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MULTIGRID METHODS FOR SECOND ORDER HAMILTON–JACOBI–BELLMAN AND HAMILTON–JACOBI–BELLMAN–ISAACS EQUATIONS*

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Abstract. We propose multigrid methods for solving the discrete algebraic equations arising from the discretization of the second order Hamilton–Jacobi–Bellman (HJB) and Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations. We propose a damped-relaxation method as a smoother for multigrid. In contrast with the standard policy iteration, the proposed damped-relaxation scheme is convergent for both HJB and HJBI equations. We show by local Fourier analysis that the damped-relaxation smoother effectively reduces high frequency error. For problems with large jumps in control, we develop restriction and interpolation methods to capture the jumps on the coarse grids as well as during the coarse grid correction. We will demonstrate the effectiveness of the proposed multigrid methods for solving HJB and HJBI equations arising from option pricing as well as problems where policy iteration does not converge or converges slowly.

Key words. multigrid methods, full approximation scheme, relaxation scheme, policy iteration, Hamilton–Jacobi–Bellman equations, Hamilton–Jacobi–Bellman–Isaacs equations, jump in control

AMS subject classifications. 65M55, 65M22, 65F10, 65F50

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1. Introduction.

Many real life problems, such as financial problems [21, 38] and stochastic games [30, 36], can be modeled as optimal control problems and formulated as Hamilton–Jacobi–Bellman (HJB) and Hamilton–Jacobi–Bellman–Isaacs (HJBI) equations. Stable discretization methods have been proposed and analyzed (e.g., [14, 21]) for solving single factor optimal control problems. Semi-Lagrangian methods have also been used [19, 20]. The discretization of these equations will lead to a set of highly nonlinear discrete equations. The primary focus of this paper is the fast solution of the discrete nonlinear algebraic equations. For the theoretical properties of the HJB equations such as the existence and regularity of viscosity solutions, we refer the interested reader to, e.g., [27, 31]. Recent developments in the analysis and approximation of differential game problems can be found in [2, 6, 15, 16, 18].

The policy iteration, also known as Howard's algorithm [8, 24, 26], is a Newton-like method [29] for solving nonlinear problems and, in particular, HJB equations. It first computes for the optimal control based on an approximate solution, linearizes the problem with the resulting control, and then solves the linear system to obtain a better approximate solution. The policy iteration is convergent for HJB equations. However, the convergence rate can be slow in the sense that the number of iterations, in general, cannot be bounded by a constant that is independent of the number of the grid points [30]. Moreover, a straightforward Newton-like extension [11] does not guarantee global convergence for HJBI equations [11, 36], which leads to the development of a different variant of policy iterations attempting to address this problem [2, 11, 13, 33, 34]. Another method for solving HJB and HJBI equations is a relaxation scheme discussed in [7], which is essentially the value iteration [25]. It is

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proven to be globally convergent to the viscosity solution, but its convergence rate can be very slow on the fine grids.

In this paper, we propose multigrid methods for solving the discrete nonlinear second order HJB and HJBI equations. Multigrid methods have been used as efficient numerical solvers for solving a wide variety of partial differential equations (PDEs) [35]. The rate of convergence is often independent of the mesh size. However, the literature on multigrid methods for HJB and HJBI equations is scarce. In [1, 3, 4, 5], portfolio selection problems are modeled as HJB equations and solved by the multigrid-Howard and the full multigrid-Howard (FMGH) algorithm. In each policy iteration, a linear multigrid method or full multigrid method is applied to solve the linearized problem. Hoppe [22] proposed two multigrid schemes for HJB equations, MGS I and MGS II, in which multigrid methods are applied directly to the nonlinear HJB equation. MGS I is based on an iterative numerical scheme which requires the solution of a unilateral variational inequality in each iteration [26] and MGS II is based on policy iteration. MGS's are similar to but different from a full approximation scheme (FAS) [35] and the main difference lies in the coarse grid problem construction. Bloss and Hoppe [10] later proposed another multigrid method, MGHJB, for HJB equations. MGHJB is an updated version of MGS II: MGHJB applies nonlinear Gauss–Seidel iteration as the smoother while MGS II applies linear relaxation methods to the linearized discrete HJB equation. Multigrid methods based on a variant of policy iteration were also applied to HJBI equations [17], in which multigrid is used to solve the linearized HJBI problem.

Fast multigrid convergence for HJB and HJBI equations is challenging to achieve. The controls couple with the solution of the partial differential equations in such a very nonlinear way that a direct application of standard multigrid methods would not work well. For instance, the MGHJB method [10] has a convergence rate of around 0.7 for a model HJB problem. In addition, we have observed that multigrid convergence deteriorates when there are jumps in the control. The issue of jumps in the control has never been discussed in the literature.

In this paper, we propose multigrid methods which are efficient for a wide variety of HJB and HJBI equations where the value functions are regular. The new smoother of our multigrid method is a damped relaxation method which has never been used as a smoother for multigrid methods. We will prove that the smoother is convergent for both HJB and HJBI equations, and show by a local Fourier analysis for a linearized problem that it is effective for removing high frequency errors. In the case when the control has jumps, we develop a restriction and interpolation method to capture the optimal control near the jumps. The resulting multigrid methods show much faster convergence than MGHJB [10], and are able to achieve fast convergence even with jumps in control. Moreover, our multigrid methods can handle the more complicated HJBI equations in the same way as the HJB equations.

We remark that multigrid methods have been proposed for complementarity problems and variation inequalities; see, e.g., [12, 23, 28, 32]. In some special cases, these problems can be formulated as HJB equations. However, HJB and especially HJBI equations, in general, cannot be formulated as complementarity problems. Thus methods for one would not generally be applicable to the other.

We will demonstrate our algorithms by mainly focusing on two example finance problems in [21, 38]. In section 2, the two example problems are described and the corresponding HJB and HJBI equations are presented. Then we will describe our new multigrid methods in section 3. A smoothing analysis will be presented in section 4, followed by numerical results on a variety of examples illustrating the convergence of different methods in section 5.

2. Model problems. HJB equations are PDEs which involve an optimal parameter generally known as the control. The optimality is typically imposed by a max or min operator in the equations. In this paper, we consider HJB equations of the form

$$V_\tau = \inf_{Q \in \hat{Q}} \left\{ a(S,\tau,Q)V_{SS} + b(S,\tau,Q)V_S - c(S,\tau,Q)V + d(S,\tau,Q) \right\}, \tag{2.1}$$

where $a(S,\tau,Q) > 0$, $b(S,\tau,Q)$, $c(S,\tau,Q)$, $d(S,\tau,Q)$ are functions of time τ, variable S, and control Q. The set of controls is denoted by $\hat{Q}$.

These equations generally arise in optimal control problems. One can also formulate obstacle problems [23, 32], linear complementarity problems [12], and American option pricing [21, 28] as HJB equations. For a single obstable problem, for instance, the control takes on only two values, one corresponding to the case when the constraint is active and the other to when the case the constraint is inactive.

HJBI equations are PDEs which involve two optimal controls imposed by a max-min or min-max operator. In this paper, we consider HJBI equations of the form

$$V_\tau = \sup_{P \in \hat{P}} \inf_{Q \in \hat{Q}} \left\{ a(S,\tau,Q,P)V_{SS} + b(S,\tau,Q,P)V_S - c(S,\tau,Q,P)V + d(S,\tau,Q,P) \right\}, \tag{2.2}$$

where $a(S,\tau,Q,P) > 0$, $b(S,\tau,Q,P)$, $c(S,\tau,Q,P)$, $d(S,\tau,Q,P)$ are functions of time τ, variable S, and controls Q and P. The two sets of controls are denoted by $\hat{Q}$ and $\hat{P}$.

These equations arise in, for instance, two-player zero-sum differential games such as the pursuit-evasion problem [6]. One player tries to maximize the payoff while the other player tries to minimize it. The players' optimal strategies correspond to the two controls in the equation. The value of the final payoff is given by the solution V of the HJBI equation.

There are two model problems we are considering in this paper which arise from nonlinear asset allocation and option pricing problems in financial modeling. They lead to an HJB and an HJBI equation, respectively, which are described briefly in the next two sections. We note that there are other methods for solving similar nonlinear asset allocation and option pricing problems. Since those methods are outside the scope of this paper, we will refer the interested reader to [9, 25, 40].

2.1. HJB case: Pension plan asset allocation problem. Suppose there are two assets in the market, one risk free and the other risky. The risky asset S follows the stochastic process

$$dS = (r + \xi\sigma)S dt + \sigma S dZ,$$

where dZ is the increment of a Wiener process, σ is volatility, r is the interest rate, and ξ is the market price of risk. The investor pays into the pension plan at a constant rate ρ in the unit time. Let $W(t)$ denote the wealth in the pension plan at time t. A proportion q of this wealth is invested in the risky asset and the rest is invested in the risk free asset. Then

$$dW = \left[(r + q\xi\sigma)W + \rho\right] dt + q\sigma W dZ.$$

Let $W_T = W(T)$, where T is the expiration time of the pension plan. The aim of the investor is to $\max_q \left\{E^{t=0}\left[W_T\right]\right\}$ such that $Var^{t=0}\left[W_T\right] = \text{constant}$, where $E\left[\cdot\right]$ is the expectation operator and $Var\left[\cdot\right]$ is the variance operator. The superscript $t = 0$ indicates that the expectation and variance are computed at $t = 0$.

