ and $\theta = 1/2$, there is a $C > 0$, such that for all $k \in \mathbb{N}$ and all $N \in \mathbb{N}$,

$$\left|a_k^N(t+h) - a_k^N(t)\right| \leqslant C \left\|u^N(t+h) - u^N(t)\right\|_2 \leqslant Ch^{1/2}.$$

Thus, for fixed $k \in \mathbb{N}$, $\{a_k^N\}_{N \in \mathbb{N}} \subset C[0, T]$ is bounded and equicontinuous. By the Arzèla–Ascoli theorem there is a convergent subsequence converging to a limit $a^k \in C[0, T]$. One may extract a diagonal subsequence N_m of N's such that the $\{a_k^N\}_{N \in \mathbb{N}}$ converge in $C[0, T]$, uniformly in k. Defining $u(t) = \sum_{k=-\infty}^{\infty} a_k(t)\psi_k$, $t \in [0, T]$, the N-uniform bound on u^N in $C[0, T; L_2(\Omega)]$ implies the weak convergence of u^{N_m} to u in $C[0, T; L_2(\Omega)]$.

For $\alpha \in (1/4, 3/4)$ and $\delta \in (0, T)$, Lemma 2.2 yields

$$\left|a_k^N(t)\right| \leqslant \lambda_k^{-\alpha} \sqrt{\sum_{m=-\infty}^{\infty} \left|a_m^N(t)\right|^2 \lambda_m^{2\alpha}} = \lambda_k^{-\alpha} \left\|A^\alpha u^N(t)\right\|_2 \leqslant C(\delta\lambda_k)^{-\alpha} \leqslant C_\delta k^{-2\alpha},$$

for all $t \in [\delta, T]$, $k \in \mathbb{Z} \setminus \{0\}$ and $N \in \mathbb{N}$. This estimate implies the strong convergence of the subsequence u^{N_m} to u in $C[\delta, T; D(A^\beta)]$, for $\delta \in (0, T)$ and $\beta \in (0, 3/4)$.

Estimating the integrals slightly differently in the proof of Lemma 2.4, we find that, for $\alpha \in (1/4, 3/4)$, there is a $C > 0$ such that for all $j \in \{1, \ldots, n\}$,

$$\left|z_j^N(t+h) - z_j^N(t)\right| \leqslant Ch^{1-\alpha}, \quad \text{for all } h > 0,\ t \in [0, T-h],\ N \in \mathbb{N}.$$

Noting Lemma 2.4, $\{z_j^N\}_{N \in \mathbb{N}}, \{y_j^N\}_{N \in \mathbb{N}} \subset C[0, T]$ are bounded and equicontinuous and thus possess subsequences converging to $z_j, y_j \in C[0, T]$ for each $j \in \{1, \ldots, n\}$. One may thus extract a convergent diagonal subsequence $(u^{N_m}, \mathbf{z}^{N_m}, \mathbf{y}^{N_m}) \to (u, \mathbf{z}, \mathbf{y})$ in

$$\left(C\left[0, T; L_2(\Omega)\right] \cap C\left[\delta, T; D\left(A^\alpha\right)\right]\right) \times C\left[0, T; \mathbb{R}^n \times \Omega^n\right],$$

for $\delta \in (0, T)$, $\alpha \in (0, 3/4)$. So the bounds of Lemmas 2.2–2.4 also apply to $(u, \mathbf{z}, \mathbf{y})$.

Integrating (2.8) and taking the L_2 inner product with some $\phi \in H^1(\Omega)$,

$$\langle u^N(t) - P^N u_0, P^N\phi\rangle + \frac{1}{R}\int_0^t \langle (u^N)', (P^N\phi)'\rangle \,\mathrm{d}s$$
$$= \int_0^t \langle f, P^N\phi\rangle \,\mathrm{d}s - \gamma \int_0^t \sum_{j=1}^n \big[u^N\big(y_j^N(s), s\big) - z_j^N(s)\big]\big(P^N\phi\big)\big(y_j^N(s)\big)\,\mathrm{d}s.$$

Taking the limit as $m \to \infty$ through the convergent subsequence above, u^{N_m} converges to u in $C[\delta, T; H^1(\Omega)]$ for any $\delta > 0$, whilst the contribution to the integrals for the second and fourth terms over $[0, \delta]$ may be bounded in terms of δ, using Lemma 2.2. Hence,

$$\langle u(t) - u_0, \phi\rangle + \frac{1}{R}\int_0^t \langle u', \phi'\rangle \,\mathrm{d}s$$
$$= \int_0^t \langle f, \phi\rangle \,\mathrm{d}s - \gamma \int_0^t \sum_{j=1}^n \big[u\big(y_j(s), s\big) - z_j(s)\big]\phi\big(y_j(s)\big)\,\mathrm{d}s. \tag{A.1}$$

For $t \in [0, T]$, we define the function $\tilde{u}(t)$ by

$$\tilde{u}(t) \equiv e^{-At}u_0 + \int_0^t e^{-A(t-s)}\left[f + u(s) - \gamma\sum_{j=1}^n S_j(s)\right]\mathrm{d}s. \tag{A.2}$$

Performing the same operations as in Lemma 2.2, we deduce that $\tilde{u}(t) \in D(A^\alpha)$, for $\alpha < 3/4$, $t \in (0, T]$. Integrating $\tilde{u}(r)$ over $[0, t]$ and exchanging the order of integration in the last term, by Fubini's theorem,

$$\int_0^t \tilde{u}(r)\,\mathrm{d}r = \int_0^t e^{-Ar}u_0\,\mathrm{d}r + \int_0^t\int_0^{t-s} e^{-Ar}\left[f + u(s) - \gamma\sum_{j=1}^n S_j(s)\right]\mathrm{d}r\,\mathrm{d}s.$$

