Lecture 4 - Features and Constraints

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Readings: Poole & Mackworth (2nd Ed.) Chapt. 4.1-4.8 (skip 4.9)

Constraint Satisfaction Problems (CSPs)

- A set of variables
- A domain for each variable
- A set of constraints **or** evaluation function
- Two kinds:
  1. Satisfiability Problems: Find an assignment that satisfies constraints (hard constraints)
  2. Optimization Problems: Find an assignment that optimises the evaluation function (soft constraints)
- A solution to a CSP is an assignment to the variables that satisfies all constraints
- A solution is a model of the constraints.

CSPs as Graph searching problems

Two ways:
Complete Assignment:

- nodes: assignment of value to all variables
- neighbors: change one variable value

Partial Assignment:

- nodes: assignment to first $k-1$ variables
- neighbors: assignment to k^{th} variable

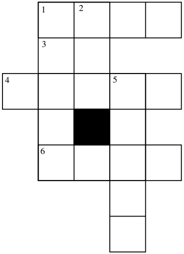
But,

- these search spaces can get extremely large (thousands of variables), so the branching factors can be big!
- path to goal is not important, only the goal is
- no predefined starting nodes

Classic CSP: Crossword Construction

Fill in all horizontal and vertical slots with words or phrases

Classic CSP: Crossword Construction



at,eta,be,hat,he,her,it,him

on,one,desk,dance,usage,easy,dove

first,else,loses,fuels,help,haste,

given,kind,sense,soon,sound,this,think

Dual Representations

Two ways to represent the crossword as a CSP

- Primal representation:
  - nodes represent word positions: 1-down. . . 6-across
  - domains are the words
  - constraints specify that the letters on the intersections must be the same.
- Dual representation:
  - nodes represent the individual squares
  - domains are the letters
  - constraints specify that the words must fit

Real World Example Domains

- Transportation Planning (Pascal Van Hentenryck)
  `https://www.youtube.com/watch?v=SxvM0jG3qLA`
- Ride-sharing scheduling
  `https://arxiv.org/pdf/2111.03204`
- Air Traffic Control
  `http://dx.doi.org/10.1017/S0269888912000215`
- Disaster Recovery
- Factory process management
- Scheduling (courses, meetings, etc)
- ...

Posing a CSP

Variables: $V_1, V_2, \ldots, V_n$
Domains: Each variable, V_i has a domain D_{V_i}
Constraints: restrictions on the values a set of variables can jointly have.
e.g.

problem	variables	domains	constraints
crosswords	letters	a-z	words in dictionary
crosswords	words	dictionary	letters match
scheduling	times events resources	times,dates types values	before, after same resource
Chess	pieces	board positions	occupied checks
party planning	guests	values	cliques
ride sharing	people/trips	locations	cars

Constraints

Constraints:

- Can be N-ary (over sets of N variables - e.g. "dual representation" for crossword puzzles with letters as domains)
- Here: Consider only Unary and Binary (e.g. "primal representation" for crossword puzzles with words as domains)

Solutions:

- Generate and test
- Backtracking
- Consistency
- Hill-Climbing
- Randomized incl. Local Search

Example

Delivery robot: activities *a,b,c,d,e*, times *1,2,3,4*.
A: variable representing the time activity a will occur
B: variable representing the time activity b will occur
etc..
Domains:
$D_A = \{1, 2, 3, 4\}$
$D_B = \{1, 2, 3, 4\}$
....
constraints :
$(B \neq 3) \wedge (C \neq 2) \wedge (A \neq B) \wedge (B \neq C) \wedge$
$(C < D) \wedge (A = D) \wedge (E < A) \wedge (E < B) \wedge (E < C)$
$\wedge (E < D) \wedge (B \neq D)$

Generate and Test

Exaustively go through all combinations, check each one
$D = D_A \times D_B \times D_C \times D_D \times D_E$
$D = \{< 1, 1, 1, 1, 1 >, < 1, 1, 1, 1, 2 >, \ldots, < 4, 4, 4, 4, 4 >\}$

test: $< 1, 1, 1, 1, 1 > \ldots$ fail $\neg(A \neq B)$
test: $< 1, 1, 1, 1, 2 > \ldots$ fail $\neg(A \neq B)$
test: $< 1, 1, 1, 1, 3 > \ldots$ fail $\neg(A \neq B)$
...
...
test: $< 1, 2, 1, 1, 1 > \ldots$ fail $\neg(C < D)$
test: $< 1, 2, 1, 1, 2 > \ldots$ fail $\neg(C < D)$
...
but ... we knew all along that $A \neq B$

Backtracking

Can use the fact that large portions of the state space can be pruned.

