Lecture 3 - States and Searching

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Readings: Poole & Mackworth Chapt. 3 (all)
Searching

Often we are not given an algorithm to solve a problem, but only a specification of what is a solution — we have to search for a solution.

A typical problem is when the agent is in one state, it has a set of deterministic actions it can carry out, and wants to get to a goal state.

Many AI problems can be abstracted into the problem of finding a path in a directed graph.

Often there is more than one way to represent a problem as a graph.
A graph consists of a set $N$ of nodes and a set $A$ of ordered pairs of nodes, called arcs.

Node $n_2$ is a neighbor of $n_1$ if there is an arc from $n_1$ to $n_2$. That is, if $\langle n_1, n_2 \rangle \in A$.

A path is a sequence of nodes $\langle n_0, n_1, \ldots, n_k \rangle$ such that $\langle n_{i-1}, n_i \rangle \in A$.

Given a set of start nodes and goal nodes, a solution is a path from a start node to a goal node.

Often there is a cost associated with arcs and the cost of a path is the sum of the costs of the arcs in the path.
The robot wants to get from outside room 103 to the inside of room 123.
cost = distance travelled
Partial Search Space for a Video Game

Grid game: collect coins $C_1$, $C_2$, $C_3$, $C_4$, don’t run out of fuel, and end up at location $(1, 1)$:
Problem space search

Partial Search Space for a Video Game
Grid game: collect coins $C_1$, $C_2$, $C_3$, $C_4$, don’t run out of fuel, and end up at location $(1, 1)$:

State:
\[\langle X\text{-pos}, Y\text{-pos}, \text{Fuel}, C_1, C_2, C_3, C_4 \rangle\]

Goal:
\[\langle 1, 1, ?, t, t, t, t \rangle\]
Generic search algorithm: given a graph, start nodes, and goal nodes, **incrementally explore paths** from the start nodes.

Maintain a **frontier** of paths from the start node that have been explored.

As search proceeds, the frontier **expands** into the unexplored nodes until a goal node is encountered.

The way in which the frontier is expanded defines the **search strategy**.
Problem Solving by Graph Searching

- start node
- explored nodes
- unexplored nodes
- ends of paths on frontier
Graph Search Algorithm

**Input:** a graph,
a set of start nodes,
Boolean procedure $goal(n)$ that tests if $n$ is a goal node.

$frontier := \{\langle s \rangle : s \text{ is a start node}\};$

**while** $frontier$ is not empty:

**select** and **remove** path $\langle n_0, \ldots, n_k \rangle$ from $frontier$;

**if** $goal(n_k)$

**return** $\langle n_0, \ldots, n_k \rangle$;

**for every** neighbor $n$ of $n_k$

**add** $\langle n_0, \ldots, n_k, n \rangle$ to $frontier$;

**end while**
We assume that after the search algorithm returns an answer, it can be asked for more answers and the procedure continues.

The neighbors define the graph.

Which value is selected from the frontier (and how the new values are added to the frontier) at each stage defines the search strategy.

goal defines what is a solution.
Types of Search

- Uninformed (blind)
- Heuristic
- More sophisticated “hacks”
Depth-first Search

- **Depth-first search** treats the frontier as a **stack**.
- It always selects the **last element** added to the frontier.
- If the list of paths on the frontier is \([p_1, p_2, \ldots]\)
  - \(p_1\) is selected. Paths that extend \(p_1\) are added to the front of the stack (in front of \(p_2\)).
  - \(p_2\) is only selected when all paths from \(p_1\) have been explored.
Illustrative Graph — Depth-first Search
Depth-first search **not guaranteed to halt** on infinite graphs or on graphs with cycles.

The **space** complexity is **linear** in the size of the path being explored.

Search is **unconstrained** by the goal until it happens to stumble on the goal (uninformed or blind).

What is the worst-case time complexity of depth-first search?
A searcher can prune a path that ends in a node already on the path.

Using depth-first methods, with the graph explicitly stored, this can be done in constant time (add a flag to each node).

