Lecture 5 - Propositions and Inference

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Readings: Poole & Mackworth 2nd ed. chapter 5.1-5.3, and 13.1-13.2
Two methods for solving problems:

- **Procedural**
  - devise an algorithm
  - program the algorithm
  - execute the program

- **Declarative**
  - identify the knowledge needed
  - encode the knowledge in a representation (knowledge base - KB)
  - use logical consequences of KB to solve the problem
Two methods for solving problems:

- **Procedural**
  - “how to” knowledge
  - programs
  - meaning of symbols is meaning of computation
  - languages: C, C++, Java ...

- **Declarative**
  - descriptive knowledge
  - databases
  - meaning of symbols is meaning in world
  - languages: propositional logic, Prolog, relational databases, ...
A logic consists of
- syntax: what is an acceptable sentence?
- semantics: what do the sentences and symbols mean?
- proof procedure: how do we construct valid proofs?

A proof: a sequence of sentences derivable using an inference rule
Logical Connectives

and (conjunction) $\land$

or (disjunction) $\lor$

not (negation) $\neg$

if . . . then . . . (implication) $\rightarrow$

. . . if and only if . . . $\leftrightarrow$
## Implication Truth Table

(A) \( \rightarrow \) (B)

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<th>( A \rightarrow B )</th>
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If it rains, then I will carry an umbrella
### Implication Truth Table

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(A)   (B)

If it rains, then I will carry an umbrella
If you don’t study, then you will fail
# Implication Truth Table

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>$A \rightarrow B$</th>
<th>$A \land \neg B$</th>
<th>$\neg(A \land \neg B)$</th>
<th>$\neg A \lor B$</th>
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\(A\) \hspace{1cm} \(B\)

- no rain or I will carry an umbrella
- study or you will fail
If and only if Truth Table

\[ A \leftrightarrow B \equiv (A \rightarrow B) \land (B \rightarrow A) \]
De Morgan’s Laws

\[ A \lor B \equiv \neg(\neg A \land \neg B) \]

it rains OR I play football
not true that ( it doesn’t rain AND I don’t play football )

\[ A \land B \equiv \neg(\neg A \lor \neg B) \]

I’m a politician AND I lie
not true that ( I’m not a politician OR I tell the truth)
**Modus Ponens**

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<th>((A → B) ∧ A) → B</th>
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Modus Ponens is a **Tautology**

If it’s raining then the grass is wet
it’s raining
therefore the grass is wet
Modus Tolens

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<th>A</th>
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Modus Tolens is a **Tautology**
If it’s raining then the grass is wet
the grass is not wet
therefore it’s not raining
Modus Bogus

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Modus Bogus is not a **Tautology**
If it’s raining then the grass is wet
the grass is wet
therefore it's raining
Logical Consequence

- \{X\} is a set of *statements*
- A set of truth assignments to \{X\} is an *interpretation*
- A *model* of \{X\} is an interpretation that makes \{X\} true.
- We say that the world in which these truth assignments hold is a *model* (a verifiable *example*) of \{X\}.
- \{X\} is *inconsistent* if it has no *model*
A statement, A, is a logical consequence of a set of statements \{X\}, if A is true in every *model* of \{X\}.

If, for every set of truth assignments that hold for \{X\} (for every *model* of \{X\}), some other statement (A) is always true, then this other statement is a *logical consequence* of \{X\}.
An argument is **valid** if any of the following is true:

- the conclusions are a logical consequence of the premises.
- the conclusions are true in every model of the premises.
- there is **no** situation in which the premises are all true, but the conclusions are false.
- argument $\rightarrow$ conclusions is a *tautology* (always true)

(These four statements are identical)
Arguments and Models

P1: If I play hockey, then I’ll score a goal if the goalie is not good
P2: If I play hockey, the goalie is not good
D: Therefore, if I play hockey, I’ll score a goal

P: I play hockey
C: I’ll score a goal
H: the goalie is good

\[ P1 : P \rightarrow (\neg H \rightarrow C) \quad P2 : P \rightarrow \neg H \]
\[ D : P \rightarrow C \]

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<tr>
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<th>( C )</th>
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D : P \rightarrow C
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Each row is an interpretation: an assignment of T/F to each proposition.