Applying the identity $A\int_0^t e^{-As}g\,\mathrm{d}s = g - e^{-At}g$, for $g \in D(A^{-1/2}) = H^{-1}(\Omega)$, see Miklavčič [9, p. 211],

$$\tilde{u}(t) - u_0 + A\int_0^t \tilde{u}(s)\,\mathrm{d}s = \int_0^t \left[f + u(s) - \gamma\sum_{j=1}^n S_j(s)\right]\mathrm{d}s.$$

Taking the inner product with ϕ as above, and subtracting from (A.1),

$$\langle u(t) - \tilde{u}(t), \phi\rangle + \frac{1}{R}\left\langle \int_0^t (u - \tilde{u})'(s)\,\mathrm{d}s, \phi'\right\rangle = -\left\langle \int_0^t u(s) - \tilde{u}(s)\,\mathrm{d}s, \phi\right\rangle.$$

Setting $\phi(t) = \int_0^t (u - \tilde{u})(s)\,\mathrm{d}s$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi(t)\|_2^2 = \langle u(t) - \tilde{u}(t), \phi\rangle \leqslant -\|\phi'\|_2^2 - \|\phi\|_2^2 \leqslant 0.$$

Thus, $\int_0^t u(s)\,\mathrm{d}s = \int_0^t \tilde{u}(s)\,\mathrm{d}s$ for all $t \in (0, T]$. We conclude that u satisfies (2.5). The other properties of a mild solution follow on integration of (2.2) and (2.3). We remark that *any* mild solution also satisfies the bounds of Lemmas 2.2–2.4, with N's removed, the proofs being identical to the Galerkin case. □

Proof of Lemma 3.3. Thanks to Thomée [10, p. 41], there is a $C > 0$ such that, for all $s \in [0, 1]$,

$$\left\| e^{-At} - e^{-A^h t} P^h \right\|_2 \leqslant C\left(h^2 t^{-1}\right)^s. \tag{A.3}$$

Applying the inverse inequality (A.3) with $s = 3/4$, and the approximation property with $s = 3/2$, we obtain

$$\begin{aligned}
&\left\| e^{-At} - e^{-A^h t} P^h \right\|_{L_2 \to H^1} \\
&\quad \leqslant \left\| P^h \left[e^{-At} - e^{-A^h t} P^h \right] \right\|_{L_2 \to H^1} + \left\| \left[I - P^h \right] e^{-At} \right\|_{L_2 \to H^1} \\
&\quad \leqslant C h^{-1} \left\| e^{-At} - e^{-A^h t} P^h \right\|_2 + C h^{-1} h^{3/2} \left\| e^{-At} \right\|_{H^{3/2}} \\
&\quad \leqslant C h^{-1} C h^{3/2} t^{-3/4} + C h^{1/2} \left\| A^{3/4} e^{-At} \right\|_2 \\
&\quad \leqslant C h^{1/2} t^{-3/4} + C h^{1/2} t^{-3/4} \leqslant C h^{1/2} t^{-3/4}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\left\| e^{-At} - e^{-A^h t} P^h \right\|_{L_2 \to L_\infty} \\
&\quad \leqslant \left\| e^{-At} - e^{-A^h t} P^h \right\|_2^{1/2} \left\| e^{-At} - e^{-A^h t} P^h \right\|_{L_2 \to H^1}^{1/2} \\
&\quad \leqslant \left(C h^{3/2} t^{-3/4} \right)^{1/2} \left(C h^{1/2} t^{-3/4} \right)^{1/2} \leqslant C h t^{-3/4},
\end{aligned}$$

which is the first part of the lemma.

For the second part of the lemma, we note from Thomée [10, p. 45] that

$$\left\| e^{-At} - e^{-A^h t} P^h \right\|_{H^1 \to L_2} \leqslant C h^{s+1} t^{-s/2}, \quad s \in [0, 1]. \tag{A.4}$$

By [10, p. 26], the approximation property and (A.4) with $s = 1$,

$$\begin{aligned}
\left\| e^{-At} - e^{-A^h t} P^h \right\|_{H^1} &\leqslant \left\| P^h \left[e^{-At} - e^{-A^h t} P^h \right] \right\|_{H^1} + \left\| \left[I - P^h \right] e^{-At} \right\|_{H^1} \\
&\leqslant C h^{-1} \left\| e^{-At} - e^{-A^h t} P^h \right\|_{H^1 \to L_2} + C h \leqslant C h t^{-1/2}.
\end{aligned}$$

Substituting this estimate and (A.4), with $s = 0$, into the inequality

$$\left\| e^{-At} - e^{-A^h t} P^h \right\|_{H^1 \to L_\infty} \leqslant \left\| e^{-At} - e^{-A^h t} P^h \right\|_{H^1}^{1/2} \left\| e^{-At} - e^{-A^h t} P^h \right\|_{H^1 \to L_2}^{1/2},$$

we deduce the second part of the lemma.

By the selfadjointness of A, the duality of $L_1(\Omega)$ and $L_\infty(\Omega)$ and Lemma 2.1,

$$\left\| e^{-At} \right\|_{L_1 \to L_2} = \left\| e^{-At} \right\|_{L_2 \to L_\infty} \leqslant C \left\| e^{-At} \right\|_2^{1/2} \left\| A^{1/2} e^{-At} \right\|_2^{1/2} \leqslant C t^{-1/4}. \tag{A.5}$$

Approximating $\delta(\cdot - y)$ by a sequence in L_1 convergent in mean,

$$\left\| e^{-At} \delta(\cdot - y) \right\|_2 \leqslant C \left\| e^{-At} \right\|_{L_1 \to L_2} \leqslant C t^{-1/4}. \tag{A.6}$$