For other methods, the cost is linear in path length, since we only have to check for cycles in the current path.
Graph Search Algorithm - with Cycle Check

- Use **Depth First Search** to get from $s$ to $g$
- Number the nodes as they are removed
- add neighbours CCW from top L,D,R,U
- Use a cycle check

**Input**: a graph,
a set of start nodes,
Boolean procedure $\text{goal}(n)$ that tests if $n$ is a goal node.

$\text{frontier} := \{\langle s \rangle : s \text{ is a start node}\}$;

while $\text{frontier}$ is not empty:
    select and remove path $\langle n_0, \ldots, n_k \rangle$ from $\text{frontier}$;
    if $\text{goal}(n_k)$
        return $\langle n_0, \ldots, n_k \rangle$;
    for every neighbor $n$ of $n_k$
        if $n \notin \langle n_0, \ldots, n_k \rangle$
            add $\langle n_0, \ldots, n_k, n \rangle$ to $\text{frontier}$;
end while
Breadth-first search treats the frontier as a queue. It always selects the earliest element added to the frontier. If the list of paths on the frontier is \([p_1, p_2, \ldots, p_r]\):

- \(p_1\) is selected. Its neighbors are added to the end of the queue, after \(p_r\).
- \(p_2\) is selected next.
Illustrative Graph — Breadth-first Search
Complexity of Breadth-first Search

- The **branching factor** of a node is the number of its neighbors.
- If the branching factor for all nodes is finite, breadth-first search is **guaranteed to find a solution if one exists**. It is guaranteed to find the path with **fewest arcs**.
- **Time** complexity is **exponential** in the path length: $b^n$, where $b$ is the **branching factor**, $n$ is path length.
- **Space** complexity is **exponential** in path length: $b^n$.
- Search is **unconstrained** by the goal.
- Not affected by cycles (remains exponential).
Multiple-path pruning:

- Prune a path to node \( n \) that any path has been found to.
- Multiple-path pruning subsumes a cycle check (because the current path is a path to the node).
- This entails storing all nodes it has found paths to.
- Want to guarantee that an optimal solution can still be found. (See slide 40)
Graph Search Algorithm - with Multiple Path Pruning

- Use **Breadth First Search** to get from \( s \) to \( g \)
- Number the nodes as they are removed
- add neighbours CW from top U,R,D,L
- Use multiple path pruning

**Input:** a graph, 
a set of start nodes, 
Boolean procedure \( \text{goal}(n) \) that tests if \( n \) is a goal node.

\[
\text{frontier} := \{ \langle s \rangle : s \text{ is a start node} \};
\]

\[
\text{has}_\text{path} := \{ \};
\]

**while** \( \text{frontier} \) is not empty:

- **select** and **remove** path \( \langle n_0, \ldots, n_k \rangle \) from \( \text{frontier} \);

  **if** \( n_k \not\in \text{has}_\text{path} \):

  - **add** \( n_k \) to \( \text{has}_\text{path} \);

  **if** \( \text{goal}(n_k) \)

  - **return** \( \langle n_0, \ldots, n_k \rangle \);

  **for every** neighbor \( n \) of \( n_k \)

  - **add** \( \langle n_0, \ldots, n_k, n \rangle \) to \( \text{frontier} \);

**end while**
Lowest-cost-first Search

- Sometimes there are costs associated with arcs. The cost of a path is the sum of the costs of its arcs.

\[
\text{cost}(\langle n_0, \ldots, n_k \rangle) = \sum_{i=1}^{k} |\langle n_{i-1}, n_i \rangle|
\]

- At each stage, lowest-cost-first search selects a path on the frontier with lowest cost.
- The frontier is a priority queue ordered by path cost.
- It finds a least-cost path to a goal node.
- When arc costs are equal \(\Rightarrow\) breadth-first search.
- Uniformed/Blind search (in that it does not take the goal into account)
- Complexity: exponential
Heuristic Search