In all the green lines, the premises are true:

these interpretations are models of $P1$ and $P2$.

Every model of $P1$ and $P2$ is a model of $D$.

Therefore, $D$ is a logical consequence of $P1$ and $P2$: $P1, P2 \models D$. 
Logical Consequence

P1: Elvis is Dead
P2: Elvis is Not Dead
D: Therefore, Gerry is Alive

Is this argument valid?
Logical Consequence

P1: Elvis is Dead
P2: Elvis is Not Dead
D: Therefore, Gerry is Alive

Is this argument valid?
Yes!
E: Elvis is Alive
G: Gerry is Alive

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An argument is valid if there is no situation in which the premises are all true, but the conclusions are false. But here, there is no model of the premises, so the argument is valid.
Given a knowledge base, we want to prove things that are true. We can use

- Truth Table
- Natural Deduction
- Semantic Tableaux
- Axiomatic Logic (Modus Ponens)
  \[(A \rightarrow B) \land A \rightarrow B\]
- Resolution Refutation (Reductio Ad Absurdum)
  \[\neg A \land \ldots \land \ldots \rightarrow \bot \rightarrow A\]
Proofs

- A **KB** is a set of axioms
- A **proof procedure** is a way of Proving Theorems
- KB ⊢ g means g can be **derived** from KB using the proof procedure
- If KB ⊢ g, then g is a **Theorem**
- A proof procedure is **sound**:
  if KB ⊢ g then KB |= g.
- A proof procedure is **complete**:
  if KB |= g then KB ⊢ g.
- Two types of proof procedures: **bottom up** and **top down**
Complete Knowledge

- we assume a *closed world*
  - the agent knows everything (or can prove everything)
  - if it can’t prove something: must be false
  - *negation as failure*

- other option is an *open world*:
  - the agent doesn’t know everything
  - can’t conclude anything from a lack of knowledge
Bottom-up proof (no variables)

also known as forward chaining - start from facts and use rules to generate all possible atoms

rain ← clouds ∧ wind.
clouds ← humid ∧ cyclone.
clouds ← near_sea ∧ cyclone.
wind ← cyclone.
near_sea.
cyclone.
also known as **forward chaining** - start from facts and use rules to generate all possible atoms

\[
\begin{align*}
\text{rain} & \leftarrow \text{clouds} \land \text{wind}. \\
\text{clouds} & \leftarrow \text{humid} \land \text{cyclone}. \\
\text{clouds} & \leftarrow \text{near\_sea} \land \text{cyclone}. \\
\text{wind} & \leftarrow \text{cyclone}. \\
\text{near\_sea}. \\
\text{cyclone}. \\
\{\text{near\_sea, cyclone}\}
\end{align*}
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also known as **forward chaining** - start from facts and use rules to generate all possible atoms

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\text{rain} \leftarrow \text{clouds} \land \text{wind}.
\]
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\text{clouds} \leftarrow \text{humid} \land \text{cyclone}.
\]
\[
\text{clouds} \leftarrow \text{near\_sea} \land \text{cyclone}.
\]
\[
\text{wind} \leftarrow \text{cyclone}.
\]
\[
\text{near\_sea}.
\]
\[
\text{cyclone}.
\]
\[
\{\text{near\_sea}\,,\text{cyclone}\}
\]
\[
\{\text{near\_sea} \,,\text{cyclone} \,,\text{wind}\}
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\text{wind} & \leftarrow \text{cyclone}. \\
\text{near\_sea} & . \\
\text{cyclone} & . \\
\{ \text{near\_sea, cyclone} \} \\
\{ \text{near\_sea, cyclone, wind} \} \\
\{ \text{near\_sea, cyclone, wind, clouds} \}
\end{align*}
\]
also known as **forward chaining** - start from facts and use rules to generate all possible atoms

```plaintext
rain ← clouds ∧ wind.
clouds ← humid ∧ cyclone.
clouds ← near_sea ∧ cyclone.
wind ← cyclone.
near_sea.
cyclone.
{near_sea,cyclone
 }  
{near_sea,cyclone
 ,wind
 }  
{near_sea,cyclone
 ,wind,cclouds
 }  
{near_sea,cyclone
 ,wind,cclouds,rain
 }
```
Bottom-up proof (no variables)