By Lemma 3.2, there is a $C > 0$, independent of h, such that for all $t \in (0, T]$,

$$\left\| e^{-A^h t} \right\|_{V_2^h \to V_\infty^h} \leqslant C \left\| e^{-A^h t} \right\|_{V_2^h}^{1/2} \left\| \left(A^h\right)^{1/2} e^{-A^h t} \right\|_{V_2^h}^{1/2} \leqslant C t^{-1/4}. \tag{A.7}$$

For $B = \{\phi^h \in V^h \mid \|\phi^h\|_2 = 1\}$, (A.7) implies that

$$\begin{aligned}\left\|e^{-A^h t} P^h \delta(\cdot - y)\right\|_{V_2^h} &= \sup_{\phi^h \in B} \left\langle e^{-A^h t} P^h \delta(\cdot - y), \phi^h \right\rangle = \sup_{\phi^h \in B} \left\langle P^h \delta(\cdot - y), e^{-A^h t} \phi^h \right\rangle \\ &= \sup_{\phi^h \in B} \left(e^{-A^h t} \phi^h\right)(y) \leqslant \left\|e^{-A^h t}\right\|_{V_2^h \to V_\infty^h} \leqslant C t^{-1/4}. \end{aligned} \tag{A.8}$$

We now consider the identity, cf. Thomée [10, p. 41],

$$\begin{aligned}&\left[e^{-At} - e^{-A^h t} P^h\right]\delta(\cdot - y) \\ &\quad = \left[e^{-At/2} - e^{-A^h t/2} P^h\right] e^{-At/2} \delta(\cdot - y) + e^{-At/2}\left[e^{-At/2} - e^{-A^h t/2} P^h\right]\delta(\cdot - y) \\ &\qquad - \left[e^{-At/2} - e^{-A^h t/2} P^h\right]^2 \delta(\cdot - y) = T_1 + T_2 + T_3. \end{aligned}$$

For T_1, by the first part of the lemma and (A.6),

$$\begin{aligned}\|T_1\|_\infty &\leqslant \left\|\left[e^{-At/2} - e^{-A^h t/2} P^h\right] e^{-At/2}\right\|_{L_1 \to L_\infty} \leqslant \left\|e^{-At/2} - e^{-A^h t/2} P^h\right\|_{L_2 \to L_\infty} \left\|e^{-At/2}\right\|_{L_1 \to L_2} \\ &\leqslant C h t^{-3/4} C t^{-1/4} \leqslant C h t^{-1}. \end{aligned}$$

For T_2, approximating the delta function in mean by a sequence in L_1, using the selfadjointness of A, A^h and the projector P^h, and the duality of L_1 and L_∞,

$$\begin{aligned}\|T_2\|_\infty &= \left\|e^{-At/2}\left[e^{-At/2} - e^{-A^h t/2} P^h\right]\delta(\cdot - y)\right\|_\infty \leqslant C \left\|e^{-At/2}\left[e^{-At/2} - e^{-A^h t/2} P^h\right]\right\|_{L_1 \to L_\infty} \\ &= C \left\|\left[e^{-At/2} - e^{-A^h t/2} P^h\right] e^{-At/2}\right\|_{L_1 \to L_\infty} \leqslant C h t^{-1}, \end{aligned}$$

where the last line follows from the argument to bound T_1.

For T_3, the first part of the lemma, (A.6) and (A.8) imply that

$$\begin{aligned}\|T_3\|_\infty &\leqslant \left\|e^{-At/2} - e^{-A^h t/2} P^h\right\|_{L_2 \to L_\infty} \left\|\left[e^{-At/2} - e^{-A^h t/2} P^h\right]\delta(\cdot - y)\right\|_2 \\ &\leqslant C h t^{-3/4}\left(\left\|e^{-At/2}\delta(\cdot - y)\right\|_2 + \left\|e^{-A^h t/2} P^h \delta(\cdot - y)\right\|_2\right) \\ &\leqslant C h t^{-3/4}\left(C t^{-1/4} + C t^{-1/4}\right) \leqslant C h t^{-1}. \end{aligned}$$

Summing the bounds for T_1, T_2 and T_3 completes the proof. □

Proof of Lemma 3.5. Taking the difference of the mild forms (2.5) and (3.8),

$$\begin{aligned} w(t) &= e^{-At} u_0 - e^{-A^h t} P^h \hat{u}_0 \\ &\quad + \int_0^t e^{-A(t-s)}\left[f + u(s) - \gamma \sum_{j=1}^n \delta\left(\cdot - y_j(s)\right)\left[u\left(y_j(s), s\right) - z_j(s)\right]\right] \mathrm{d}s \\ &\quad - \int_0^t e^{-A^h(t-s)}\left[P^h f + u^h(s) - \gamma \sum_{j=1}^n P^h \delta\left(\cdot - y_j^h(s)\right)\left[u^h\left(y_j^h(s), s\right) - z_j^h(s)\right]\right] \mathrm{d}s. \end{aligned}$$