1. Order all variables
2. Evaluate constraints into the order as soon as they are grounded

e.g. Assignment $A = 1 \wedge B = 1$ is inconsistent with constraint $A \neq B$ regardless of the value of the other variables.

Backtracking - Example

test: $< 1, \_, \_, \_, \_ >$ ... ok
test: $< 1, 1, \_, \_, \_ >$ ... fail $\neg(A \neq B)$
test: $< 1, 2, 1, \_, \_ >$ ... ok
test: $< 1, 2, 1, 1, \_ >$ ... fail $\neg(C < D)$
test: $< 1, 2, 1, 2, \_ >$ ... fail $\neg(A = D)$
test: $< 1, 2, 1, 3, \_ >$ ... fail $\neg(A = D)$
test: $< 1, 2, 1, 4, \_ >$ ... fail $\neg(A = D)$
backtrack
test: $< 1, 2, 2, \_, \_ >$ ... fail $\neg(C \neq 2)$
test: $< 1, 2, 3, \_, \_ >$ ... ok
test $< 1, 2, 3, 1, \_ >$ ... fail $\neg(C < D)$
...
...
test: $< 2, \_, \_, \_, \_ >$ ok
...
(draw the search tree using the partial assignment method)

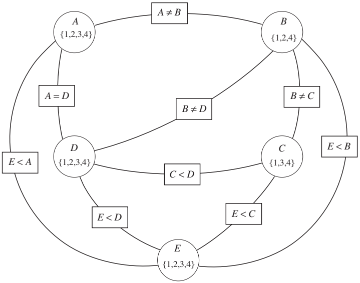
Backtracking

- Efficiency depends on order of variables!
- Finding optimal ordering is as hard as solving the problem
- idea: push failures as high as possible
- cut off large branches of the tree as soon as possible

Consistency

- More general approach
- look for inconsistencies.
- e.g. C=4 in example inconsistent with any value of D ($C < D$)
- backtracking will "re-discover" this for every value of A,B
- graphical representation

Constraint Satisfaction: Graphically

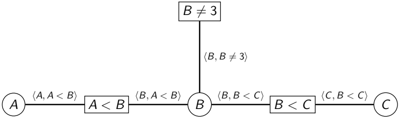


Goal: each domain has a single element, and all constraints are satisfied.

Consistency:

- Constraint Network (CN)
- domain constraint is unary constraint on values in a domain, written $\langle X, c(X)\rangle$.
- A node in a CN is domain consistent if no domain value violates any domain constraint.
- A CN is domain consistent if all nodes are domain consistent
- Arc $\langle X, c(X, Y)\rangle$ is a constraint on X.
- An arc $\langle X, c(X, Y)\rangle$ is arc consistent if for each $X \in \mathrm{D}_X$, there is some $Y \in \mathrm{D}_Y$ such that $c(X, Y)$ is satisfied.
- A CN is arc consistent if all arcs are arc consistent
- A set of variables $\{X_1, X_2, X_3, \ldots, X_N\}$ is path consistent if all arcs and domains are consistent.

Constraint Satisfaction: Graphically (formal)



AC-3

- Alan Mackworth 1977!
- Makes a CN arc consistent (and domain consistent)
- To-Do Arcs Queue (TDA) has all inconsistent arcs

1. Make all domains domain consistent
2. Put all arcs $\langle Z, c(Z, \_)\rangle$ in TDA
3. repeat
   a. Select and remove an arc $\langle X, c(X, Y)\rangle$ from TDA
   b. Remove all values of domain of X that don't have a value in domain of Y that satisfies the constraint $c(X, Y)$
   c. If any were removed, Add all arcs $\langle Z, c'(Z, X)\rangle$ to TDA $\forall Z \neq Y$

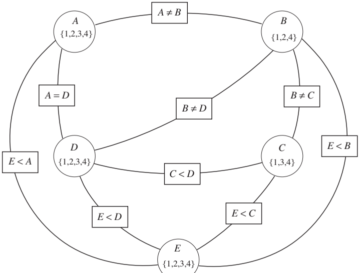
until TDA is empty

When AC-3 Terminates

AC-3 always terminates with one of these three conditions:

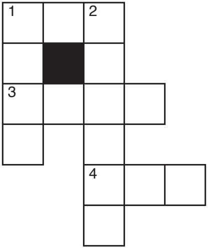
- Every domain is empty: there is no solution
- Every domain has a single value: solution!
- Some domain has more than one value: split it in two, run AC-3 recursively on two halves. Don't have to start from scratch - only have to put back all arcs $\langle Z, c'(Z, X)\rangle$ if X was the domain that was split.
- Connection between domain splitting and search.

Constraint Satisfaction: Example



Goal: each domain has a single element, and all constraints are satisfied.

