We made two crucial observations in Chapter I:

(1) that the MU-puzzle has depth largely because it involves the interplay of lengthening and shortening rules;

(2) that hope nevertheless exists for cracking the problem by employing a tool which is in some sense of adequate depth to handle matters of that complexity: the theory of numbers.

We did not analyze the MU-puzzle in those terms very carefully in Chapter I; we shall do so now. And we will see how the second observation (when generalized beyond the insignificant MIU-system) is one of the most fruitful realizations of all mathematics, and how it changed mathematicians' view of their own discipline.

For your ease of reference, here is a recapitulation of the MIU-system:

SYMBOLS: M, I, U

AXIOM: MI

RULES:

I. If xI is a theorem, so is xIU.
II. If Mx is a theorem, so is Mxx.
III. In any theorem, III can be replaced by U.
IV. UU can be dropped from any theorem.

Mumon Shows Us How to Solve the MU-puzzle

According to the observations above, then, the MU-puzzle is merely a puzzle about natural numbers in typographical disguise. If we could only find a way to transfer it to the domain of number theory, we might be able to solve it. Let us ponder the words of Mumon, who said, "If any of you has one eye, he will see the failure on the teacher's part." But why should it matter to have one eye?

If you try counting the number of I's contained in theorems, you will soon notice that it seems never to be 0. In other words, it seems that no matter how much lengthening and shortening is involved, we can never work in such a way that all I's are eliminated. Let us call the number of I's in any string the *I-count* of that string. Note that the I-count of the axiom MI is 1. We can do more than show that the I-count can't be 0—we can show that the I-count can never be any multiple of 3.

To begin with, notice that rules I and IV leave the I-count totally undisturbed. Therefore we need only think about rules II and III. As far as rule III is concerned, it diminishes the I-count by exactly 3. After an application of this rule, the I-count of the output might conceivably be a multiple of 3—but only if the I-count of the *input* was also. Rule III, in short, never creates a multiple of 3 from scratch. It can only create one when it began with one. The same holds for rule II, which doubles the I-count. The reason is that if 3 divides $2n$, then—because 3 does not divide 2—it must divide n (a simple fact from the theory of numbers). Neither rule II nor rule III can create a multiple of 3 from scratch.

But this is the key to the MU-puzzle! Here is what we know:

(1) The I-count begins at 1 (not a multiple of 3);

(2) Two of the rules do not affect the I-count at all;

(3) The two remaining rules which do affect the I-count do so in such a way as never to create a multiple of 3 unless given one initially.

The conclusion—and a typically hereditary one it is, too—is that the I-count can never become any multiple of 3. In particular, 0 is a forbidden value of the I-count. Hence, MU *is not a theorem of the MIU-system.*

Notice that, even as a puzzle about I-counts, this problem was still plagued by the crossfire of lengthening and shortening rules. Zero became the goal; I-counts could increase (rule II), could decrease (rule III). Until we analyzed the situation, we might have thought that, with enough switching back and forth between the rules, we might eventually hit 0. Now, thanks to a simple number-theoretical argument, we know that that is impossible.

Gödel-Numbering the MIU-System

Not all problems of the the type which the MU-puzzle symbolizes are so easy to solve as this one. But we have seen that at least one such puzzle could be embedded within, and solved within, number theory. We are now going to see that there is a way to embed *all* problems about *any* formal system, in number theory. This can happen thanks to the discovery, by Gödel, of a special kind of isomorphism. To illustrate it, I will use the MIU-system.

We begin by considering the notation of the MIU-system. We shall map each symbol onto a new symbol:

M $\Longleftrightarrow$ 3
I $\Longleftrightarrow$ 1
U $\Longleftrightarrow$ 0

The correspondence was chosen arbitrarily; the only rhyme or reason to it is that each symbol looks a little like the one it is mapped onto. Each number is called the *Gödel number* of the corresponding letter. Now I am sure you can guess what the Gödel number of a multiletter string will be:

MU $\Longleftrightarrow$ 30
MIIU $\Longleftrightarrow$ 3110
etc.