- **Idea:** don’t ignore the goal when selecting paths.
- Often there is extra knowledge that can be used to guide the search: **heuristics.**
- $h(n)$ is an **estimate** of the cost of the **shortest path** from node $n$ to a goal node.
- $h(n)$ uses only **readily obtainable** information (that is easy to compute) about a node.
- computing the heuristic must be **much easier** than solving the problem
- $h$ can be **extended** to paths: $h(\langle n_0, \ldots, n_k \rangle) = h(n_k)$.
- $h(n)$ is an **underestimate** if there is no path from $n$ to a goal that has path length less than $h(n)$.
Example Heuristic Functions

- If the nodes are points on a Euclidean plane and the cost is the distance, we can use the **straight-line distance** from \( n \) to the closest goal as the value of \( h(n) \).
- If the nodes are locations and cost is **time**, we can use the distance to a goal divided by the maximum speed.
- If nodes are locations on a grid and cost is distance, we can use the **Manhattan Distance**: distance by taking horizontal and vertical moves only.
- Think of heuristics for your favorite games: chess? go? starcraft?
Greedy Best-first Search

- **Idea:** select the path whose end is closest to a goal according to the heuristic function.
- **Best-first search** selects a path on the frontier with minimal $h$-value.
- It treats the frontier as a priority queue ordered by $h$. 
Illustrative Example — Best First Search

best first: S-A-C-G (not optimal)
Graph Search Algorithm - with Multiple Path Pruning

- Use **Best First Search** to get from \( s \) to \( g \)
- Number the nodes as they are removed
- Use multiple path pruning
- break ties arbitrarily
- Use **Manhattan Distance** as heuristic

**Input:** a graph + start nodes, 
Boolean procedure \( \text{goal}(n) \) that tests if \( n \) is a goal node.

\[
\text{frontier} := \{ \langle s \rangle : s \text{ is a start node} \};
\]

\[
\text{has\_path} := \{ \};
\]

**while** \( \text{frontier} \) is not empty:

**select** and **remove** path \( \langle n_0, \ldots, n_k \rangle \) from \( \text{frontier} \);

**if** \( n_k \notin \text{has\_path} \):

**add** \( n_k \) to \( \text{has\_path} \);

**if** \( \text{goal}(n_k) \)

**return** \( \langle n_0, \ldots, n_k \rangle \);

**for every** neighbor \( n \) of \( n_k \)

**add** \( \langle n_0, \ldots, n_k, n \rangle \) to \( \text{frontier} \);

**end while**
Heuristic Depth-first Search

- **Idea:** Do a depth-first search, but add paths to the stack ordered according to $h$.
- **Locally** does a best-first search, but aggressively pursues the best looking path (even if it ends up being worse than one higher up).
- Suffers from the same problems as depth-first search.
- Is often used in practice.
cost of an arc is its length
heuristic: euclidean distance
red nodes all look better than green nodes
a challenge for heuristic depth first search
Graph Search Algorithm - with Multiple Path Pruning

- Use **Heuristic Depth-First Search**
- Number the nodes as they are removed
- Use multiple path pruning
- break ties arbitrarily
- Use **Manhattan Distance** as heuristic

**Input**: a graph + start nodes

Boolean procedure $goal(n)$ that tests if $n$ is a goal node.

$frontier := \{\langle s \rangle : s \text{ is a start node}\}$;

$has\_path := \{\}$;

while $frontier$ is not empty:

- **select** and **remove** path $\langle n_0, \ldots, n_k \rangle$ from $frontier$;
  - if $n_k \notin has\_path$:
    - add $n_k$ to $has\_path$;
  - if $goal(n_k)$
    - return $\langle n_0, \ldots, n_k \rangle$;
  - for every neighbor $n$ of $n_k$
    - add $\langle n_0, \ldots, n_k, n \rangle$ to $frontier$;

end while
$A^*$ Search

- $A^*$ search uses both path **cost and heuristic** values.
- $\text{cost}(p)$ is the cost of path $p$.
- $h(p)$ estimates the cost from the end of $p$ to a goal.
- Let $f(p) = \text{cost}(p) + h(p)$. $f(p)$ estimates the **total path cost** of going from a start node to a goal via $p$.
A* Search Algorithm