\[ C := \{\}; \]

repeat

\[
\text{select } r \in KB \text{ such that } \\
\quad \cdot r \text{ is } h \leftarrow b_1 \land \ldots \land b_m \\
\quad \cdot b_i \in C \quad \forall \quad i \\
\quad \cdot h \notin C
\]

\[ C := C \cup \{h\} \]

until no more clauses can be selected

Sound and Complete
Top-Down Proof (no variables)

start from query and work backwards

rain ← clouds ∧ wind.
clouds ← humid ∧ cyclone.
clouds ← near Sea ∧ cyclone.
wind ← cyclone.
near Sea.
cyclone.
Top-Down Proof (no variables)

start from query and work backwards

\[
\begin{align*}
\text{rain} & \leftarrow \text{clouds} \land \text{wind}. \\
\text{clouds} & \leftarrow \text{humid} \land \text{cyclone}. \\
\text{clouds} & \leftarrow \text{near\_sea} \land \text{cyclone}. \\
\text{wind} & \leftarrow \text{cyclone}. \\
\text{near\_sea} & . \\
\text{cyclone} & . \\
\text{yes} & \leftarrow \text{rain}.
\end{align*}
\]
Top-Down Proof (no variables)

start from query and work backwards

rain ← clouds ∧ wind.
clouds ← humid ∧ cyclone.
clouds ← near_sea ∧ cyclone.
wind ← cyclone.
near_sea.
 cyclone.

yes ← rain.
yes ← clouds ∧ wind
Top-Down Proof (no variables)

start from query and work backwards

rain ← clouds ∧ wind.
clouds ← humid ∧ cyclone.
clouds ← near_sea ∧ cyclone.
wind ← cyclone.
near_sea.
cyclone.

yes ← rain.
yes ← clouds ∧ wind
yes ← near_sea ∧ cyclone ∧ wind
Top-Down Proof (no variables)

start from query and work backwards

\[ \text{rain} \leftarrow \text{clouds} \land \text{wind}. \]
\[ \text{clouds} \leftarrow \text{humid} \land \text{cyclone}. \]
\[ \text{clouds} \leftarrow \text{near\_sea} \land \text{cyclone}. \]
\[ \text{wind} \leftarrow \text{cyclone}. \]
\[ \text{near\_sea}. \]
\[ \text{cyclone}. \]

\[ \text{yes} \leftarrow \text{rain}. \]
\[ \text{yes} \leftarrow \text{clouds} \land \text{wind} \]
\[ \text{yes} \leftarrow \text{near\_sea} \land \text{cyclone} \land \text{wind} \]
\[ \text{yes} \leftarrow \text{near\_sea} \land \text{cyclone} \land \text{cyclone} \]
Top-Down Proof (no variables)

start from query and work backwards

rain ← clouds ∧ wind.
clouds ← humid ∧ cyclone.
clouds ← near Sea ∧ cyclone.
wind ← cyclone.
near Sea.
cyclone.