For the convenience of computation, we will follow the common practice in the literature to introduce a parameter $\tau = T - t$. Let w be in a set of all admissible wealth $W(t)$ for $0 \le t \le T$. Define an intermediate variable

$$V(w, \tau) = \inf_{Q \in \hat{Q}} \left\{ E\left[\left(W_T - \frac{\gamma}{2} \right)^2 | W(T - \tau) = w \right] \right\}$$

that has terminal condition $V(w, 0) = (w - \frac{\gamma}{2})^2$, where γ is a predetermined constant. The pension problem can be simplified to two steps [21]: first solve for $V(w, T)$, which satisfies an HJB equation

$$V_\tau = \inf_{q \in \hat{Q}} \left\{ \frac{1}{2}(q\sigma w)^2 V_{ww} + [\rho + w(r + q\sigma\xi)V_w] \right\}, \tag{2.3}$$

and then compute the expected wealth by solving a Black–Scholes-like equation. Equation (2.3) will be used as an example to illustrate our method for the HJB case.

2.2. HJBI case: American options and stock borrowing fees. In this problem, $V(S, t)$ is the value of an American option written on asset S. Let V^* be the payoff. Then the price of the option can be written in penalty form as

$$V_t + \sup_{\mu \in \{0,1\}} \left\{ \frac{\sigma^2 S^2}{2} V_{SS} + rSV_S - rV + \mu \frac{V^* - V}{\eta} \right\} = 0,$$

where σ is the volatility, r is the interest rate, and $\eta \ll 1$ is a small positive number. Extend this model to include unequal borrowing rates (r_b), lending rates (r_l), and stock borrowing fees (r_f). The holder of a short position will receive $r_l - r_f$ on the proceeds of the short sale. This gives rise to the equation

$$\begin{aligned} V_\tau = \sup_{\mu \in \hat{P}} \inf_{Q \in \hat{Q}} \Bigg\{ & \frac{\sigma^2 S^2}{2} V_{SS} + q_3 q_1 (SV_S - V) \\ & + (1 - q_3)[(r_l - r_f)SV_S - q_2 V] + \mu \frac{V^* - V}{\eta} \Bigg\}, \end{aligned} \tag{2.4}$$

where $Q = (q_1, q_2, q_3)$, $\hat{Q} = (\{r_l, r_b\}, \{r_l, r_b\}, \{0, 1\})$, and $\hat{P} = \{0, 1\}$. Equation (2.4) is an HJBI equation, which we will use as an example to illustrate our method for the HJBI case.

2.3. Discretization. We will briefly discuss the discretization for the PDE in the general form of HJB and HJBI equations in this section. A positive coefficient discretization scheme, which will ensure the convergence to the viscosity solution, is applied to both HJB and HJBI cases. It is shown that near quadratic convergence can be achieved as the grid size is reduced. The details for the positive coefficient discretization can be found in [21].

Define a grid $\{S_0, S_1, \ldots, S_M\}$ with $S_M = S_{max}$. Let V_i^n be a discrete approximation to $V(S_i, \tau^n)$ and let $V^n = [V_0^n, \ldots, V_M^n]^T$. The objective function that depends on V in (2.1) at (S_i, τ^{n+1}) is discretized using a combination of forward, backward, or central differencing methods, giving

$$\begin{aligned} &(a(S,\tau,Q)V_{SS} + b(S,\tau,Q)V_S - c(S,\tau,Q)V)_i \\ &= \alpha_i^{n+1}(Q)V_{i-1}^{n+1} + \beta_i^{n+1}(Q)V_{i+1}^{n+1} - (\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q))V_i^{n+1}, \end{aligned} \tag{2.5}$$

where α_i and β_i are defined such that a positive coefficient scheme is resulted; see [21] for details. Assuming central differencing and uniform grid, we obtain

$$\alpha(Q) = \frac{2a(Q)}{2h^2} - \frac{b(Q)}{2h}, \quad \beta(Q) = \frac{2a(Q)}{2h^2} + \frac{b(Q)}{2h}, \tag{2.6}$$

where h is the grid size.

We will consider fully implicit timestepping and (2.5) to discretize (2.1),

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \inf_{Q\in\hat{Q}} \left\{ \left[A(Q)V^{n+1}\right]_i + [D(Q)]_i^{n+1} \right\}, \quad i < M, \tag{2.7}$$

where $[A(Q)V^{n+1}]_i$ is the matrix form of the operator defined in (2.5) and $[D(Q)]_i^{n+1}$ is the vector form of $d_i^{n+1}(Q)$. The first and last row of matrix A and vector D are modified accordingly to handle the boundary conditions. Also we note that higher order timestepping, such as Crank–Nicolson timestepping, can be used [21, 38]. The HJBI equation can be discretized in a similar way, yielding

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \sup_{P\in\hat{P}} \inf_{Q\in\hat{Q}} \left\{ \left[A(Q,P)V^{n+1}\right]_i + [D(Q,P)]_i^{n+1} \right\}. \tag{2.8}$$

Since the discretized equations (2.7) and (2.8) are highly nonlinear, iterative methods are usually used to solve the equations. Policy iteration is commonly used for solving HJB equations [22, 24]. It consists of an iterative algorithm on the control and the value functions, and generates an improving sequence of controls to the nonlinear problem. Let $\hat{V}^k$ be an approximate solution. The idea of policy iteration is to compute the optimal control Q^k from $\hat{V}^k$. Then an improved approximation $\hat{V}^{k+1}$ is obtained from Q^k. The procedure is repeated until convergence. Policy iteration is a form of Newton-like iteration and it is globally convergent for HJB equations. However, examples show that the convergence of policy iteration for discrete HJB problems can depend on the number of grid points. In [30], a discrete HJB problem defined on a grid with M grid points and a control set of size 2 requires $M-1$ policy iterations to converge. In addition, policy iteration does not guarantee global convergence for HJBI equations. Pathological cases in [11, 36] show that the Newton-like policy iteration does not converge for these problems. Thus, we do not consider the policy iteration scheme for solving the HJBI equations. Instead, we propose a multigrid method which is efficient in the sense that it is independent of the grid size, and which can be applied to both HJB and HJBI equations.

3. Multigrid methods for HJB and HJBI equations.

In this section, we propose multigrid methods to solve HJB and HJBI equations. One approach is to solve the linear system $\left[I - \Delta\tau A^{n+1}\left(Q^k\right)\right]\hat{V}^{k+1} = V^n + \Delta\tau D^{n+1}\left(Q^k\right)$ for $\hat{V}^{k+1}$ using standard multigrid in each policy iteration [3]. Our approach, however, is to solve the discrete nonlinear HJB and HJBI equations directly using the full approximation scheme (FAS). Write the discrete HJB equation as

$$N_h^Q\left(V^{n+1}\right) = B_h, \tag{3.1}$$

where

$$B_h \equiv V^n, \quad N_h^Q(V^{n+1}) \equiv V^{n+1} - \Delta\tau \inf_{Q\in\hat{Q}} \left\{\mathcal{L}^Q V^{n+1}\right\}, \tag{3.2}$$

and

$$\mathcal{L}^Q V^{n+1} = A^{n+1}(Q)V^{n+1} + D^{n+1}(Q). \tag{3.3}$$

The V-cycle is used with the FAS and the problem on the coarsest grid is solved by the smoother or the policy iteration. The remaining components needing to be defined are the smoother and the intergrid transfer operators, which will be described in section 3.1 and 3.4, respectively. We remark that if there is no jump in the control, linear interpolation and full weighting restriction could be used for the intergrid transfer. However, when there is a jump in the control, special care is needed; otherwise, the multigrid convergence can be very slow or even divergent due to the jump. In section 3.4, we propose a restriction and interpolation method for capturing the optimal control on the coarse and fine grids.