Example: Crossword Puzzle



Words:

ant, big, bus, car, has
book, buys, hold,
lane, year
beast, ginger, search,
symbol, syntax

Variable Elimination

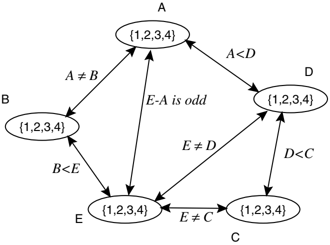
- Idea: eliminate the variables one-by-one passing their constraints to their neighbours
- When there is a single variable remaining, if it has no values, the network was inconsistent.
- The variables are eliminated according to some elimination ordering
- Different elimination orderings result in different size intermediate constraints.

Variable elimination (cont.)

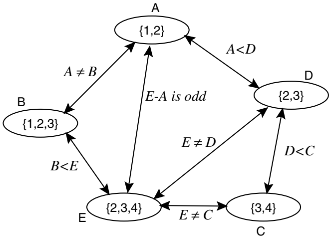
Variable Elimination Algorithm:

- If there is only one variable, return the intersection of the (unary) constraints that contain it
- Select a variable X
  - Join the constraints in which X appears, forming constraint R
  - Project R onto its variables other than X: call this R_2
  - Place new constraint R_2 between all variables that were connected to X
  - Remove X
  - Recursively solve the simplified problem
  - Return R joined with the recursive solution

Example network



Example: arc-consistent network



Example: eliminating C

$r_1 : C \neq E$	C	E
	3	2
	3	4
	4	2
	4	3

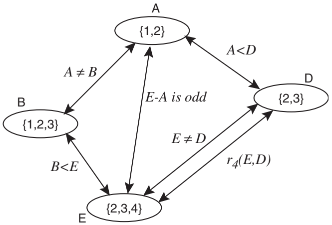
$r_2 : C > D$	C	D
	3	2
	4	2
	4	3

$r_3 : r_1 \bowtie r2$	C	D	E
	3	2	2
	3	2	4
	4	2	2
	4	2	3
	4	3	2
	4	3	3

$r_4 : \pi_{\{D,E\}} r_3$	D	E
	2	2
	2	3
	2	4
	3	2
	3	3

$\hookrightarrow$ new constraint

Resulting network after eliminating C



Local Search

Back to CSP as Search (Local Search):

- Maintain an assignment of a value to each variable.
- At each step, select a neighbor of the current assignment (e.g., one that improves some heuristic value).
- Stop when a satisfying assignment is found, or return the best assignment found.

Requires:

- What is a neighbor?
- Which neighbor should be selected?

(Some methods maintain multiple assignments.)

Local Search for CSPs

- Aim is to find an assignment with zero unsatisfied constraints.
- Given an assignment of a value to each variable, a conflict is an unsatisfied constraint.
- The goal is an assignment with zero conflicts.
- Heuristic function to be minimized: the number of conflicts.

Greedy Descent Variants

- Find the variable-value pair that minimizes the number of conflicts at every step.
- Select a variable that participates in the most number of conflicts. Select a value that minimizes the number of conflicts.
- Select a variable that appears in any conflict. Select a value that minimizes the number of conflicts.
- Select a variable at random. Select a value that minimizes the number of conflicts.
- Select a variable and value at random; accept this change if it doesn't increase the number of conflicts.

GSAT (Greedy SATisfyability)

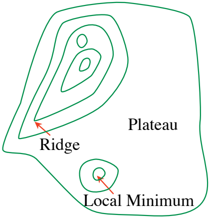
Let n be random assignment of values to all variables
$h(n)$ is number of un-satisfied constraints

repeat
  evaluate neighbors, n' of n.
  can't change the same variable twice in a row
  $n = n^*$, where $n^* = \arg\min_{n'}(h(n'))$
  (even if $h(n^*) > h(n)$!)
Until stopping criteria is reached

e.g. start with $A = 2, B = 2, C = 3, D = 2, E = 1$ .... h= 3
change B to 4 ... h = 1
local minimum
change D to 4 (h=2)
change A to 4 (h=2)
change B to 2 (h=0)

Problems with Greedy Descent

- a local minimum that is not a global minimum
- a plateau where the heuristic values are uninformative
- a ridge is a local minimum where *n*-step look-ahead might help



Randomized Greedy Descent

As well as downward steps we can allow for:

- Random steps: move to a random neighbor.
- Random restart: reassign random values to all variables.

Which is more expensive computationally?
A mix of the two = stochastic local search

1-Dimensional Ordered Examples

Two 1-dimensional search spaces; step right or left:

- Which method would most easily find the global minimum?
- What happens in hundreds or thousands of dimensions?
- What if different parts of the search space have different structure?