yes ← rain.
yes ← clouds ∧ wind
yes ← near Sea ∧ cyclone ∧ wind
yes ← near Sea ∧ cyclone ∧ cyclone
yes ← near Sea ∧ cyclone
Top-Down Proof (no variables)

start from query and work backwards

rain ← clouds ∧ wind.
clouds ← humid ∧ cyclone.
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\text{yes} & \leftarrow \text{rain}. \\
\text{yes} & \leftarrow \text{clouds} \land \text{wind} \\
\text{yes} & \leftarrow \text{near~sea} \land \text{cyclone} \land \text{wind} \\
\text{yes} & \leftarrow \text{near~sea} \land \text{cyclone} \land \text{cyclone} \\
\text{yes} & \leftarrow \text{near~sea} \land \text{cyclone} \\
\text{yes} & \leftarrow \text{cyclone} \\
\text{yes} & \leftarrow
\end{align*}
\]
Top-Down Interpreter (no variables)

\[
solve(q_1 \land \ldots \land q_k):
\]
\[
ac := \text{"yes} \leftarrow q_1 \land \ldots \land q'_k
\]
\[
\text{repeat}
\]
\[
\text{select} \text{ a conjunct } a_i \text{ from body of } ac
\]
\[
\text{choose} \text{ a clause } C \text{ from KB with } a_i \text{ as head}
\]
\[
\text{replace } a_i \text{ in body of } ac \text{ by body of } C
\]
\[
\text{until } ac \text{ is an answer}
\]
Top-Down Proof (no variables)

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
water.
bread.
start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.

house.
water.
bread.

yes ← health.
Top-Down Proof (no variables)

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
water.
bread.

yes ← health.
yes ← sustenance ∧ shelter
Top-Down Proof (no variables)

start from query and work backwards

health ← sustenance \& shelter.
sustenance ← macdonalds \& cocacola.
sustenance ← bread \& water.
shelter ← house.

yes ← health.
yes ← sustenance \& shelter
yes ← bread \& water \& shelter
Top-Down Proof (no variables)

start from query and work backwards

health ← sustenance \(\land\) shelter.
sustenance ← macdonalds \(\land\) cocacola.
sustenance ← bread \(\land\) water.
shelter ← house.

house.
water.
bread.

yes ← health.
yes ← sustenance \(\land\) shelter
yes ← bread \(\land\) water \(\land\) shelter
yes ← bread \(\land\) water \(\land\) house
start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.

yes ← health.
yes ← sustenance ∧ shelter
yes ← bread ∧ water ∧ shelter
yes ← bread ∧ water ∧ house
yes ← bread ∧ water
Top-Down Proof (no variables)

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
water.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← bread ∧ water ∧ shelter
yes ← bread ∧ water ∧ house
yes ← bread ∧ water
yes ← bread
Top-Down Proof (no variables)

start from query and work backwards

\( \text{health} \leftarrow \text{sustenance} \land \text{shelter}. \)
\( \text{sustenance} \leftarrow \text{macdonalds} \land \text{cocacola}. \)
\( \text{sustenance} \leftarrow \text{bread} \land \text{water}. \)
\( \text{shelter} \leftarrow \text{house}. \)
\( \text{house}. \)
\( \text{water}. \)
\( \text{bread}. \)

\( \text{yes} \leftarrow \text{health}. \)
\( \text{yes} \leftarrow \text{sustenance} \land \text{shelter} \)
\( \text{yes} \leftarrow \text{bread} \land \text{water} \land \text{shelter} \)
\( \text{yes} \leftarrow \text{bread} \land \text{water} \land \text{house} \)
\( \text{yes} \leftarrow \text{bread} \land \text{water} \)
\( \text{yes} \leftarrow \text{bread} \)
\( \text{yes} \leftarrow \)
solve($q_1 \land \ldots \land q_k$):
   $ac := “yes \leftarrow q_1 \land \ldots \land q’_k$  
   repeat
      select a conjunct $a_i$ from body of $ac$
      choose a clause $C$ from KB with $a_i$ as head
      replace $a_i$ in body of $ac$ by body of $C$
   until $ac$ is an answer

**select**: “don’t care nondeterminism”  
(If one doesn’t give a solution, no point trying others!)

**choose**: “don’t know nondeterminism”  
(if one doesn’t give a solution, others may)
Top-Down Proof - selection failure

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.
Top-Down Proof - selection failure

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
Top-Down Proof - selection failure

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
Top-Down Proof - selection failure