3.1. Damped relaxation smoother for HJB. In [3, 22], the nonlinear HJB problem is linearized and Gauss–Seidel is used as a smoother for the linearized operator. We would like to use a smoother directly applied to the nonlinear HJB equation. The policy iteration could be a possible smoother but it does not guarantee convergence for HJBI equations. Instead, our proposed smoother is based on a relaxation scheme [7], also known as the value iteration method [25]. Consider the HJB equation. The discrete equation (2.7) can be written as

$$\begin{aligned} V_i^{n+1} = \Delta\tau \inf_{Q\in\hat{Q}} \big\{ &\alpha_i^{n+1}(Q)V_{i-1}^{n+1} + \beta_i^{n+1}(Q)V_{i+1}^{n+1} \\ &- \left(\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)\right) V_i^{n+1} + d_i^{n+1}(Q)\big\} + V_i^n. \end{aligned} \tag{3.4}$$

Since V_i^{n+1} does not depend on the control, then Q, V_i^n, and $\Delta\tau$ are constants. Rearranging (3.4), we obtain

$$\begin{aligned} 0 = \inf_{Q\in\hat{Q}} \big\{ &\Delta\tau \left(\alpha_i^{n+1}(Q)V_{i-1}^{n+1} + \beta_i^{n+1}(Q)V_{i+1}^{n+1} + d_i^{n+1}(Q)\right) + V_i^n \\ &- \left[1 + \Delta\tau\left(\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)\right)\right] V_i^{n+1} \big\}, \end{aligned}$$

which can be written as

$$\begin{aligned} 0 = \inf_{Q\in\hat{Q}} \Bigg\{ &\left[1 + \Delta\tau\left(\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)\right)\right] \\ &\left[-V_i^{n+1} + \frac{\Delta\tau\left(\alpha_i^{n+1}(Q)V_{i-1}^{n+1} + \beta_i^{n+1}(Q)V_{i+1}^{n+1} + d_i^{n+1}(Q)\right) + V_i^n}{1 + \Delta\tau\left(\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)\right)} \right] \Bigg\}. \end{aligned} \tag{3.5}$$

Note that α_i^{n+1}, β_i^{n+1}, and c_i^{n+1} are all nonnegative. Letting $\hat{V}^k$ be the kth estimate for V^{n+1}, a relaxation scheme can be derived from (3.5),

$$\hat{V}_i^{k+1} = \inf_{Q\in\hat{Q}} \left\{ \frac{\Delta\tau\left(\alpha_i^{n+1}(Q)\hat{V}_{i-1}^k + \beta_i^{n+1}(Q)\hat{V}_{i+1}^k + d_i^{n+1}(Q)\right) + V_i^n}{1 + \Delta\tau\left(\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)\right)} \right\}. \tag{3.6}$$

This relaxation scheme, however, does not, in general, reduce high frequency errors, and hence is not an effective smoother. To achieve a better smoothing effect, we introduce a damping factor to the relaxation scheme, as it is used for the damped-Jacobi method. The damped-relaxation smoother is defined as

$$\begin{aligned} \hat{V}_i^{k+1} &= (1-\omega)\hat{V}_i^k \\ &\quad + \omega \inf_{Q\in\hat{Q}} \left\{ \frac{\Delta\tau\left(\alpha_i^{n+1}(Q)\hat{V}_{i-1}^k + \beta_i^{n+1}(Q)\hat{V}_{i+1}^k + d_i^{n+1}(Q)\right) + V_i^n}{1+\Delta\tau\left(\alpha_i^{n+1}(Q)+\beta_i^{n+1}(Q)+c_i^{n+1}(Q)\right)} \right\}, \end{aligned} \tag{3.7}$$

where ω is the damping factor.

A damped relaxation smoother can be defined similarly for HJBI equations; see section 3.3.

THEOREM 3.1. *Suppose that the discretization* (2.7) *satisfies a positive coefficient condition* [21]. *Then the iteration scheme* (3.7) *is globally convergent for any initial guess if* $0<\omega<2/(1+\gamma)$. *Furthermore,*

$$\left\|\hat{V}^{k+1}-\hat{V}^k\right\|_\infty \le (|1-\omega|+|\omega|\gamma)\left\|\hat{V}^k-\hat{V}^{k-1}\right\|_\infty,$$

where

$$\gamma = \max_i \sup_{Q\in\hat{Q}} \left\{ \frac{\alpha_i^{n+1}(Q)+\beta_i^{n+1}(Q)}{1+\Delta\tau\left[\alpha_i^{n+1}(Q)+\beta_i^{n+1}(Q)+c_i^{n+1}(Q)\right]} \right\}. \tag{3.8}$$

Proof. By (3.7) and the properties of the inf operator,

$$\begin{aligned} &|\hat{V}_i^{k+1}-\hat{V}_i^k| \\ &\le |1-\omega||\hat{V}_i^k-\hat{V}_i^{k-1}| \\ &\quad + |\omega|\sup_{Q\in\hat{Q}}\left\{\left|\frac{\Delta\tau\left(\alpha_i^{n+1}(Q)\hat{V}_{i-1}^k+\beta_i^{n+1}(Q)\hat{V}_{i+1}^k+d_i^{n+1}(Q)\right)+V_i^n}{1+\Delta\tau\left(\alpha_i^{n+1}(Q)+\beta_i^{n+1}(Q)+c_i^{n+1}(Q)\right)}\right.\right. \\ &\qquad\qquad \left.\left. - \frac{\Delta\tau\left(\alpha_i^{n+1}(Q)\hat{V}_{i-1}^{k-1}+\beta_i^{n+1}(Q)\hat{V}_{i+1}^{k-1}+d_i^{n+1}(Q)\right)+V_i^n}{1+\Delta\tau\left(\alpha_i^{n+1}(Q)+\beta_i^{n+1}(Q)+c_i^{n+1}(Q)\right)}\right|\right\} \\ &= |1-\omega||\hat{V}_i^k-\hat{V}_i^{k-1}| \\ &\quad + |\omega|\sup_{Q\in\hat{Q}}\left\{\frac{\alpha_i^{n+1}(Q)+\beta_i^{n+1}(Q)}{1+\Delta\tau\left[\alpha_i^{n+1}(Q)+\beta_i^{n+1}(Q)+c_i^{n+1}(Q)\right]}\right\}|\hat{V}_i^k-\hat{V}_i^{k-1}|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\hat{V}^{k+1}-\hat{V}^k\|_\infty &\le |1-\omega|\|\hat{V}^k-\hat{V}^{k-1}\|_\infty + |\omega|\gamma\|\hat{V}^k-\hat{V}^{k-1}\|_\infty \\ &= (|1-\omega|+|\omega|\gamma)\|\hat{V}^k-\hat{V}^{k-1}\|_\infty. \end{aligned}$$

Since $\alpha_i^{n+1}(Q)$, $\beta_i^{n+1}(Q)$, and $c_i^{n+1}(Q)$ are nonnegative for all $Q\in\hat{Q}$, we have $\gamma < 1$. Hence, the iteration converges if the damping factor ω satisfies $0<\omega<2/(1+\gamma)$. ∎

We note that the convergence of the relaxation scheme alone can be very slow for small grid sizes. However, with a carefully chosen damping factor, a smoothing factor close to 0.5 can be achieved; see the smoothing analysis in section 4 for more details.

3.2. Difference between our FAS method and MGS. The MGS scheme in [22] is also based on the FAS, but differs from ours in several ways. First, MGS uses W-cycles while our FAS uses V-cycles. Second, different smoothing procedures are applied. MGS first linearizes the HJB problem by finding the optimal control based on the current approximate solution, and then applies common smoothers in standard multigrid for a few iterations to the linearized problem. Thus the approximate solution given by the smoother approximates the solution of the linearized system. Our approach, however, applies the nonlinear damped-relaxation smoother in section 3.1 to the HJB problem directly. In other words, the approximate solution given by our smoother approximates the nonlinear HJB problem. Third, the construction procedures of the coarse grid problem are different. Suppose the discrete HJB equation on the fine grid is given by $\inf_{Q\in\hat{Q}}\{A_h^Q(V_h) - f_h^Q\} = B_h$ with $B_h \equiv 0$, and the coarse grid problem is given by $\inf_{Q\in\hat{Q}}\{A_H^Q(V_H) - f_H^Q\} = B_H$, where the operator $A_H^Q(\cdot)$ is obtained from direct discretization. In MGS, $A_H^Q(V_H)$ and f_H^Q are considered separately. f_H^Q is obtained from

$$f_H^Q = A_H^Q(R\cdot V_h) + R\cdot\left(f_h^Q - A_h^Q(V_h)\right)$$

for all $Q \in \hat{Q}$, where R is the restriction operator. B_H is simply set to 0. In our approach, $N_h^Q \equiv \inf_{Q\in\hat{Q}}\{A_h^Q(V_h) - f_h^Q\}$ is considered as one nonlinear term. Let $N_H^Q(V_H) \equiv \inf_{Q\in\hat{Q}}\{A_H^Q(V_H) - f_H^Q\}$. $N_H^Q(\cdot)$ is obtained from direct discretization and so is f_H^Q. B_H is then defined as

$$B_H = N_H^Q(R\cdot V_h) + R\cdot\left(B_h - N_h^Q(V_h)\right).$$

These differences in algorithm design result in different convergence results, which will be shown in section 5.