Thus,

$$
\begin{aligned}
w(t) = {} & e^{-At} w(0) + \left[e^{-At} - e^{-A^h t} P^h\right]\hat{u}_0 + \int_0^t \left[e^{-A(t-s)} - e^{-A^h(t-s)} P^h\right] f \, \mathrm{d}s \\
& + \int_0^t \left[e^{-A(t-s)} - e^{-A^h(t-s)} P^h\right] u(s) \, \mathrm{d}s + \int_0^t e^{-A^h(t-s)} P^h w(s) \, \mathrm{d}s \\
& - \int_0^t e^{-A(t-s)} \gamma \sum_{j=1}^n \delta\big(\cdot - y_j(s)\big)\left[u\big(y_j(s), s\big) - u\big(y_j^h(s), s\big)\right] \mathrm{d}s \\
& - \int_0^t e^{-A(t-s)} \gamma \sum_{j=1}^n \left[\delta\big(\cdot - y_j(s)\big) - \delta\big(\cdot - y_j^h(s)\big)\right] \\
& \qquad \times \left\{\left[u\big(y_j^h(s), s\big) - u\big(y_j^h(t), t\big)\right] - \left[z_j(s) - z_j(t)\right]\right\} \mathrm{d}s \\
& - \int_0^t \left[e^{-A(t-s)} - e^{-A^h(t-s)} P^h\right] \gamma \sum_{j=1}^n \delta\big(\cdot - y_j^h(s)\big) \\
& \qquad \times \left\{\left[u\big(y_j^h(s), s\big) - u\big(y_j^h(t), t\big)\right] - \left[z_j(s) - z_j(t)\right]\right\} \mathrm{d}s \\
& - \int_0^t e^{-A^h(t-s)} \gamma \sum_{j=1}^n P^h \delta\big(\cdot - y_j^h(s)\big)\left[w\big(y_j^h(s), s\big) - \zeta_j(s)\right] \mathrm{d}s \\
& - \int_0^t e^{-A(t-s)} \gamma \sum_{j=1}^n \left[u\big(y_j^h(t), t\big) - z_j(t)\right] \\
& \qquad \times \left\{\delta\big(\cdot - y_j(s)\big) - \delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j(t)\big) + \delta\big(\cdot - y_j^h(t)\big)\right\} \mathrm{d}s \\
& - \int_0^t e^{-A(t-s)} \gamma \sum_{j=1}^n \left[u\big(y_j^h(t), t\big) - z_j(t)\right]\left\{\delta\big(\cdot - y_j(t)\big) - \delta\big(\cdot - y_j^h(t)\big)\right\} \mathrm{d}s \\
& - \int_0^t \left[e^{-A(t-s)} - e^{-A^h(t-s)} P^h\right] \gamma \sum_{j=1}^n \left[u\big(y_j^h(t), t\big) - z_j(t)\right]\left\{\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\right\} \mathrm{d}s \\
& - \int_0^t \left[e^{-A(t-s)} - e^{-A^h(t-s)} P^h\right] \gamma \sum_{j=1}^n \left[u\big(y_j^h(t), t\big) - z_j(t)\right] \delta\big(\cdot - y_j^h(t)\big) \, \mathrm{d}s \\
= {} & T_1 + \cdots + T_{13}.
\end{aligned}
\tag{A.9}
$$

For T_1, Lemma 2.1 implies that

$$
\|T_1\|_\infty = \left\|e^{-At} w(0)\right\|_\infty \leqslant C \left\|A^{1/2} e^{-At} w(0)\right\|_2^{1/2} \left\|e^{-At} w(0)\right\|_2^{1/2} \leqslant C t^{-1/4} \left\|w(0)\right\|_2.
$$

For T_2, Lemma 3.3 implies that

$$\|T_2\|_\infty \leqslant \left\|e^{-At} - e^{-A^h t}P^h\right\|_{L_2\to L_\infty}\|\hat{u}_0\|_2 \leqslant Cht^{-3/4}.$$

$$\begin{aligned}\|T_3\|_\infty &\leqslant \int_0^t \left\|e^{-A(t-s)} - e^{-A^h(t-s)}P^h\right\|_{L_2\to L_\infty}\|f\|_2\,\mathrm{d}s \\ &\leqslant \int_0^t Ch(t-s)^{-3/4}\|f\|_2\,\mathrm{d}s \leqslant Ch.\end{aligned}$$

$$\begin{aligned}\|T_4\|_\infty &\leqslant \int_0^t \left\|e^{-A(t-s)} - e^{-A^h(t-s)}P^h\right\|_{L_2\to L_\infty}\left\|u(s)\right\|_2\,\mathrm{d}s \\ &\leqslant \int_0^t Ch(t-s)^{-3/4}\,\mathrm{d}s \leqslant Ch.\end{aligned}$$

For T_5, Lemma 3.3 implies that

$$\begin{aligned}\left\|e^{-A^h t}P^h\right\|_\infty &\leqslant C\left\|e^{-A^h t}P^h\right\|_{L_2\to L_\infty} \leqslant C\left\|e^{-A^h t}P^h\right\|_{L_2\to L_2}^{1/2}\left\|e^{-A^h t}P^h\right\|_{L_2\to H^1}^{1/2} \\ &\leqslant C\left\|\left(A^h\right)^{1/2}e^{-A^h t}\right\|_{V_2^h}^{1/2} \leqslant Ct^{-1/4}.\end{aligned}$$

Hence,

$$\|T_5\|_\infty \leqslant \int_0^t \left\|e^{-A^h(t-s)}P^h\right\|_\infty\left\|w(s)\right\|_\infty\,\mathrm{d}s \leqslant C\int_0^t (t-s)^{-1/4}\left\|w(s)\right\|_\infty\,\mathrm{d}s.$$

For T_6, Lemma 2.1 with $\delta = 1/16$ and Lemma 2.7 imply

$$\begin{aligned}\|T_6\|_\infty &\leqslant C\left\|A^{5/16}T_6\right\|_\infty \\ &\leqslant \int_0^t \left\|A^{5/8}e^{-A(t-s)}\right\|_2 \gamma\sum_{j=1}^n \left\|A^{5/16}\delta\big(\cdot - y_j(s)\big)\right\|_2\left\|u(s)\right\|_{C^{0,1}}\left|\eta_j(s)\right|\,\mathrm{d}s \\ &\leqslant C\int_0^t (t-s)^{-5/8}s^{-3/4}\left(\left\|\boldsymbol{\eta}(0)\right\|_{l_1} + \left\|\boldsymbol{\zeta}(0)\right\|_{l_1} + \int_0^s \left\|w(\sigma)\right\|_\infty\,\mathrm{d}\sigma\right)\mathrm{d}s \\ &\leqslant C\left(\left\|\boldsymbol{\eta}(0)\right\|_{l_1} + \left\|\boldsymbol{\zeta}(0)\right\|_{l_1} + \int_0^t \left\|w(s)\right\|_\infty\,\mathrm{d}s\right).\end{aligned}$$

