First Order Logic

A proof consists of a sequence of numbered steps, each of which is an axiom, or follows from preceding steps by a rule of inference. For example:

1. $S \vdash S$	[$\texttt{ASSUME}$]
2. $S \vdash S \lor \neg S$	[$\texttt{DISJ1}$ 1]
3. $\neg S \vdash \neg S$	[$\texttt{ASSUME}$]
4. $\neg S \vdash S \lor \neg S$	[$\texttt{DISJ2}$ 3]
5. $\vdash S \lor \neg S$	[$\texttt{CASES}$ 2,4]

The Assumption Axiom [ASSUME]

$$S \vdash S$$

Cases (elimination of assumption) [CASES]

$$\frac{\Gamma_1, S \vdash S' \qquad \Gamma_2, \neg S \vdash S'}{\Gamma_1 \cup \Gamma_2 \vdash S'}$$

Truth axiom [TRUTH]

$$\vdash \texttt{T}$$

Falsity axiom [NOT_FALSE]

$$\vdash \neg\texttt{F}$$

Right $\vee$-introduction [`DISJ1`]

$$\frac{\Gamma \ \vdash \ S_1}{\Gamma \ \vdash \ S_1 \ \vee \ S_2}$$

Left $\vee$-introduction [`DISJ2`]

$$\frac{\Gamma \ \vdash \ S_2}{\Gamma \ \vdash \ S_1 \ \vee \ S_2}$$

$\vee$-elimination [`DISJ_CASES`]

$$\frac{\Gamma \ \vdash \ S_1 \ \vee \ S_2 \qquad \Gamma_1, \ S_1 \ \vdash \ S \qquad \Gamma_2, \ S_2 \ \vdash \ S}{\Gamma \cup \Gamma_1 \cup \Gamma_2 \ \vdash \ S}$$

Right $\vee$-introduction [`DISJ1`]

$$\frac{\Gamma \ \vdash \ S_1}{\Gamma \ \vdash \ S_1 \ \vee \ S_2}$$

Left $\vee$-introduction [`DISJ2`]

$$\frac{\Gamma \ \vdash \ S_2}{\Gamma \ \vdash \ S_1 \ \vee \ S_2}$$

$\wedge$-introduction [`CONJ`]

$$\frac{\Gamma_1 \ \vdash \ S_1 \qquad\qquad \Gamma_2 \ \vdash \ S_2}{\Gamma_1 \cup \Gamma_2 \ \vdash \ S_1 \ \wedge \ S_2}$$

$\wedge$-elimination [`CONJUNCT1`, `CONJUNCT2`]

$$\frac{\Gamma \ \vdash \ S_1 \ \wedge \ S_2}{\Gamma \ \vdash \ S_1 \qquad\qquad \Gamma \ \vdash \ S_2}$$

Discharging (⇒-introduction) [DISCH]

$$\frac{\Gamma \ \vdash \ S_2}{\Gamma - \{S_1\} \ \vdash \ S_1 \Rightarrow S_2}$$

- Where $\Gamma - \{S_1\}$ is the set subtraction of $\{S_1\}$ from Γ.

Modus Ponens (⇒-elimination) [MP]

$$\frac{\Gamma_1 \ \vdash \ S_1 \Rightarrow S_2 \qquad\qquad \Gamma_2 \ \vdash \ S_1}{\Gamma_1 \cup \Gamma_2 \ \vdash \ S_2}$$

Negation introduction [NEG_INTRO]

$$\frac{\Gamma_1, S_1 \ \vdash \ S_2 \qquad\qquad \Gamma_2, S_1 \ \vdash \ \neg S_2}{\Gamma_1 \cup \Gamma_2 \ \vdash \ \neg S_1}$$

Negation elimination [NEG_ELIM]

$$\frac{\Gamma_1 \ \vdash \ S_1 \qquad\qquad \Gamma_2 \ \vdash \ \neg S_1}{\Gamma_1 \cup \Gamma_2 \ \vdash \ S_2}$$

Double negation introduction [NEG_NEG_INTRO]

$$\frac{\Gamma \ \vdash \ S}{\Gamma \ \vdash \ \neg\neg S}$$

Double negation elimination [NEG_NEG_ELIM]

$$\frac{\Gamma \ \vdash \ \neg\neg S}{\Gamma \ \vdash \ S}$$

Generalization (∀-introduction) [GEN]

$$\frac{\Gamma \ \vdash \ S}{\Gamma \ \vdash \ \forall x.\ S}$$

- Where x is not free in Γ.

Specialization ($\forall$-elimination) [`SPEC`]

$$\frac{\Gamma \;\vdash\; \forall x.\ S}{\Gamma \;\vdash\; S[t/x]}$$

- $S[t/x]$ denotes the result of substituting t for free occurrences of x in S, with the restriction that no free variables in t become bound after substitution.

$\exists$-introduction [`EXISTS`]

$$\frac{\Gamma \;\vdash\; S[t]}{\Gamma \;\vdash\; \exists x.\ S[x]}$$

- Where $S[t]$ denotes a statement S with some free occurrences of t singled out, and $S[x]$ denotes the result of replacing these occurrences of t by x, subject to the restriction that x doesn't become bound after substitution.

$\exists$-elimination [`CHOOSE`]

$$\frac{\Gamma_1 \;\vdash\; \exists x.\ S[x] \qquad\qquad \Gamma_2,\ S[v] \;\vdash\; S'}{\Gamma_1 \cup \Gamma_2 \;\vdash\; S'}$$

- Where $S[v]$ denotes a statement containing a variable v, all of whose occurrences are free, and $S[x]$ denotes the result of replacing these occurrences of v by x.
- Side conditions on the rule are:
  - x doesn't become bound after substitution;
  - v does not occur in Γ_1, Γ_2 or S'.

Equivalence introduction [`EQV_INTRO`]

$$\frac{\Gamma_1 \;\vdash\; S_1 \Rightarrow S_2 \qquad\qquad \Gamma_2 \;\vdash\; S_2 \Rightarrow S_1}{\Gamma_1 \cup \Gamma_2 \;\vdash\; S_1 = S_2}$$

Equivalence elimination [`EQV_ELIM1`, `EQV_ELIM2`]

$$\frac{\Gamma \;\vdash\; S_1 = S_2}{\Gamma \;\vdash\; S_1 \Rightarrow S_2 \qquad\qquad \Gamma \;\vdash\; S_2 \Rightarrow S_1}$$