- **A*** is a **mix** of **lowest-cost-first** and **best-first search**.
- It treats the frontier as a **priority queue ordered by** \( f(p) \).
- It always selects the node on the frontier with the **lowest estimated distance** from the start to a goal node constrained to go via that node.
Illustrative Example — $A^*$ sxearch

recall best first: S-A-C-G (not optimal)
$A^*$ : S-A-B-C-G (optimal)
If there is a solution, $A^*$ always finds an **optimal** solution —the first path to a goal selected— if

- the branching factor is **finite**
- arc costs are **bounded above zero** (there is some $\epsilon > 0$ such that all of the arc costs are greater than $\epsilon$), and
- $h(n)$ is a **lower bound** on the length (cost) of the shortest path from $n$ to a goal node.

Admissible heuristics never overestimate the cost to the goal.
Why is $A^*$ with admissible $h$ optimal?

- assume: $paths \rightarrow p \rightarrow g$ is the optimal
- $f(p) = \text{cost}(s, p) + h(p) < \text{cost}(s, g)$ due to $h$ being a lower bound
- $\text{cost}(s, g) < \text{cost}(s, p') + \text{cost}(p', g)$ due to optimality of $path$
- therefore $\text{cost}(s, p) + h(p) = f(p) < \text{cost}(s, p') + \text{cost}(p', g)$
- therefore, we will never choose $path'$ while $path$ is unexplored.
- $A^*$ halts, as the costs of the paths on the frontier keeps increasing, and will eventually exceed any finite number.
Graph Search Algorithm - with Multiple Path Pruning

- Use **A* search**
- Number the nodes as they are removed
- Use multiple path pruning
- break ties arbitrarily
- Use **Manhattan Distance** as heuristic

**Input:** a graph and set of start nodes, Boolean procedure \( \text{goal}(n) \) that tests if \( n \) is a goal node.

\[
\text{frontier} := \{ \langle s \rangle : \text{s is a start node} \};
\]

\[
\text{has\_path} := \{ \};
\]

**while** frontier is not empty:

- select and **remove** path \( \langle n_0, \ldots, n_k \rangle \) from frontier;

  **if** \( n_k \notin \text{has\_path} \):

  - **add** \( n_k \) to \( \text{has\_path} \);

  **if** \( \text{goal}(n_k) \)

  - **return** \( \langle n_0, \ldots, n_k \rangle \);

  **for every** neighbor \( n \) of \( n_k \)

  - **add** \( \langle n_0, \ldots, n_k, n \rangle \) to frontier;

**end while**
How do we construct a heuristic?

“magic square” tiles can move into adjacent empty slot only

Relax the game (make it simpler, easier)

1. Can move tile from position A to position B if A is next to B (ignore whether or not position is blank)
2. Can move tile from position A to position B if B is blank (ignore adjacency)
3. Can move tile from position A to position B
How do we construct a heuristic?

“magic square” tiles can move into adjacent empty slot only

```
Start State
7 2 4
5 1 6
8 3 1
```

```
Goal State
1 2
3 4 5
6 7 8
```

Relax the game (make it simpler, easier)

1. Can move tile from position A to position B if A is next to B (ignore whether or not position is blank)

- leads to **manhattan distance heuristic**
- To solve the puzzle need to slide each tile into its final position
- **Admissible**
How do we construct a heuristic?

“magic square” tiles can move into adjacent empty slot only

Relax the game (make it simpler, easier)

3. Can move tile from position A to position B

- leads to misplaced tile heuristic
- To solve this problem need to move each tile into its final position
- Number of moves = number of misplaced tiles
- Admissible
## Summary of Search Strategies