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← bread ∧ water ∧ shelter
Top-Down Proof - selection failure

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← bread ∧ water ∧ shelter
yes ← bread ∧ water ∧ house
Top-Down Proof - selection failure

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← bread ∧ water ∧ shelter
yes ← bread ∧ water ∧ house
yes ← bread ∧ water
Top-Down Proof - selection failure

start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← bread ∧ water ∧ shelter
yes ← bread ∧ water ∧ house
yes ← bread ∧ water
yes ← water
start from query and work backwards

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← bread ∧ water ∧ shelter
yes ← bread ∧ water ∧ house
yes ← bread ∧ water
yes ← water

Failure with this choice, try the other
(have to try them all)
Top-Down Proof - selection failure

start again part way through

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← macdonalds ∧ cocacola ∧ shelter
start again part way through

\[
\begin{align*}
\text{health} & \leftarrow \text{sustenance} \land \text{shelter}. \\
\text{sustenance} & \leftarrow \text{macdonalds} \land \text{cocacola}. \\
\text{sustenance} & \leftarrow \text{bread} \land \text{water}. \\
\text{shelter} & \leftarrow \text{house}. \\
\text{house}. \\
\text{cocacola}. \\
\text{bread}. \\
\text{yes} & \leftarrow \text{health}. \\
\text{yes} & \leftarrow \text{sustenance} \land \text{shelter} \\
\text{yes} & \leftarrow \text{macdonalds} \land \text{cocacola} \land \text{shelter} \\
\text{yes} & \leftarrow \text{macdonalds} \land \text{cocacola} \land \text{house}
\end{align*}
\]
Top-Down Proof - selection failure

start again part way through

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← macdonalds ∧ cocacola ∧ shelter
yes ← macdonalds ∧ cocacola ∧ house
yes ← macdonalds ∧ cocacola
Top-Down Proof - selection failure

start again part way through

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← macdonalds ∧ cocacola ∧ shelter
yes ← macdonalds ∧ cocacola ∧ house
yes ← macdonalds ∧ cocacola
yes ← macdonalds
start again part way through

health ← sustenance ∧ shelter.
sustenance ← macdonalds ∧ cocacola.
sustenance ← bread ∧ water.
shelter ← house.
house.
cocacola.
bread.

yes ← health.
yes ← sustenance ∧ shelter
yes ← macdonalds ∧ cocacola ∧ shelter
yes ← macdonalds ∧ cocacola ∧ house
yes ← macdonalds ∧ cocacola
yes ← macdonalds

Failure with this selection, so we’re done
(since one conjunct is going to fail, the whole thing fails)
Top-Down Interpreter (no variables)

solve($q_1 \land \ldots \land q_k$):

$$ac := \text{"yes $\leftarrow q_1 \land \ldots \land q_k"\}$$

repeat

select a conjunct $a_i$ from body of $ac$

choose a clause $C$ from KB with $a_i$ as head

replace $a_i$ in body of $ac$ by body of $C$

until $ac$ is an answer

select: “don’t care nondeterminism”

any one will do, but be careful: some selections will lead more quickly to solutions!

choose: “don’t know nondeterminism”

have to do them all: can determine the complexity of the problem

Read more about negation as failure in Section 5.6 (not course material)
Towards Automated Methods

- A proof procedure gives us a method for deriving theorems
- Therefore, given a knowledge base of assumptions, we can 'prove' things and know they are tautologies (they are logical consequences of our knowledge base)

but ....
The method is difficult and requires some know-how - how could we make it work more automatically?
A well-formed formula is in *conjunctive normal form* (CNF) if it is a conjunction of disjunctions of atoms.

\[(p_1 \lor p_2) \land (p_3 \lor p_4 \lor p_5) \land (p_6 \lor p_7 \lor \ldots) \ldots \land (p_{n-1} \lor p_n)\]

Convert a propositional formula to CNF:

1. Eliminate $\leftrightarrow$ using $A \leftrightarrow B \equiv (A \rightarrow B) \land (B \rightarrow A)$
2. Eliminate $\rightarrow$ using $A \rightarrow B \equiv \neg A \lor B$
3. Use deMorgan’s laws to push $\neg$ into atoms
4. Use $\neg\neg A \equiv A$ to eliminate double negatives
5. Use distributive law to complete $A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$

write

\[(p_1 \lor p_2) \land (p_3 \lor p_4 \lor p_5) \land (p_6 \lor p_7 \lor \ldots) \ldots \land (p_{n-1} \lor p_n)\]

as

\[
\{\{p_1, p_2\}, \{p_3, p_4, p_5\}, \{p_6, p_7, \ldots\} \ldots, \{p_{n-1}, p_n\}\}
\]
Refutation of Modus Ponens

\[ A \land (A \rightarrow B) \vdash B \]

show a contradiction

\[ A \land (A \rightarrow B) \land \neg B \models \bot \]

1. \[ A \land (\neg A \lor B) \land (\neg B) \]

2. \{\{A\}, \{\neg A, B\}, \{\neg B\}\}
Conjunctive Normal Form - Example 2

Transitivity of Implication

\(((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C)\)

try to show a contradiction

\((A \rightarrow B) \land (B \rightarrow C) \land \neg(A \rightarrow C) \models \bot\)

1. \((\neg A \lor B) \land (\neg B \lor C) \land \neg(\neg A \lor C)\)
2. \((\neg A \lor B) \land (\neg B \lor C) \land (\neg \neg A \land \neg C)\)
3. \((\neg A \lor B) \land (\neg B \lor C) \land A \land \neg C\)
4. \{\{\neg A, B\}, \{\neg B, C\}, \{A\}, \{\neg C\}\}
A complementary pair of propositions is $p_i, \neg p_i$

Can show that two clauses with a complementary pair:
\[
\{\{A, B\}, \{C, \neg B\}\} \equiv \{\{A, B\}, \{C, \neg B\}, \{A, C\}\}
\]

That is, since $B$ and $\neg B$ cannot both be true, one of $A$ or $C$ has to be true, otherwise the whole formula is false.

Therefore, we can “resolve” \{\{A, B\}, \{C, \neg B\}\} into \{\{A, C\}\}

This means that \{\{A, C\}\} is true whenever \{\{A, B\}, \{C, \neg B\}\} is true.

So we can *add* \{\{A, C\}\} to the statement without changing the truth value.
Proof by *resolution refutation*: deny the conclusions and show
a resolution to $\bot$.

Resolve clauses - adds new clauses that are true whenever the
existing clauses are true.

If you can find a contradiction, then
  ▶ the existing clauses cannot all be true
  ▶ If the premises are all true, the refutation of the conclusion
    **must** be false,
  ▶ so the argument is valid

If you cannot find a contradiction after resolving all clauses
  ▶ the refutation of the conclusion **must** be true
  ▶ so the argument is invalid
Refutation of Modus Ponens

\[ A \land (A \rightarrow B) \vdash B \]

show a contradiction

\[ A \land (A \rightarrow B) \land \neg B \vdash \bot \]

1. \[ A \land (\neg A \lor B) \land (\neg B) \]
2. \[ \{\{A\}, \{\neg A, B\}, \{\neg B\}\} \]
3. \[ \{\{A\}, \{\neg A, B\}, \{B\}, \{\neg B\}\} \]
4. \[ \bot \]
Transitivity of Implication (again)

\[ ((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C) \]

try to show a contradiction

\[ (A \rightarrow B) \land (B \rightarrow C) \land \neg(A \rightarrow C) \models \bot \]