3.3. Multigrid for HJBI equations. We do not consider policy iteration for the HJBI equations due to its uncertainty in convergence. As for the HJB equations, we will apply the multigrid scheme we proposed in the previous section to the HJBI problem. Applying the FAS to HJBI equations is very similar to the FAS for HJB. The original nonlinear problem is rewritten in a similar way as in the HJB case with one more control added. Thus (2.8) becomes $N_h^{Q,P}(V^{n+1}) = B_h$. Following a similar derivation as in section 3.1, the relaxation scheme for HJBI problems is given by

(3.9)

$$\hat{V}_i^{k+1} = \sup_{P\in\hat{P}}\inf_{Q\in\hat{Q}}\left\{\frac{\Delta\tau\left(\alpha_i^{n+1}(Q,P)\hat{V}_{i-1}^k + \beta_i^{n+1}(Q,P)\hat{V}_{i+1}^k + d_i^{n+1}(Q,P)\right) + V_i^n}{1+\Delta\tau\left(\alpha_i^{n+1}(Q,P) + \beta_i^{n+1}(Q,P) + c_i^{n+1}(Q,P)\right)}\right\}.$$

A damped version of the relaxation scheme is used as a smoother. By a similar argument as in Theorem 3.1, it can be shown that the relaxation scheme (3.9) is globally convergent. The proof is omitted here.

The smoothing iteration is similar to the HJB case except that both Q^k and P^k need to be determined. More precisely, define

$$\left(\mathcal{F}^{Q,P}\hat{V}^k\right)_i \equiv \frac{\Delta\tau\left(\alpha_i^{n+1}(Q,P)\hat{V}_{i-1}^k + \beta_i^{n+1}(Q,P)\hat{V}_{i+1}^k + d_i^{n+1}(Q,P)\right) + V_i^n}{1+\Delta\tau\left(\alpha_i^{n+1}(Q,P) + \beta_i^{n+1}(Q,P) + c_i^{n+1}(Q,P)\right)}.$$

To find the optimal control values Q and P at grid point i in the kth iteration, we compute the value of $(\mathcal{F}^{Q,P}\hat{V}^k)_i$ for every Q with a fixed $P \in \hat{P}$ to obtain an infimum $\mathcal{I}(P) = (\mathcal{F}^{Q^*_P,P}\hat{V}^k)_i$, where $Q^*_P \in \arg\inf_{Q\in\hat{Q}}\{(\mathcal{F}^{Q,P}\hat{V}^k)_i\}$. Then compute the infimum $\mathcal{I}(P)$ for every $P \in \hat{P}$ to obtain the supremum of all $\mathcal{I}(P)$'s and its corresponding optimal P^*_i. The corresponding optimal Q^*_i is given by $Q^*_{P^*}$.

This process can be easily implemented by two nested loops:

```
Optimal Control at Grid Point i for HJBI Smoother
Let Sup = -∞
For all P ∈ P̂
    Let Inf = +∞
    For all Q ∈ Q̂
        If (F^{Q,P} V̂^k)_i < Inf
        Then Inf = (F^{Q,P} V̂^k)_i, Q*_P = Q
    If Inf > Sup
    Then Sup = Inf, Q*_i = Q*_P, P*_i = P.
```

Linear restriction and interpolation are used for intergrid transfer between fine and coarse levels when there is no jump. On the coarsest level, the nonlinear problem is solved by applying the relaxation scheme.

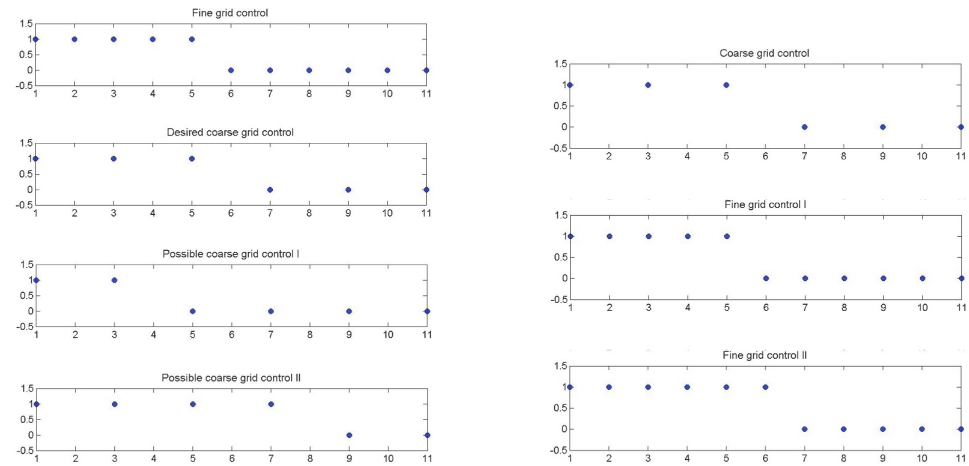
3.4. Jumps in control. The optimal control values P^*_i and Q^*_i can vary significantly from one grid point to another. Take the American options with stock borrowing fees problem in section 2.2, for example. In (2.4), there are two sets of controls. $\hat{Q}$ is composed by different combinations of r_l, r_b, 0, and 1, whose values do not change significantly; therefore, no special care is required. The other control μ has two possible values, 0 and 1. Note that μ is used with the penalty term: $\mu\frac{V-V^*}{\eta}$. Thus one can think that the control $\hat{P}$ is effectively $\{0, 10^8\}$, which will create a large jump when the optimal control P changes from one grid point to another. This issue can also appear when the control is not bounded. Ignoring such "jumps" in the optimal control values could slow down the convergence or even render a diverging result.

As a result, special care for the intergrid transfer is needed. We use a two-grid FAS for an HJB problem (3.1) with $|\hat{Q}| = 2$ to illustrate the modified FAS scheme for "jumps" in control. The procedures for coarser grid problem construction and coarse grid correction we proposed are presented in this section.

3.4.1. Coarser grid problem construction. The optimal control on the coarse grid might not be consistent with the optimal control on the fine grid. Suppose the coarse grid function $N^Q_H(\cdot)$ is a direct discretization of $N^Q_h(\cdot)$ on the coarse grid. The only information passed from the fine grid to the coarse grid is the approximate coarse grid solution V_H, which is the restricted approximate fine grid solution. In the standard approach, the coarse grid vector $B_H = R \cdot r_h + N^Q_H(V_H)$. While the fine grid residual r_h depends on the optimal control on the fine grid Q^*_h, $N^Q_H(V_H)$ depends on the optimal control on the coarse grid Q^*_H, which is obtained from (3.2). One can visualize the consistency issue by the diagram in Figure 3.1. Suppose the fine grid solution V_h has optimal control Q^*_h. The restriction of V_h on the coarse grid gives V_H, whose corresponding optimal control is Q^*_H. Consistency refers to whether Q^*_H is a restriction of Q^*_h, which is equivalent to whether the diagram commutes. When $Q^*_H \neq Q^*_h$ at a coarse grid point, the two components of vector B_H are inconsistent

$$\begin{tikzcd} V_h \arrow[r, leftrightarrow] \arrow[d] & Q_h^* \arrow[d, dashed] \\ V_H \arrow[r, leftrightarrow] & Q_H^* \end{tikzcd}$$

FIG. 3.1. *Diagram showing the relation between the fine grid solution V_h, fine grid optimal control Q_h^*, coarse grid solution V_H, and coarse grid optimal control Q_H^*.*



(a) Construct coarse problem (b) Optimal control interpolation

FIG. 3.2. *The different possible optimal control values near the jump during the intergrid transfers.*

with each other. Such a discrepancy is introduced by the restriction process and is mostly visible near the "jump".

To be more specific, consider the plot for the optimal controls of an HJB problem in Figure 3.2(a). The x-axis represents the grid point indices and the y-axis represents the optimal control values. The first plot shows the fine grid optimal control Q_h^* on each grid point: the first five fine grid points have optimal control 10^8 while the others have optimal control 0. There is a "jump" in Q_h^* between grid points 5 and 6. The desired Q_H^*, which would be consistent with the Q_h^* in the top plot, is shown in the second plot, where $Q_H^* = Q_h^*$ at all coarse grid points. The tricky part is that Q_H^* is not something we can choose directly. The optimal control is determined by the restricted coarse grid solution V_H. Depending on the values of V_H, the optimal control Q_H^* may not necessary be the second plot. It could be the third plot or the last plot in which the "jump" position is off by one grid point. As a result, B_H might have one term $R \cdot r_h$, which has the optimal control Q_h^* as shown in the first plot, and the other term $N_H^Q(V_H)$, which has the optimal control Q_H^* as shown in the third or last plot, rendering inconsistent optimal control values in the computation of B_H. When the "jump" size is large, the inconsistency will be very significant and the convergence of the multigrid method will be slowed down.