For T_7, applying Lemma 2.1 with $\delta = 1/16$,

$$\|T_7\|_\infty \leqslant C\left\|A^{5/16}T_7\right\|_2$$

$$\leqslant C\int_0^t \sum_{j=1}^n \|A^{9/8}e^{-A(t-s)}\|_2 \|A^{-13/16}[\delta(\cdot - y_j(s)) - \delta(\cdot - y_j^h(s))]\|_2$$
$$\times \left(\left|u\left(y_j(s),s\right) - u\left(y_j^h(t),t\right)\right| + \left|z_j(s) - z_j(t)\right|\right)\mathrm{d}s.$$

Taking $B = \{\phi \in D(A^{13/16}) \mid \|A^{13/16}\phi\|_2 = 1\}$ and noting Lemma 2.1,

$$\begin{aligned}
&\|A^{-13/16}[\delta(\cdot - y_j(s)) - \delta(\cdot - y_j^h(s))]\|_2 \\
&\quad = \sup_{\phi\in B}\langle \delta(\cdot - y_j(s)) - \delta(\cdot - y_j^h(s)), \phi\rangle = \sup_{\phi\in B}\left(\phi(y_j(s)) - \phi(y_j^h(s))\right) \\
&\quad \leqslant \Big(\sup_{\phi\in B}\|\phi\|_{C^{0,1}}\Big)\left|\eta_j(s)\right| \leqslant C\left|\eta_j(s)\right|, \quad j \in \{1,\dots,n\}.
\end{aligned}$$

For $j \in \{1,\dots,n\}$ and $0 \leqslant s \leqslant t \leqslant T$, Lemma 3.1 implies that

$$\left|y_j^h(t) - y_j^h(s)\right| = \left|\int_s^t z_j^h(\sigma)\,\mathrm{d}\sigma\right| \leqslant (t-s)\sup_{\sigma\in[0,T]}\left|z_j^h(\sigma)\right| \leqslant C(t-s).$$

So, for $j \in \{1,\dots,n\}$ and $0 \leqslant s \leqslant t \leqslant T$, Lemmas 2.3 and 2.5 imply

$$\begin{aligned}
&\left|u\left(y_j^h(s),s\right) - u\left(y_j^h(t),t\right)\right| \\
&\quad \leqslant \left|u\left(y_j^h(s),s\right) - u\left(y_j^h(t),s\right)\right| + \left|u\left(y_j^h(t),s\right) - u\left(y_j^h(t),t\right)\right| \\
&\quad \leqslant C\left\|u(s)\right\|_{C^{0,1}}\left|y_j^h(t) - y_j^h(s)\right| + C\left\|A^{5/16}\left(u(t) - u(s)\right)\right\|_2 \\
&\quad \leqslant Cs^{-3/4}(t-s) + Cs^{-9/16}(t-s)^{1/4}.
\end{aligned}$$

Thus, by Lemmas 2.4 and 3.4,

$$\begin{aligned}
\|T_7\|_\infty &\leqslant C\int_0^t \sum_{j=1}^n (t-s)^{-9/8}\left|\eta_j(s)\right|\left(s^{-3/4}(t-s) + s^{-9/16}(t-s)^{1/4} + s^{-5/16}(t-s)\right)\mathrm{d}s \\
&\leqslant C\int_0^t (t-s)^{-7/8}s^{-9/16}\left(\left\|\boldsymbol{\eta}(0)\right\|_{l_1} + \left\|\boldsymbol{\zeta}(0)\right\|_{l_1} + \int_0^s \left\|w(\sigma)\right\|_\infty \mathrm{d}\sigma\right)\mathrm{d}s \\
&\leqslant C\left(\left\|\boldsymbol{\eta}(0)\right\|_{l_1} + \left\|\boldsymbol{\zeta}(0)\right\|_{l_1} + \int_0^t \left\|w(s)\right\|_\infty \mathrm{d}s\right).
\end{aligned}$$

Similarly for T_8, but this time using Lemma 3.1 as well,

$$\begin{aligned}
\|T_8\|_\infty &\leqslant C\int_0^t \sum_{j=1}^n \left\|\left[e^{-A(t-s)} - e^{-A^h(t-s)}P^h\right]\delta\left(\cdot - y_j^h(s)\right)\right\|_\infty \\
&\qquad \times \left\{\left|\left(u\left(y_j^h(s),s\right) - u\left(y_j^h(t),t\right)\right)\right| + \left|z_j(s) - z_j(t)\right|\right\}\mathrm{d}s \\
&\leqslant C\int_0^t h(t-s)^{-1}\left(s^{-3/4}(t-s) + s^{-9/16}(t-s)^{1/4}\right)\mathrm{d}s \leqslant Ch.
\end{aligned}$$

For T_9, the derivation of (A.8) and Lemma 3.1 yield

$$\begin{aligned}\|T_9\|_\infty &\leqslant C\int_0^t \sum_{j=1}^n \left\|e^{-A^h(t-s)/2}\right\|_{V_2^h\to V_\infty^h}\left\|e^{-A^h(t-s)/2}P^h\delta\big(\cdot - y_j^h(s)\big)\right\|_{V_2^h}\left[\left\|w(s)\right\|_\infty + \left|\zeta_j(s)\right|\right]\mathrm{d}s\\ &\leqslant C\int_0^t (t-s)^{-1/2}\left(\left\|\boldsymbol{\eta}(0)\right\|_{l_1} + \left\|\boldsymbol{\zeta}(0)\right\|_{l_1} + \left\|w(s)\right\|_\infty + \int_0^s \left\|w(\sigma)\right\|_\infty \mathrm{d}\sigma\right)\mathrm{d}s\\ &\leqslant C\left(\left\|\boldsymbol{\eta}(0)\right\|_{l_1} + \left\|\boldsymbol{\zeta}(0)\right\|_{l_1} + \int_0^t (t-s)^{-1/2}\left\|w(s)\right\|_\infty \mathrm{d}s\right).\end{aligned}$$