1. \[ (\neg A \lor B) \land (\neg B \lor C) \land \neg(\neg A \lor C) \]
2. \[ (\neg A \lor B) \land (\neg B \lor C) \land (\neg \neg A \land \neg C) \]
3. \[ (\neg A \lor B) \land (\neg B \lor C) \land A \land \neg C \]
4. \[ \{\{\neg A, B\}, \{\neg B, C\}, \{A\}, \{\neg C\}\} \]
5. \[ \{\{\neg A, B\}, \{\neg B, C\}, \{\neg A, C\}, \{A\}, \{\neg C\}\} \]
6. \[ \{\{\neg A, B\}, \{\neg B, C\}, \{\neg A, C\}, \{A\}, \{C\}, \{\neg C\}\} \]
7. \[ \bot \]
Resolution - Example 3

P1: If I play hockey, then I’ll score a goal if the goalie is not good
P2: If I play hockey, the goalie is not good
D: if I play hockey, I’ll score a goal

P: I play hockey, C: I’ll score a goal, H: the goalie is good

$P1 : P \rightarrow (\neg H \rightarrow C) \quad P2 : P \rightarrow \neg H$

$D : P \rightarrow C$  \hspace{1cm} \text{test (refutation of } D) : P1 \wedge P2 \wedge \neg D$

\[
\begin{align*}
(P \rightarrow (\neg H \rightarrow C)) \wedge (P \rightarrow \neg H) \wedge \neg(P \rightarrow C) \\
(\neg P \lor (\neg H \rightarrow C)) \wedge (\neg P \lor \neg H) \wedge \neg(\neg P \lor C) \\
(\neg P \lor (\neg \neg H \lor C)) \wedge (\neg P \lor \neg H) \wedge \neg(\neg P \lor C) \\
(\neg P \lor \neg \neg H \lor C) \wedge (\neg P \lor \neg H) \wedge (\neg \neg P \land \neg C) \\
(\neg P \lor H \lor C) \wedge (\neg P \lor \neg H) \wedge (P) \wedge (\neg C) \\
\left\{\{\neg P, H, C\}, \{\neg P, \neg H\}, \{P\}, \{\neg C\}\right\} \\
\left\{\{\neg P, H, C\}, \{\neg P, \neg H\}, \{\neg P, C\}, \{P\}, \{\neg C\}\right\} \\
\left\{\{\neg P, H, C\}, \{\neg P, \neg H\}, \{\neg P, C\}, \{P\}, \{C\}, \{\neg C\}\right\}
\end{align*}
\]
\[\bot\]
Many problems can be formulated as a CNF

- Satisfiability
- Logic circuits
- Gene decoding
- Scheduling
- Air traffic control
- ...

Combinatorial Search Problems
Constraint Satisfaction as CNF

- A CSP variable $Y$ with domain $\{v_1, \ldots, v_k\}$ can be converted into $k$ Boolean variables $\{Y_1, \ldots, Y_k\}$ where $Y_i$ is true when $Y$ has value $v_i$ and false otherwise.

- Thus, $k$ atoms $y_1, \ldots, y_k$ are used to represent the CSP variable.

- Constraints:
  - $y_i$ and $y_j$ cannot both be true when $i \neq j$: $\neg y_i \lor \neg y_j$ for $i < j$
  - At least one of the $y_i$ must be true: $y_1 \lor \ldots \lor y_k$
  - There is a clause for each false assignment in each constraint, which specifies which assignments are not allowed.
  - Thus, if there are two variables $Y$ and $Z$, and a constraint $Y \neq Z$, then we have clauses $\neg y_i \lor \neg z_i$ for all $i$ (Assuming $Y$ and $Z$ have the same domains).
Example Delivery robot: activities \( a, b \), times \( 1,2,3,4 \).

Constraints:
\[(A \neq 2) \land (B \neq 1) \land (A < B)\]

Write down the CNF for this CSP
Where propositions stop...