To avoid such a situation, our approach is to make the diagram in Figure 3.1 commute by forcing Q_H^* to match Q_h^* on the coarse grid points. Since Q_H^* is determined by V_H, the idea is to alter V_H. More precisely, suppose Q_H^* and Q_h^* are different at a particular grid point, i.e. $Q_h^* = q_1$ and $Q_H^* = q_2$ and $q_1 \neq q_2$. From (3.2), since q_2 is the optimal control for V_H, it gives the smallest value of $\mathcal{L}^{q_2} V_H$ and, in particular, $\mathcal{L}^{q_1} V_H > \mathcal{L}^{q_2} V_H$. Notice that $\mathcal{L}^{q_1} V_H$ depends on both q_1 and V_H. By changing the value of V_H, we also change the value of $\mathcal{L}^{q_1} V_H$. The trick is to modify V_H in such a way that $\mathcal{L}^{q_1} V_H < \mathcal{L}^{q_2} V_H$, which will then result in $Q_H^* = q_1$. To define an appropriate value for V_H, we solve

$$\mathcal{L}^{q_1} V_H + \delta = \mathcal{L}^{q_2} V_H, \tag{3.10}$$

where δ is a very small positive number, e.g., 10^{-10}. Equation (3.10) is a linear PDE since q_1 and q_2 are fixed. Also it is defined on one grid point, and hence it is a small linear problem which is easy to solve. The new V_H will ensure that $\mathcal{L}^{q_1} V_H < \mathcal{L}^{q_2} V_H$, yielding $Q_H^* = q_1$; therefore $Q_H^* = Q_h^*$. After handling all the grid points where the controls are different, the coarse grid problem becomes consistent.

For problems with a control set that has more than two values, i.e., $\hat{Q} = \{q_1, q_2, \dots, q_n\}$, change (3.10) to

$$\mathcal{L}^{q_1} V_H + \delta = \min_{Q \in \{q_2, \dots, q_n\}} \left\{ \mathcal{L}^Q V_H \right\}$$

and keep the other steps the same.

3.4.2. Coarse grid correction. Unlike the function V, the control does not always have a continuous control set. As such, it is not clear how to interpolate, or more precisely, how to define the control on the fine grid from the control on the coarse grid. Consider Figure 3.2(b). The plots show the optimal control for an HJB problem with a control set $\hat{Q} = \{q_1, q_2\}$, where the x-axis represents the grid points and the y-axis represents the value of the optimal control in 10^8 scale. From the coarse grid solution shown in the first plot, the optimal control is determined for every other fine grid point (grids with odd indices in this example). When there is no "jump" in the coarse grid control, the fine grid control is obtained from linear interpolation of coarse grid control; thus grids with even indices will have the same control as their neighboring grids have in the example. However, for grid point 6, its neighboring grids have different controls due to the "jump." Thus there are two possible scenarios for the optimal control on the fine grid: Fine grid controls I and II. In other words, the optimal control at grid point 6 can be the same as either its left neighbor or its right neighbor. It is not clear which of the two possible controls we should use. The linear interpolation is not applicable at this point since the "jump" size is large and there is no intermediate control value between the two in this example. If the control we choose to use is different from the one it should be, the convergence rate can be significantly slowed down or even yield divergence.

To address this issue, let i denote the fine grid index where the optimal control is different on its left and its right grid point ($i = 6$ in Figure 3.2(b)), and let $(Q_h^*)_i$ denote the optimal fine grid control at grid point i. Since there are two possible $(Q_h^*)_i$'s, we will consider them separately.

Case A. Assume the fine grid optimal control is taken as Fine grid control I in Figure 3.2(b). Let $Q'_j = (Q_h^*)_j$ for all $j \neq i$ and $Q'_i = (Q_h^*)_{i+1}$. Let V'_h be the updated solution after the standard coarse grid correction, i.e.,

$$V'_h = V_h + P \cdot (V_H - R \cdot V_h),$$

where V_h is the approximate fine solution after presmoothing, V_H is the coarse grid solution, and P and R are the interpolation and restriction operators. Due to the "jump" in the control, the error of V_h' near the grid point i could be large. Note that the control on the fine grid is now fixed, so $N_h^{Q'}(\cdot)$ becomes a linear operator. As both $N_h^{Q'}(\cdot)$ and B_h are now deterministic, we can compute an improved fine grid solution $\tilde{V}_h'$ corresponding to Q' by solving the linear system

$$N_h^{Q'}\left(\tilde{V}_h'\right) = B_h.$$

Since the "jump" in control mainly affects its neighboring grid points, a small local problem is considered in order to simplify the computation. With a predetermined small positive integer m, which usually lies between 2 and 5, the local linear problem is defined as

$$\left[N_h^{Q'}(\tilde{V}_h')\right]_j = (B_h)_j, \quad j = i-m,\ldots,i+m, \tag{3.11}$$

which is centered at the grid point i with size $2m+1$. Note that when we substitute $(\tilde{V}_h')_j$ back into (3.2), the resulting optimal control might not be the same as Q_j'. Hence we force the control to be the same as Q_j' by applying to the local problem the same technique used in section 3.4.1, and then update V_h' by setting $(V_h')_j = (\tilde{V}_h')_j$.

Case B. Assume the fine grid optimal control is taken as Fine grid control II in Figure 3.2(b). Let $Q_j'' = (Q_h^*)_j$ for all $j \neq i$ and $Q_i'' = (Q_h^*)_{i-1}$. Let

$$V_h' = V_h + P \cdot (V_H - R \cdot V_h).$$

Then repeat the process of Case A and obtain the updated solution $(V_h'')_j$.

V_h' and V_h'' are two possible fine grid updated solutions. In general, it is difficult to tell which is the desired solution. Assuming the correct fine grid control will yield a solution with smaller residual norm, that is the one we choose.

We remark that the construction of interpolation by a local PDE solve in (3.11) shares a similar flavor as the matrix dependent interpolation (e.g., [39]) and energy-minimizing interpolation (e.g., [37]) for solving PDEs with "jumps" in coefficients. However, with the existence of the control for the HJB and HJBI equations, it is not clear whether the proposed interpolation will still possess any energy minimizing property.

4. Smoothing analysis.

4.1. Smoothing analysis for HJB equations.

We investigate the smoothing property of the damped-relaxation smoother by applying a local Fourier analysis (LFA) on the grid functions $\varphi(\theta, x) = e^{\mathbf{i}\theta x}$ [35]. Let the exact solution for time step $n+1$ be $\bar{V}$ and let the approximate solution after the kth smoothing iteration be $\hat{V}^k = \bar{V} + \epsilon^k$, where ϵ^k is the error after the kth iteration. Let Q_i^k be the optimal control for $\hat{V}_i^k$ and let $\alpha_i^* = \alpha_i^{n+1}(Q_i^k)$, $\beta_i^* = \beta_i^{n+1}(Q_i^k)$, $c_i^* = c_i^{n+1}(Q_i^k)$, $d_i^* = d_i^{n+1}(Q_i^k)$, $l_i^* = \alpha_i^* + \beta_i^* + c_i^*$. By (3.6), we obtain

$$\epsilon_i^{k+1} = \begin{bmatrix} \frac{\Delta\tau\cdot\alpha_i^*}{1+\Delta\tau l_i^*} & 0 & \frac{\Delta\tau\cdot\beta_i^*}{1+\Delta\tau l_i^*} \end{bmatrix} \cdot \begin{bmatrix} \epsilon_{i-1}^k \\ \epsilon_i^k \\ \epsilon_{i+1}^k \end{bmatrix} + C(Q_i^k) \tag{4.1}$$

and

$$C\left(Q_i^k\right) = \begin{bmatrix} \frac{\Delta\tau\cdot\alpha_i^*}{1+\Delta\tau l_i^*} & -1 & \frac{\Delta\tau\cdot\beta_i^*}{1+\Delta\tau l_i^*} \end{bmatrix} \cdot \begin{bmatrix} \bar{V}_{i-1} \\ \bar{V}_i \\ \bar{V}_{i+1} \end{bmatrix} + \frac{V_i^n + \Delta\tau\cdot d_i^*}{1+\Delta\tau l_i^*}.$$

Since it is difficult to evaluate the smoothing effect on a nonlinear smoother, we will assume the optimal control for every grid point will not change from iteration to iteration in order to simplify the analysis. Let $\bar{Q}_i$ be the optimal control corresponding to the exact solution $\bar{V}$. Suppose $Q_i^k = \bar{Q}_i$ for all i. From (3.4), we deduce that $C\left(Q^k\right)$ is a zero vector. Thus (4.1) can be written as

$$\epsilon^{k+1} = S^k \cdot \epsilon^k,$$

where

$$S_{i,i}^k = 0, \quad S_{i,i-1}^k = \frac{\Delta\tau \cdot \alpha_i^*}{1+\Delta\tau l_i^*}, \quad S_{i,i+1}^k = \frac{\Delta\tau \cdot \beta_i^*}{1+\Delta\tau l_i^*}, \quad i = 2,\ldots,M-1.$$

Then the symbol of the smoothing operator for the relaxation scheme is

$$\widetilde{S_i^k}(\theta) = \frac{\Delta\tau \cdot \alpha_i^* \cdot e^{-\mathbf{i}\theta} + \Delta\tau \cdot \beta_i^* \cdot e^{\mathbf{i}\theta}}{1+\Delta\tau l_i^*}. \tag{4.2}$$

As we use the damped-relaxation scheme (3.6) for smoothing, the symbol of the HJB smoother can be obtained by introducing damping factor ω to (4.2),

$$\widetilde{S_i^k}(\theta,\omega) = \frac{\Delta\tau \cdot \alpha_i^* \cdot e^{-\mathbf{i}\theta} - (1-\frac{1}{\omega})\left[1+\Delta\tau l_i^*\right] + \Delta\tau \cdot \beta_i^* \cdot e^{\mathbf{i}\theta}}{\frac{1}{\omega}\cdot(1+\Delta\tau l_i^*)}. \tag{4.3}$$

Simplifying (4.3), we obtain

$$\widetilde{S_i^k}(\theta,\omega) = \frac{\omega\Delta\tau\left(\alpha_i^*+\beta_i^*\right)\cos\theta + (1-\omega)\left[1+\Delta\tau l_i^*\right] + \mathbf{i}\omega\Delta\tau\left(\beta_i^*-\alpha_i^*\right)\sin\theta}{1+\Delta\tau l_i^*}. \tag{4.4}$$

4.1.1. Smoothing factors of the example HJB equation. Since generating a useful analytical expression for $\widetilde{S_i^k}(\theta,\omega)$ is very complicated, we will consider $\widetilde{S_i^k}(\theta,\omega)$ for specific values of $\theta \approx -\pi$, which represent the high frequency components. Also, though we assume the optimal control is fixed for each grid point from iteration to iteration, we will examine many possible values of optimal controls for each grid point to make sure that the smoother is efficient even with the worst case optimal control for all grid points.