For T_{10}, Lemma 2.1 with $\delta = 1/8$ implies that

$$\begin{aligned}\|T_{10}\|_\infty &\leqslant C\left\|A^{3/8}T_9\right\|_2\\ &\leqslant C\int_0^t \left\|A^{7/4}e^{-A(t-s)}\right\|_2\left\|A^{-(11/8)}\right.\\ &\quad\times\left\{\left[\delta\big(\cdot - y_j(s)\big) - \delta\big(\cdot - y_j^h(s)\big)\right] - \left[\delta\big(\cdot - y_j(t)\big) - \delta\big(\cdot - y_j^h(t)\big)\right]\right\}\Big\|_2\\ &\quad\times\left(\left\|u(t)\right\|_\infty + \left|z_j(t)\right|\right)\mathrm{d}s.\end{aligned}$$

Now, for $B = \{\phi \in D(A^{11/8}) \mid \|A^{11/8}\phi\|_2 = 1\}$, applying Lemma 3.4,

$$\begin{aligned}&\left\|A^{-11/8}\left\{\left[\delta\big(\cdot - y_j(s)\big) - \delta\big(\cdot - y_j^h(s)\big)\right] - \left[\delta\big(\cdot - y_j(t)\big) - \delta\big(\cdot - y_j^h(t)\big)\right]\right\}\right\|_2\\ &\quad= \sup_{\phi\in B}\left\langle\left[\delta\big(\cdot - y_j(s)\big) - \delta\big(\cdot - y_j^h(s)\big)\right] - \left[\delta\big(\cdot - y_j(t)\big) - \delta\big(\cdot - y_j^h(t)\big)\right], \phi\right\rangle\\ &\quad= \sup_{\phi\in B}\left\{\left[\phi\big(y_j(s)\big) - \phi\big(y_j^h(s)\big)\right] - \left[\phi\big(y_j(t)\big) - \phi\big(y_j^h(t)\big)\right]\right\}\\ &\quad= \sup_{\phi\in B}\left\{\int_{y_j^h(t)}^{y_j(t)+[y_j(s)-y_j^h(s)-y_j(t)+y_j^h(t)]} \phi'\big(\theta + y_j^h(s) - y_j^h(t)\big)\,\mathrm{d}\theta - \int_{y_j^h(t)}^{y_j(t)} \phi'(\theta)\,\mathrm{d}\theta\right\}\\ &\quad= \sup_{\phi\in B}\left\{\int_{y_j(t)}^{y_j(t)+[y_j(s)-y_j^h(s)-y_j(t)+y_j^h(t)]} \phi'\big(\theta + y_j^h(s) - y_j^h(t)\big)\,\mathrm{d}\theta\right.\\ &\qquad\left. + \int_{y_j^h(t)}^{y_j(t)} \left[\phi'\big(\theta + y_j^h(s) - y_j^h(t)\big) - \phi'(\theta)\right]\mathrm{d}\theta\right\}\\ &\quad\leqslant \sup_{\phi\in B}\left\{\left|y_j(s) - y_j^h(s) - y_j(t) + y_j^h(t)\right|\left\|\phi'\right\|_\infty + \left\|\phi''\right\|_\infty \int_{y_j^h(t)}^{y_j(t)} \left|y_j^h(t) - y_j^h(s)\right|\mathrm{d}\theta\right\}\end{aligned}$$

$$\leqslant \sup_{\phi\in B}\left\{\left|\int_s^t \left(z_j^h - z_j\right)(\theta)\,\mathrm{d}\theta\right|\|\phi'\|_\infty + \|\phi''\|_\infty\left|y_j^h(t) - y_j^h(s)\right|\left|\eta_j(t)\right|\right\}$$
$$\leqslant \sup_{\phi\in B}\left\{\|\phi'\|_\infty \max_{\sigma\in[s,t]}(t-s)\left|\zeta_j(\sigma)\right| + C\|\phi''\|_\infty(t-s)\left|\eta_j(t)\right|\right\}$$
$$\leqslant C(t-s)\Big(\left|\eta_j(t)\right| + \max_{\sigma\in[s,t]}\left|\zeta_j(\sigma)\right|\Big), \quad j\in\{1,\dots,n\},$$

using $\|\phi'\|_\infty, \|\phi''\|_\infty \leqslant C$ for $\phi\in B\subset C^2(\Omega)$, by Lemma 2.1.

Recalling $\|u(t)\|_\infty \leqslant Ct^{-1/4}$ from Lemmas 2.2 and 2.5 and using Lemma 3.1,

$$\|T_{10}\|_\infty \leqslant Ct^{-1/4}\int_0^t (t-s)^{-3/4}\sum_{j=1}^n\Big(\max_{\sigma\in[s,t]}\left|\zeta_j(\sigma)\right| + \left|\eta_j(t)\right|\Big)\,\mathrm{d}s$$
$$\leqslant Ct^{-1/4}\int_0^t (t-s)^{-3/4}\left(\|\boldsymbol{\eta}(0)\|_{l_1} + \|\boldsymbol{\zeta}(0)\|_{l_1} + \int_0^t \|w(\sigma)\|_\infty\,\mathrm{d}\sigma\right)\mathrm{d}s$$
$$\leqslant C\|\boldsymbol{\eta}(0)\|_{l_1} + \|\boldsymbol{\zeta}(0)\|_{l_1} + C\int_0^t \|w(s)\|_\infty\,\mathrm{d}s.$$

For T_{11}, since all terms in the integrand are independent of s,

$$T_{11} = -\gamma\sum_{j=1}^n\left[I - e^{-At}\right]A^{-1}\left[\delta\big(\cdot - y_j(t)\big) - \delta\big(\cdot - y_j^h(t)\big)\right]\left[u\big(y_j^h(t),t\big) - z_j(t)\right].$$