High Dimensional Search Spaces

- In high dimensions the search space is less easy to visualize
- Often consists of long, nearly flat "canyons"
- Hard to optimize using local search
- Step-size can be adjusted

Stochastic Local Search

Stochastic local search is a mix of:

- Greedy descent: move to a lowest neighbor
- Random walk: taking some random steps
- Random restart: reassigning values to all variables

Variant: Simulated Annealing

- Pick a variable at random and a new value at random.
- If it is an improvement, adopt it.
- If it isn't an improvement, adopt it probabilistically depending on a temperature parameter, T.
  - With current assignment n and proposed assignment n' we move to n' with probability $e^{-(h(n')-h(n))/T}$
- Temperature can be reduced.

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  - With current assignment n and proposed assignment n' we move to n' with probability $e^{-(h(n')-h(n))/T}$
- Temperature can be reduced.

Probability of accepting a change:

Temperature	1-worse	2-worse	3-worse
10	0.91	0.81	0.74
1	0.37	0.14	0.05
0.25	0.02	0.0003	0.000005
0.1	0.00005	0	0

Simulated Annealing

```
Let n be random assignment of values to all variables
Let T be a (high) temperature
repeat
    Select neighbor n' of n at random
    If h(n') < h(n) then
       n = n'
    else
       n = n' with probability e^{-(h(n')-h(n))/T}
    reduce T
until stopping criteria is reached
```

Tabu lists

- recall GSAT: never choose same variable twice.
- To prevent cycling we can maintain a tabu list of the k last assignments.
- Don't allow an assignment that is already on the tabu list.
- If $k = 1$, we don't allow an assignment of to the same value to the variable chosen.
- We can implement it more efficiently than as a list of complete assignments.
- It can be expensive if k is large.

Parallel Search

A total assignment is called an individual.

- maintain a population of k individuals instead of one.
- At every stage, update each individual in the population.
- Whenever an individual is a solution, it can be reported.
- Like k restarts, but uses k times the minimum number of steps.

Beam Search

- Like parallel search, with k individuals, but choose the k best out of all of the neighbors (all if there are less than k).
- When $k = 1$, it is greedy descent.
- The value of k lets us limit space and parallelism.

Stochastic Beam Search

- Like beam search, but it probabilistically chooses the k individuals at the next generation.
- The probability that a neighbor is chosen is proportional to its heuristic value : $e^{-h(n)/T}$.
- This maintains diversity amongst the individuals.
- The heuristic value reflects the fitness of the individual.
- Like asexual reproduction: each individual mutates and the fittest ones survive.

Genetic Algorithms

- Like stochastic beam search, but pairs of individuals are combined to create the offspring:
- For each generation:
  - Randomly choose pairs of individuals where the fittest individuals are more likely to be chosen.
  - For each pair, perform a cross-over: form two offspring each taking different parts of their parents:
  - Mutate some values.
- Stop when a solution is found.

Crossover

- Given two individuals:

$$X_1 = a_1, X_2 = a_2, \ldots, X_m = a_m$$

$$X_1 = b_1, X_2 = b_2, \ldots, X_m = b_m$$

- Select i at random.
- Form two offspring:

$$X_1 = a_1, \ldots, X_i = a_i, X_{i+1} = b_{i+1}, \ldots, X_m = b_m$$

$$X_1 = b_1, \ldots, X_i = b_i, X_{i+1} = a_{i+1}, \ldots, X_m = a_m$$

- The effectiveness depends on the ordering of the variables.
- Many variations are possible.

Comparing Stochastic Algorithms

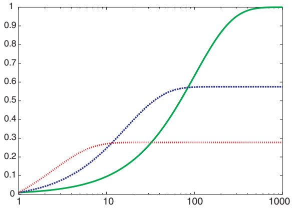
- How can you compare three algorithms when
  - one solves the problem 30% of the time very quickly but doesn't halt for the other 70% of the cases
  - one solves 60% of the cases reasonably quickly but doesn't solve the rest
  - one solves the problem in 100% of the cases, but slowly?

Comparing Stochastic Algorithms

- How can you compare three algorithms when
  - one solves the problem 30% of the time very quickly but doesn't halt for the other 70% of the cases
  - one solves 60% of the cases reasonably quickly but doesn't solve the rest
  - one solves the problem in 100% of the cases, but slowly?
- Summary statistics, such as mean run time, median run time, and mode run time don't make much sense.

Runtime Distribution

- Plots runtime (or number of steps) and the proportion (or number) of the runs that are solved within that runtime.



Next:

- Inference (Poole & Mackworth (2nd Ed.) chapter 5.1-5.3 and 13.1-13.2)