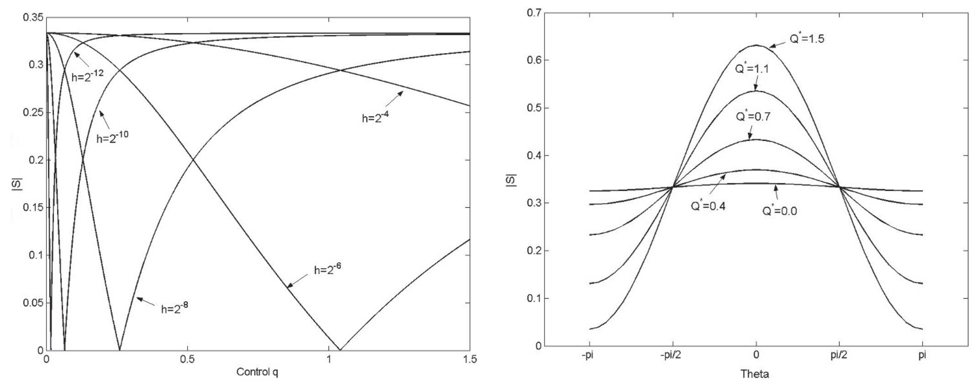
For simplicity of the analysis, we transform the equation to the log scale, which is a common practice in option pricing literature. Let $X = \log w$. Substituting $w = e^X$ into (2.3), the HJB example problem on the log grid can be written as

$$V_\tau = \inf_{Q\in\hat{Q}}\left\{\frac{1}{2}q^2\sigma^2 V_{XX} + \left(r + q\sigma\xi - \frac{1}{2}q^2\sigma^2 + \frac{\rho}{e^X}\right)V_X\right\}.$$

For simplicity, we will assume $\rho = 0$. The coefficients for the example HJB problem on the log grid in (2.1) are

$$a(\tau,Q) = \frac{1}{2}q^2\sigma^2, \quad b(\tau,Q) = r + Q\sigma\xi - \frac{1}{2}q^2\sigma^2, \quad c(\tau,Q) = 0, \quad d(\tau,Q) = 0.$$

We note that the coefficients on the log grid do not depend on X or S, which is a desirable property for LFA. For the pension plan asset allocation problem, the typical values for the parameters are $\sigma = 0.15$, $\xi = 0.33$, $r = 0.03$, $\Delta\tau = 0.01$, and $q \in [0, 1.5]$.



(a) $|\widetilde{S}(-\pi, \frac{2}{3})|$ with different values of h.

(b) $|\widetilde{S}(\theta, \frac{2}{3})|$ with $h = 0.025$ and different values of optimal control.

FIG. 4.1. *The amplification factor for the example HJB equation with* $\omega = \frac{2}{3}$.

Consider $\theta \approx -\pi$. Then $\sin\theta \approx 0$ and $\cos\theta \approx -1$. Substituting (2.6) into (4.4), we obtain

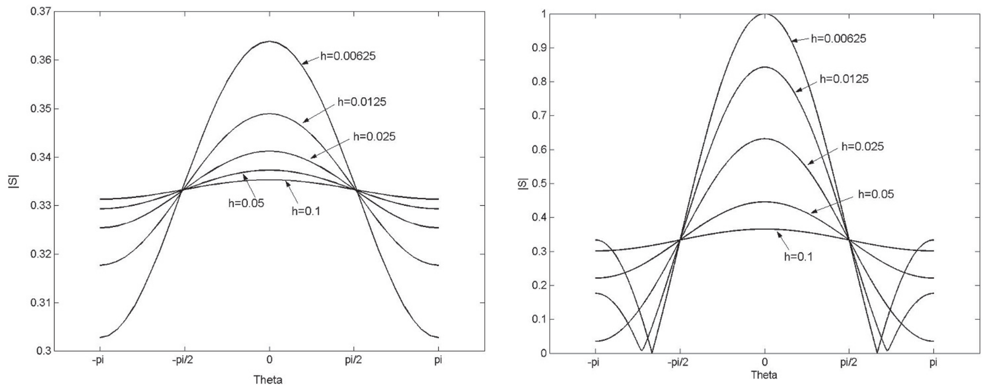
$$\begin{aligned} \widetilde{S_i^k}(\theta, \omega) &\approx \frac{-\omega\Delta\tau(\alpha_i^* + \beta_i^*) + (1-\omega)\left[1 + \Delta\tau(\alpha_i^* + \beta_i^*)\right]}{1 + \Delta\tau(\alpha_i^* + \beta_i^*)} \\ &= 1 - 2\omega + \frac{\omega}{1 + \Delta\tau \cdot \frac{q^2\sigma^2}{h^2}}. \end{aligned} \tag{4.5}$$

Considering the parameter values for σ, ξ, r, $\Delta\tau$, since $q \in [0, 1.5]$, we have $\widetilde{S_i^k}(\theta, \omega) \in [1 - 2\omega + \frac{\omega}{1+0.01\cdot\frac{1.5^2 0.15^2}{h^2}}, 1 - \omega]$. For $h \to 0$, $\widetilde{S_i^k}(\theta, \omega) \in [1 - 2\omega, 1 - \omega]$ and $|\widetilde{S_i^k}(\theta, \omega)| \leq \max(|1 - 2\omega|, |1 - \omega|)$ for all $\omega \in [0, 2]$. It is easy to show that the upper bound is less than 1 when $\omega \in (0, 1)$, and the upper bound is minimized when $\omega^* = \frac{2}{3}$. Using this ω^*, $|\widetilde{S_i^k}(\theta, \omega^*)| \leq \frac{1}{3}$. Plots for $|\widetilde{S_i^k}(\theta, \omega^*)|$ with different values of h are shown in Figure 4.1(a), which verifies the above analysis.

Also, by substituting $\theta \approx \frac{\pi}{2}$ and $\theta \approx 0$ into (4.4), we obtain that with $\omega \in (0, 2)$, $|\widetilde{S_i^k}(\theta, \omega)| < 1$ for medium and low frequency components on practical grid sizes. Therefore, from the above analysis, $\omega = \frac{2}{3}$ is an eligible damping factor and it yields an efficient smoothing effect among different values of ω's.

Since we are only able to compute the theoretical value of amplification factor on some particular frequencies, it is desirable to evaluate its actual value by making plots of $|\widetilde{S}(\theta, \omega)|$ in (4.3) for all frequencies, with different values of ω, grid size, and optimal control.

Previous analysis shows that ω has to lie between 0 and 1 to reach convergence, and $\omega = \frac{2}{3}$ appears to produce the optimal smoothing effect. To examine the smoothing effect of $\omega = \frac{2}{3}$, let $h = 0.025$, which is a practical value for the grid size. Plotting $|\widetilde{S}(\theta, \frac{2}{3})|$ against θ which varies from $-\pi$ to π, we obtain Figure 4.1(b). There are five curves in the plot, corresponding to different values of optimal control Q^* varying from 0 to 1.5. $|\widetilde{S}(\theta, \frac{2}{3})|$ has the properties of an effective smoother: low for high frequency components and bounded by 1 for low frequency components. It reaches its minimal when $\theta = \pm\pi$ and it is bounded above by $\frac{1}{3}$ when $|\theta| \geq \frac{\pi}{2}$. We then alter the grid size h to further investigate the relationship between $|\widetilde{S}(\theta, \frac{2}{3})|$ and the parame-



(a) $|\widetilde{S}(\theta, \frac{2}{3})|$ with $Q^* = 0$ and different values of h.

(b) $|\widetilde{S}(\theta, \frac{2}{3})|$ with $Q^* = 1.5$ and different values of h.