$
\begin{align*}
\text{peter is smart} & \land \text{peter is a student} \rightarrow \text{peter will pass} \\
\text{ann is smart} & \land \text{ann is a student} \rightarrow \text{ann will pass} \\
\text{lou is smart} & \land \text{lou is a student} \rightarrow \text{lou will pass}
\end{align*}
$
Where propositions stop...

\[
\text{peter is smart} \land \text{peter is a student} \rightarrow \text{peter will pass}
\]
\[
\text{ann is smart} \land \text{ann is a student} \rightarrow \text{ann will pass}
\]
\[
\text{lou is smart} \land \text{lou is a student} \rightarrow \text{lou will pass}
\]
\[
\underline{\text{peter is smart and peter is a student}}
\]
\[
\underline{\text{peter will pass}}
\]
Where propositions stop...

\[\text{peter is smart } \land \text{ peter is a student } \rightarrow \text{ peter will pass}\]
\[\text{ann is smart } \land \text{ ann is a student } \rightarrow \text{ ann will pass}\]
\[\text{lou is smart } \land \text{ lou is a student } \rightarrow \text{ lou will pass}\]
\[\text{ann is smart and ann is a student}
\]

\[\text{ann will pass}\]
Where propositions stop...

\[
\begin{align*}
\text{peter is smart} & \land \text{peter is a student} \implies \text{peter will pass} \\
\text{ann is smart} & \land \text{ann is a student} \implies \text{ann will pass} \\
\text{lou is smart} & \land \text{lou is a student} \implies \text{lou will pass}
\end{align*}
\]

\underline{\text{lou is smart and lou is a student}}

\underline{\text{lou will pass}}
Where propositions stop...

\[
\begin{align*}
\text{peter is smart} \land \text{peter is a student} & \rightarrow \text{peter will pass} \\
\text{ann is smart} \land \text{ann is a student} & \rightarrow \text{ann will pass} \\
\text{lou is smart} \land \text{lou is a student} & \rightarrow \text{lou will pass}
\end{align*}
\]

**All smart students pass**

\[
\begin{align*}
\text{ann is smart and ann is a student} \\
\hline
\text{ann will pass}
\end{align*}
\]
Where propositions stop...

- Peter is smart $\land$ Peter is a student $\rightarrow$ Peter will pass
- Ann is smart $\land$ Ann is a student $\rightarrow$ Ann will pass
- Lou is smart $\land$ Lou is a student $\rightarrow$ Lou will pass

**All smart students pass**

- Ann is smart and Ann is a student $\rightarrow$ Ann will pass

-smart(Ann) $\land$ student(Ann) $\rightarrow$ will_pass(Ann)
Where propositions stop...

peter is smart \land peter is a student \rightarrow peter will pass
ann is smart \land ann is a student \rightarrow ann will pass
lou is smart \land lou is a student \rightarrow lou will pass

**All smart students pass**

ann is smart and ann is a student

\[ \text{ann will pass} \]

\[ \forall X(\text{smart}(X) \land \text{student}(X) \rightarrow \text{will_pass}(X)) \]
Where propositions stop...

- Peter is smart and Peter is a student → Peter will pass
- Ann is smart and Ann is a student → Ann will pass
- Lou is smart and Lou is a student → Lou will pass

**All smart students pass**

Ann is smart and Ann is a student

---

Ann will pass

\[ \forall X (\text{smart}(X) \land \text{student}(X) \rightarrow \text{will_pass}(X)) \]

Smart(Ann) and student(Ann)

---

\[ \rightarrow \text{will_pass}(ann) \]
Beyond propositions: Individuals and Relations

\[
\text{in(kim, r123).} \\
\text{part of (r123, cs.building).} \\
\text{in(X, Y) } \leftarrow \text{ part of (Z, Y) } \land \text{ in(X, Z).}
\]
Symbols: \( M, I, U \)

Axiom: \( MI \)

Rules:

- if \( xI \) is a theorem, so is \( xIU \)
- \( Mx \) is a theorem, so is \( Mxx \)
- in any theorem, \( III \) can be replaced by \( U \)
- \( UU \) can be dropped from any string

Starting from \( MI \), can you generate \( MU \)? (use top-down or bottom-up)
Next:

- Planning under certainty (Poole & Mackworth 2nd ed. Chapter 6.1-6.4)
- Supervised Learning (Poole & Mackworth 2nd ed. Chapter 7.1-7.6)