For $j\in\{1,\dots,n\}$, the Mean Value Theorem implies that

$$\left|A^{-1}\left[\delta\big(\cdot - y_j(t)\big) - \delta\big(\cdot - y_j^h(t)\big)\right]\right|(x)$$
$$= \left|\frac{\sqrt{R}}{2}\left\{\exp\big(-\sqrt{R}\left|x - y_j(t)\right|\big) - \exp\big(-\sqrt{R}\left|x - y_j^h(t)\right|\big)\right\} + C_1(t)e^{\sqrt{R}x} + C_2(t)e^{-\sqrt{R}x}\right|$$
$$\leqslant \frac{R}{2}e^{-\sqrt{R}\theta_j(t)}\left|\left|x - y_j(t)\right| - \left|x - y_j^h(t)\right|\right| + \left|C_1(t)\right|e^{\sqrt{R}x} + \left|C_2(t)\right|e^{-\sqrt{R}x}$$
$$\leqslant \frac{R}{2}\left|\eta_j(t)\right| + \left|C_1(t)\right|e^{\sqrt{R}x} + \left|C_2(t)\right|e^{-\sqrt{R}x} \leqslant C\|\boldsymbol{\eta}(t)\|_{l_1},$$

since $C_1(t)$ and $C_2(t)$ depend sublinearly on $|\eta_j(t)|$. Hence,

$$\left|\sum_{j=1}^n A^{-1}\left[\delta\big(\cdot - y_j(t)\big) - \delta\big(\cdot - y_j^h(t)\big)\right]\left[u\big(y_j^h(t),t\big) - z_j(t)\right]\right|$$
$$\leqslant C\sum_{j=1}^n\big(\|u(t)\|_\infty + \left|z_j(t)\right|\big)\left|\eta_j(t)\right| \leqslant t^{-1/4}\|\boldsymbol{\eta}(t)\|_{l_1}.$$

The parabolic maximum principle implies that $\|e^{-At}\|_\infty \leqslant 1$, and thus,

$$\|T_{11}\|_\infty \leqslant Ct^{-1/4}\big\|\boldsymbol{\eta}(t)\big\|_{l_1} \leqslant Ct^{-1/4}\left(\big\|\boldsymbol{\eta}(0)\big\|_{l_1} + t\big\|\boldsymbol{\zeta}(0)\big\|_{l_1} + \int_0^t \big\|w(s)\big\|_\infty \,\mathrm{d}s\right).$$

For T_{12}, we split part of the integrand for each $j \in \{1,\dots,n\}$:

$$\begin{aligned}
&\big[e^{-A(t-s)} - e^{-A^h(t-s)t}P^h\big]\big\{\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\}\\
&\quad = e^{-A(t-s)/2}\big[e^{-A(t-s)/2} - e^{-A^h(t-s)/2}P^h\big]\big\{\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\}\\
&\qquad + \big[e^{-A(t-s)/2} - e^{-A^h(t-s)/2}P^h\big]e^{-A^h(t-s)/2}P^h\big\{\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\}\\
&\quad = I_j^1 + I_j^2.
\end{aligned}$$

For $B = \{\phi \in L_1 \mid \|\phi\|_1 = 1\}$, the estimates of Lemma 3.2 imply that

$$\begin{aligned}
\big\|I_j^1\big\|_\infty &= \sup_{\phi\in B}\big\langle e^{-A(t-s)/2}\big[e^{-A(t-s)/2} - e^{-A^h(t-s)/2}P^h\big]\big\{\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\}, \phi\big\rangle\\
&= \sup_{\phi\in B}\big\langle \delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big), \big[e^{-A(t-s)/2} - e^{-A^h(t-s)/2}P^h\big]e^{-A(t-s)/2}\phi\big\rangle\\
&\leqslant \big\|\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\|_{H^{-1}}\big\|e^{-A(t-s)/2} - e^{-A^h(t-s)/2}P^h\big\|_{L_2\to H^1}\sup_{\phi\in B}\big\|e^{-A(t-s)/2}\phi\big\|_2\\
&\leqslant \big\|\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\|_{H^{-1}}Ch(t-s)^{-1}(t-s)^{-1/4},
\end{aligned}$$

where in the last line we have used (A.5). From (2.16),

$$\big\|\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\|_{H^{-1}} \leqslant \big\|A^{-1/2}\big[\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big]\big\|_2 \leqslant C(t-s)^{1/2}. \quad \text{(A.10)}$$

Consequently,

$$\big\|I_j^1\big\|_\infty \leqslant Ch(t-s)^{-5/4}\big\|\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\|_{H^{-1}} \leqslant Ch(t-s)^{-3/4}.$$

For I_j^2, Lemma 3.2 and (2.16) imply that

$$\begin{aligned}
\big\|I_j^2\big\|_\infty &\leqslant \big\|e^{-A(t-s)/2} - e^{-A^h(t-s)/2}P^h\big\|_{L_2\to L_\infty}\big\|e^{-A^h(t-s)/2}P^h\big\|_{H^{-1}\to L_2}\\
&\quad\times \big\|\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\|_{H^{-1}}\\
&\leqslant Ch(t-s)^{-3/4}C(t-s)^{1/2}\big\|e^{-A^h(t-s)/2}P^h\big\|_{L_2\to H^1}\\
&\leqslant Ch(t-s)^{-1/4}C\big\|\big(A^h\big)^{1/2}e^{-A^h(t-s)/2}P^h\big\|_2 \leqslant Ch(t-s)^{-3/4}.
\end{aligned}$$