FIG. 4.2. *The amplification factor of the example HJB equation with $\omega = \frac{2}{3}$ and different values of h.*

ters. As shown in Figure 4.1(b), all the curves are bounded above by $Q^* = 1.5$ and $Q^* = 0$. Hence we will focus on these two control values. Figures 4.2(a) and 4.2(b) show the amplification factors for $Q^* = 0$ and $Q^* = 1.5$. It verifies that $\omega = \frac{2}{3}$ yields an efficient smoothing effect for different values of h.

4.2. Smoothing analysis for HJBI equations. Similar to the HJB case, we assume the exact solution for an HJBI problem at time step $n+1$ is $\bar{V}$ and the approximation solution after the kth iteration is $\hat{V}^k = \bar{V} + \epsilon^k$, where ϵ^k is the error after the kth smoothing iteration. Assume the optimal control for every grid point will not change from iteration to iteration. Applying to the HJBI case a similar deduction procedure as in section 4.1, we obtain

$$\widetilde{S_i^k}(\theta, \omega) = \frac{\omega\Delta\tau(\alpha_i^* + \beta_i^*)\cos\theta + (1-\omega)\left[1 + \Delta\tau l_i^*\right] + \mathbf{i}\omega\Delta\tau(\beta_i^* - \alpha_i^*)\sin\theta}{1 + \Delta\tau l_i^*}, \tag{4.6}$$

where

$$\begin{aligned} &\alpha_i^* = \alpha_i^{n+1}(Q_i^k, P_i^k), \quad \beta_i^* = \beta_i^{n+1}(Q_i^k, P_i^k), \quad c_i^* = c_i^{n+1}(Q_i^k, P_i^k), \\ &d_i^* = d_i^{n+1}(Q_i^k, P_i^k), \quad l_i^* = \alpha_i^* + \beta_i^* + c_i^*, \end{aligned}$$

with Q_i^k and P_i^k as the optimal control for $\hat{V}^k$ at grid point i.

4.2.1. Smoothing factors of the example HJBI equation. As in the HJB case, let $X = \log S$. The example HJBI problem on the log grid can then be written as

$$\begin{aligned} V_\tau = \sup_{P\in\hat{P}}\inf_{Q\in\hat{Q}} \Bigg\{ &\frac{\sigma^2}{2}V_{XX} + \left[q_3q_1 + (1-q_3)(r_l - r_f) - \frac{\sigma^2}{2}\right]V_X \\ &- \left[q_3q_1 + q_2(1-q_3) + \frac{\mu}{\eta}\right]V + \frac{\mu}{\eta}V^* \Bigg\}. \end{aligned}$$

The coefficients for the example HJBI problem on the log grid (2.2) are

$$a(\tau, Q, P) = \frac{1}{2}\sigma^2, \quad b(\tau, Q, P) = q_3 q_1 + (1 - q_3)(r_l - r_f) - \frac{\sigma^2}{2},$$
$$c(\tau, Q, P) = q_3 q_1 + (1 - q_3) q_2 + \frac{\mu}{\eta}, \quad d(\tau, Q, P) = \frac{\mu}{\eta} V^*.$$

The typical values for the parameters are $r_b = 0.05$, $r_l = 0.03$, $r_f = 0.004$, $\sigma^2 = 0.09$, $\Delta\tau = 0.01$, and $\eta = 10^{-6}\Delta\tau$. As in the HJB case, we will examine the amplification factor for high frequency components on different values of optimal controls.

Consider $\theta \approx -\pi$. Then $\sin\theta \approx 0$ and $\cos\theta \approx -1$. Substituting (2.6) into (4.6), we obtain

$$\begin{aligned}\widetilde{S_i^k}(\theta, \omega) &\approx \frac{-\omega\Delta\tau(\alpha_i^* + \beta_i^*) + (1 - \omega)\left[1 + \Delta\tau l_i^*\right]}{1 + \Delta\tau l_i^*} \\ &= 1 - \omega - \frac{\omega\Delta\tau\frac{\sigma^2}{h^2}}{1 + \Delta\tau\left(\frac{\sigma^2}{h^2} + \frac{\mu}{\eta}\right) + \Delta\tau(q_3 q_1 + (1 - q_3) q_2)}.\end{aligned}$$

When $\mu = 1$, $\widetilde{S_i^k}(\theta, \omega) \to 1 - \omega$ as $h \to 0$ due to the penalty term, despite other parameters. Therefore, to ensure convergence, it is required that $\omega \in (0, 2)$, and ω close to 1 is preferred. When $\mu = 0$,

$$\widetilde{S_i^k}(\theta, \omega) \to 1 - 2\omega \quad \text{as} \quad h \to 0.$$

Therefore, ω has to be smaller than 1 to ensure the convergence of the smoother. Furthermore, in the limit $h \to 0$, $|\widetilde{S_i^k}(\theta, \omega)|$ is bounded by $\max(|1 - \omega|, |1 - 2\omega|)$, which reaches its minimal when $\omega^* = \frac{2}{3}$. Using this $\omega^* = \frac{2}{3}$, $|\widetilde{S_i^k}(\theta, \omega^*)| \le \frac{1}{3}$.

By substituting $\theta \approx \frac{\pi}{2}$ and $\theta \approx 0$ into (4.6), we obtain that with $\omega \in (0, 2)$, $|\widetilde{S_i^k}(\theta, \omega)| < 1$ for medium and low frequency components on different grid sizes. Therefore, from the above analysis, $\omega = \frac{2}{3}$ is an eligible damping factor for the HJBI example and it yields an efficient smoothing effect among other ω's.

To justify the conclusion about the value for ω and the smoothing factor, Figure 4.3 shows the amplification factor when $\mu = 0$. The four curves represent $|\widetilde{S^k}(\theta, \omega)|$

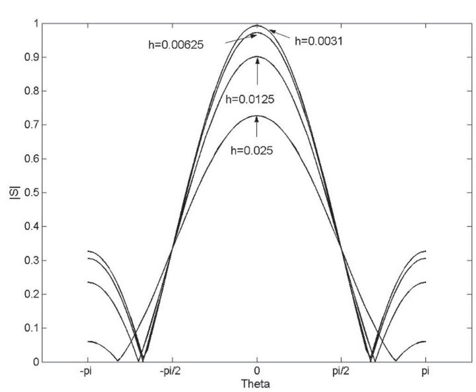


FIG. 4.3. $|\widetilde{S^k}(\theta, \frac{2}{3})|$ *with* $\mu = 0$ *and different values of* h.

on different values of h with θ varying from $-\pi$ to π. Although as the grid size decreases, $|\widetilde{S^k}(\theta,\omega)|$ of low frequency components move up to 1, high frequency components have small $|\widetilde{S^k}(\theta,\omega)|$ bounded by $\frac{1}{3}$. It verifies that $\omega = \frac{2}{3}$ will yield efficient smoothing for different values of h. The situation is similar when $\mu = 1$, which is not shown here.

5. Numerical results. In this section, we will illustrate the convergence of the proposed multigrid methods by a variety of HJB and HJBI problems which include the pension plan asset allocation problem and the American option problem, where there is a jump in control (cf. section 2). A comparison with the multigrid method proposed by Hoppe [10, 22] is given in one of the examples. For all the results in this section, the stopping criteria is that the residual norm of the nonlinear problem is smaller than 10^{-6}. The smoother is as defined in section 3.1. Two pre- and two postsmoothings are used for the multigrid methods.

Example 1. This example shows that the policy iteration, while convergent for HJB equations, can take up to the number of grid points to converge. The proposed multigrid method, on the other hand, can still be very fast. Consider the Markovian dynamic programming (MDP) problem in [30] which can be written as

$$V_i = \max\left\{V_{i-1} + f_i^1,\, V_{i+1} + f_i^2\right\}, \quad i = 0, \dots, M,$$

where $f_0^1 = f_0^2 = f_M^1 = f_M^2 = 0$, $f_i^1 = -1$, $f_i^2 = -2$ for all $i = 1, \dots, M-1$, and $f_{M-1}^1 = -1$, $f_{M-1}^2 = 2M$. Suppose the initial guess is $V_0 = 0$. The Newton-like policy iteration will correct the optimal control one by one, from grid $M-1$ to grid 1. Hence the number of iteration $= M-1$. We apply our multigrid method to solve this problem, increasing M from 128 to 1028. The number of multigrid iterations stays between 2 and 3.

Example 2. We compare the convergence of our method with MGHJB proposed in [10]. We use the same example as in their paper. The HJB problem is defined as

$$\begin{cases} \max\limits_{1\le v\le 2} \left[A^v u(x,y) - f^v(x,y)\right] = 0, & x, y \in (0,\,1), \\ u(x,y) = 0, & x, y \in \{0, 1\}, \end{cases}$$

where

$$A^1 = -\frac{\partial^2}{\partial x^2} - 0.5\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2}, \quad A^2 = -0.5\frac{\partial^2}{\partial x^2} - 0.1\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y^2},$$

and

$$f^1 = f^2 = \max\left(A^1\bar{u},\, A^2\bar{u}\right), \quad \bar{u} = x(1-x)y(1-y).$$

$\bar{u}$ turns out to be the exact solution of the corresponding HJB equation. We apply a standard finite difference discretization to the second order derivatives

$$\frac{\partial^2}{\partial x^2} \approx h^{-2} D_{h,x}^+ D_{h,x}^-,\ \frac{\partial^2}{\partial y^2} \approx h^{-2} D_{h,y}^+ D_{h,y}^-,\ \frac{\partial^2}{\partial x \partial y} \approx \frac{1}{2} h^{-2}\left[D_{h,x}^+ D_{h,y}^+ + D_{h,x}^- D_{h,y}^-\right],$$

where $D_{h,x}^{\pm}$ and $D_{h,y}^{\pm}$ denote the forward and backward difference in x and y, respectively.

Table 5.1 shows the number of levels, the number of multigrid iterations in one time step, and the rate of convergence for different grid sizes. We can observe that the

Table 5.1
Convergence of FAS scheme on the HJB example problem in [10].

h	Level	MG	Rate of conv.
$\frac{1}{8}$	2	5	0.02236
$\frac{1}{16}$	3	6	0.03914
$\frac{1}{32}$	4	7	0.04366
$\frac{1}{64}$	5	7	0.05039

Table 5.2
Parameters used in the HJB example.

r	σ	T	ξ	π	γ	*tol*	$\Delta\tau$	$\hat{Q}$
0.03	0.15	20 yrs	0.33	0.10	9.125	10^{-6}	0.01	[0, 1.5]

Table 5.3
Convergence result for policy iterations with multigrid.

Grid size (h)	Nonlinear iterations	Multigrid iterations per policy iteration		
		1st policy	2nd policy	3rd policy
0.02	2	2	1	N/A
0.01	2	3	1	N/A
0.005	3	4	3	1
0.0025	3	6	4	1

convergence of our multigrid method does not depend on the grid size. The rate of convergence presented in the last column is computed by averaging $\frac{\|r^{k+1}\|}{\|r^k\|}$ over all the iterations, where r^k is the residual of the HJB equation after the kth FAS iteration. Figures 1 and 2 in [10] show that the convergence rate of MGHJB is approximately 0.7 for this example problem. As shown in Table 5.1, the convergence rates of the proposed multigrid method for different grid sizes and the number of levels are all smaller than 0.1.

Example 3. In this example, we apply our multigrid method for solving the pension plan asset allocation problem (2.3) with parameters specified in Table 5.2. Meanwhile, we will compare our method with the approach in [3], where the policy iteration to solve the HJB equation is applied. In each policy iteration, a standard multigrid is used for the linearized problem with Gauss–Seidel as smoother.

Table 5.3 presents the convergence results for policy iterations with multigrid. Column 2 shows the number of nonlinear policy iterations required for one time step with different values of grid size. Columns 3–5 show the number of multigrid iterations required for solving the linear problem in each nonlinear policy iteration stated in the second column.

Table 5.4 shows the convergence results for the relaxation scheme as a solver and our multigrid method. The damping factor ω of the smoother is chosen to be $\frac{2}{3}$. The convergence rate of the relaxation scheme becomes unacceptably slow as grid size decreases. On the other hand, the number of multigrid iterations is insensitive to the grid size. Compared to Table 5.3, our multigrid is slightly more efficient than the policy iteration plus multigrid approach in terms of the total number of multi-

Table 5.4
Convergence of relaxation scheme and FAS scheme for the HJB case on uniform grid.

h	Relaxation iter.	MG iter.
0.02	90	4
0.01	330	5
0.005	1320	6
0.0025	$\approx$ 4000	8

Table 5.5
Parameters used in the HJBI example.

r_b	r_l	r_f	μ	σ	T	K	$\Delta\tau$	*tol*	penalty term η
0.05	0.03	0.004	$\{0,1\}$	0.30	1 yr	100	0.01	10^{-6}	$10^{-6}\Delta\tau$

grid iterations. Using either of these methods would yield satisfactory convergence. However, there is no obvious extension of the approach in [3] for HJBI equations.

Example 4. We will apply the proposed multigrid method to HJBI equations. The methods by Hoppe [10, 22] are for HJB equations only. The approach by [3] is based on the policy iteration, but an example presented in [11] has shown that the policy iteration may not converge for HJBI equations, in general. Consider the following discrete double-obstacle problem: find $V^* = (U_i)_{1\le i\le N} \in \mathbb{R}^N$ such that

$$\begin{cases} \max\left(\min\left(-\frac{U_{i-1}-2U_i+U_{i+1}}{\Delta s^2},\, \gamma(U_i - g(s_i))\right),\, \gamma(U_i - h(s_i))\right) = 0, \quad i = 1,\dots,N, \\ U_0 = 1,\ U_{N+1} = 0.8, \end{cases}$$

where $\Delta s = \frac{1}{N+1}$, $s_i = i\Delta s$, $g(s) = \max(0,\, 1.2 - ((s-0.6)/0.1)^2)$, and $h(s) = \min(2,\, 0.3 + ((s-0.2)/0.1)^2)$, with $\gamma = 1000$, $N = 127$, and a starting point V^0 such that $V^0 \notin (g,h)$. The policy iteration as shown in [11] does not converge. An iterative method that converges in 16 iterations for this problem is proposed in [11]. However, it involves solving an $N \times N$ linear system for 95 times in total. The relaxation iteration for this problem is

$$\hat{V}_i^{k+1} = -\max_{P\in\{1,2\}}\min_{Q\in\{1,2\}} \left\{ \frac{a_i(P,Q)\cdot\left(\hat{V}_{i-1}^k + \hat{V}_{i+1}^k\right) + c_i(P,Q)}{b_i(P,Q)} \right\},$$

where

$$a_i = \begin{bmatrix} -\frac{1}{\Delta s^2} & 0 \\ 0 & 0 \end{bmatrix},\ b_i = \begin{bmatrix} \frac{2}{\Delta s^2} & \gamma \\ \gamma & \gamma \end{bmatrix}, \quad \text{and} \quad c_i = \begin{bmatrix} 0 & -\gamma g(s_i) \\ -\gamma h(s_i) & -\gamma h(s_i) \end{bmatrix}.$$

While convergence is guaranteed, it will take 9457 iterations for the relaxation scheme to converge. On the other hand, the proposed multigrid method with three damped-relaxation pre- and postsmoothings will converge in 14 iterations.

Example 5. We apply the proposed multigrid method to solve the American options with stock borrowing fees (2.4) with parameters specified in Table 5.5. In this example, there is a jump in the control due to the penalty term η.

Table 5.6 shows the convergence results. It is obvious that the relaxation scheme converges very slowly on the fine grid in each time step while the number of multigrid iterations is insensitive to the grid size. In column 4, the smallest value for m is

TABLE 5.6
Convergence of relaxation scheme and FAS scheme for the HJBI case on uniform grid.

h	Relaxation iter.	MG iter.	m
$10^{-2}K$	319	6	1
$5 \times 10^{-3}K$	610	7	3
$2.5 \times 10^{-3}K$	2500	7	3
$1.25 \times 10^{-3}K$	≈ 10000	8	3

presented (section 3.4.2). It is a predetermined small positive integer and $2m + 1$ is the size of the local problem we solve to find the correct fine grid control at the jump locations. Here we list only the smallest possible value for m that will result in a convergent scheme. A larger m will also guarantee convergence, but the convergence rate will not improve much. It is clear that the multigrid method is more efficient than just the relaxation scheme alone for the example HJBI problem. Also, we note that if the linear interpolation and full weighting restriction were used instead of the interpolation and restriction described in section 3.4, the resulting multigrid method would diverge due to the jumps in control.

Example 6. We apply the multigrid method to solve a two-dimensional HJBI problem. It is based on a pursuit game example in [17] which can be written as a stationary HJBI equation

$$-\rho + \epsilon\Delta V + \max_{\alpha\in A}(\alpha \cdot \nabla V) + \min_{\beta\in B}(\beta \cdot \nabla V) + \|x\|_2^2 = 0$$

on $(-0.5, 0.5)^2$ with $\epsilon = 0.5$, Neumann boundary conditions, and $A = \{(a_1, a_2)|\, a_i = \pm 1, 0\}$, $B = \{(0, 0), (1, 2), (2, 1)\}$. Having ρ set to constant 0.194, we solve the problem using the proposed multigrid method. For fine grid size of 2^{-4}, 2^{-5}, 2^{-6}, and 2^{-7}, the multigrid method converges in three iterations for all four cases.

6. Conclusion.

In this paper, we propose solving the discretized HJB and HJBI equations by applying multigrid with damped-relaxation smoother. Unlike policy iteration, the relaxation scheme is convergent for both HJB and HJBI equations. Damping factor is appropriately chosen to damp away high frequency errors. Smoothing analysis based on two financial problems shows the efficiency of the smoother. Special restriction and interpolation techniques have been developed to handle the case when there are jumps in the optimal control. Our variation of FAS is applied to real life problems and examples in which policy iteration will not converge or converges slowly. Numerical results show that our multigrid method converges in small numbers of iterations for those examples.

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