Hence, by Lemma 3.2,

$$\begin{aligned}
\|T_{12}\|_\infty &\leqslant \int_0^t \gamma\sum_{j=1}^n\big[e^{-A(t-s)} - e^{-A^h(t-s)}P^h\big]\big\{\delta\big(\cdot - y_j^h(s)\big) - \delta\big(\cdot - y_j^h(t)\big)\big\}\big[u\big(y_j(t),t\big) - z_j(t)\big]\,\mathrm{d}s\\
&\leqslant C\int_0^t \sum_{j=1}^n\big(\big\|I_j^1(s)\big\|_\infty + \big\|I_j^2(s)\big\|_\infty\big)\big[\big\|u(t)\big\|_\infty + \big|z_j(t)\big|\big]\,\mathrm{d}s
\end{aligned}$$

$$\leqslant Ct^{-1/4}\int_0^t h(t-s)^{-3/4}\,\mathrm{d}s \leqslant Ch.$$

$$\|T_{13}\|_\infty \leqslant \sum_{j=1}^n\big(\|u(t)\|_\infty + |z_j(t)|\big)\left\|\int_0^t\big[e^{-A(t-s)} - e^{-A_h(t-s)}P^h\big]\delta\big(\cdot - y_j^h(t)\big)\,\mathrm{d}s\right\|_\infty$$

$$\leqslant Ct^{-1/4}\sum_{j=1}^n\big\|\big\{\big[I-e^{-At}\big]A^{-1} - \big[I-e^{-A^h t}\big]\big(A^h\big)^{-1}P^h\big\}\delta\big(\cdot - y_j^h(t)\big)\big\|_\infty. \tag{A.11}$$

Approximating the delta function by convergent sequences in $L_1(\Omega)$, applying (A.7), Lemmas 2.1 and 3.2, we may bound the jth term in (A.11) by

$$C\big\|\big[e^{-At} - e^{-A^h t}P^h\big]A^{-1/2}\big\|_{L_2\to L_\infty}\big\|A^{-1/2}\delta\big(\cdot - y_j^h(t)\big)\big\|_2$$

$$+\,C\big\|I - e^{-A^h t}P^h\big\|_\infty\big\|\big[A^{-1} - \big(A^h\big)^{-1}P^h\big]\delta\big(\cdot - y_j^h(t)\big)\big\|_\infty$$

$$\leqslant C\big\|e^{-At} - e^{-A^h t}P^h\big\|_{H^1\to L_\infty} + \big\|e^{-A^h t}\big\|_{V_2^h\to V_\infty^h}C\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to L_\infty}$$

$$\leqslant Cht^{-1/4} + Ct^{-1/4}\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to L_\infty}. \tag{A.12}$$

From Thomée [10, Chapter 1],

$$\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_2 \leqslant Ch^2, \qquad \big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_2\to H^1} \leqslant Ch.$$

Since $L_1(\Omega)$ and $L_\infty(\Omega)$ are dual and A, A^h and P^h are selfadjoint,

$$\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to L_2} = \big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_2\to L_\infty}$$

$$\leqslant C\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_2^{1/2}\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_2\to H^1}^{1/2} \leqslant Ch^{3/2}.$$

Using duality, the inverse and approximation properties of V^h,

$$\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to H^1}$$

$$\leqslant \big\|P^hA^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to H^1} + \big\|\big(I-P^h\big)A^{-1}\big\|_{L_1\to H^1}$$

$$\leqslant Ch^{-1}\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to L_2} + \big\|A^{-1}\big(I-P^h\big)\big\|_{H^{-1}\to L_\infty}$$

$$\leqslant Ch^{1/2} + C\big\|A^{-1}\big(I-P^h\big)\big\|_{H^{-1}\to L_2}^{1/2}\big\|A^{-1}\big(I-P^h\big)\big\|_{H^{-1}\to H^1}^{1/2}$$

$$\leqslant Ch^{1/2} + C\big\|\big(I-P^h\big)A^{-1}\big\|_{L_2\to H^1}^{1/2}\big\|A^{-1/2}\big(I-P^h\big)\big\|_{H^{-1}\to L_2}^{1/2}$$

$$\leqslant Ch^{1/2} + Ch^{1/2}\big\|\big(I-P^h\big)A^{-1/2}\big\|_{L_2\to H^1}^{1/2}$$

$$\leqslant Ch^{1/2}. \tag{A.13}$$

Hence,

$$\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to L_\infty} \leqslant \big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to H^1}^{1/2}\big\|A^{-1} - \big(A^h\big)^{-1}P^h\big\|_{L_1\to L_2}^{1/2} \leqslant Ch.$$

Substituting this estimate into (A.12) and then (A.12) into (A.11), we deduce that

$$\|T_{13}\|_\infty \leqslant Cht^{-1/2}.$$

Collecting together the estimates for $\{\|T_k\|_\infty\}_{k=1}^{13}$ in (A.9), we conclude that for $t \in (0, T]$,

$$\|w(t)\|_\infty \leqslant C\big[ht^{-3/4} + \|\boldsymbol{\zeta}(0)\|_{l_1} + t^{-1/4}\big(\|\boldsymbol{\eta}(0)\|_{l_1} + \|w(0)\|_\infty\big)\big] + C\int_0^t (t-s)^{-1/2}\|w(s)\|_\infty \, \mathrm{d}s.$$

Thanks to a modified Gronwall Lemma [4, p. 188], for $t \in [0, T]$

$$\|w(t)\|_\infty \leqslant C\big[ht^{-3/4} + \|\boldsymbol{\zeta}(0)\|_{l_1} + t^{-1/4}\big(\|\boldsymbol{\eta}(0)\|_{l_1} + \|w(0)\|_2\big)\big] + C\int_0^t (t-s)^{-1/2} hs^{-3/4} \, \mathrm{d}s,$$

from which we deduce the required bound for $\|w(t)\|_\infty$. The last part of the lemma follows by inserting the $\|w(t)\|_\infty$ bound into those of Lemma 3.2. □

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