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Frontier Selection</th>
<th>Halts?</th>
<th>Space</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth-first</td>
<td>Last node added</td>
<td>No</td>
<td>Linear</td>
<td>Exp</td>
</tr>
<tr>
<td>Breadth-first</td>
<td>First node added</td>
<td>Yes</td>
<td>Exp</td>
<td>Exp</td>
</tr>
<tr>
<td>Heuristic depth-first</td>
<td>Local(^1) min (h(n))</td>
<td>No</td>
<td>Linear</td>
<td>Exp</td>
</tr>
<tr>
<td>Best-first</td>
<td>Global(^2) min (h(n))</td>
<td>No</td>
<td>Exp</td>
<td>Exp</td>
</tr>
<tr>
<td>Lowest-cost-first</td>
<td>Minimal (cost(n))</td>
<td>Yes</td>
<td>Exp</td>
<td>Exp</td>
</tr>
<tr>
<td>A(^*)</td>
<td>Minimal (f(n))</td>
<td>Yes</td>
<td>Exp</td>
<td>Exp</td>
</tr>
</tbody>
</table>

\(^1\)Locally in some region of the frontier

\(^2\)Globally for all nodes on the frontier
Multiple-Path Pruning & Optimal Solutions

**Problem:** what if a subsequent path to $n$ is shorter than the first path to $n$?

- **remove** all paths from the frontier that use the longer path.
- **change** the initial segment of the paths on the frontier to use the shorter path.
- **ensure this doesn’t happen**. Make sure that the shortest path to a node is found first (lowest-cost-first search)
Multiple-Path Pruning & $A^*$

- Suppose path $p$ to $n$ was selected, but there is a shorter path to $n$. Suppose this shorter path is via path $p'$ on the frontier.
- Suppose path $p'$ ends at node $n'$.
- $\text{cost}(p) + h(n) \leq \text{cost}(p') + h(n')$ because $p$ was selected before $p'$.
- $\text{cost}(p') + \text{cost}(n', n) < \text{cost}(p)$ because the path to $n$ via $p'$ is shorter (by assumption).

$$\text{cost}(n', n) < \text{cost}(p) - \text{cost}(p') \leq h(n') - h(n).$$

You can ensure this doesn’t occur if $h(n') - h(n) \leq \text{cost}(n', n)$. 
Monotone Restriction

- Heuristic function $h$ satisfies the **monotone restriction** if $h(m) - h(n) \leq \text{cost}(m, n)$ for every arc $\langle m, n \rangle$.

- $h(m) - h(n)$ is the heuristic estimate of the path cost from $m$ to $n$.

- The heuristic estimate of the path cost is **always less than** the actual cost.

- If $h$ satisfies the monotone restriction, $A^*$ with multiple path pruning **always finds the shortest path** to a goal.
Monotonicity and Admissibility

- This is a strengthening of the admissibility criterion.
- if $n = g$ so $h(n) = 0$ and $cost(n', n) = cost(n')$, then we can derive from
  \[
  h(n') \leq cost(n', n) + h(n)
  \]
  that
  \[
  h(n') \leq cost(n')
  \]
  which is **admissibility**
- So Monotonicity is like Admissibility but between any two nodes
Iterative Deepening

- So far all search strategies that are guaranteed to halt use \textbf{exponential space}.
- \textbf{Idea:} let’s \textbf{recompute} elements of the frontier rather than saving them.
- Look for paths of depth 0, then 1, then 2, then 3, etc.
- You need a \textbf{depth-bounded depth-first} searcher.
- If a path cannot be found at depth $B$, look for a path at depth $B + 1$. Increase the depth-bound when the search fails \textbf{unnaturally} (depth-bound was reached).
Iterative Deepening Complexity

Complexity with solution at depth $k$ & branching factor $b$:

<table>
<thead>
<tr>
<th>level</th>
<th>breadth-first</th>
<th>iterative deepening</th>
<th># nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$k$</td>
<td>$b$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$k - 1$</td>
<td>$b^2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$k - 1$</td>
<td>1</td>
<td>2</td>
<td>$b^{k-1}$</td>
</tr>
<tr>
<td>$k$</td>
<td>1</td>
<td>1</td>
<td>$b^k$</td>
</tr>
</tbody>
</table>

$\geq b^k \leq b^k \left( \frac{b}{b-1} \right)^2$

\[ b^k + 2b^{k-1} + 3b^{k-2} + \ldots = b^k \sum_{n=1}^{k} n \left( \frac{1}{b} \right)^{n-1} \] rewrite

\[ < b^k \sum_{n=1}^{\infty} n \left( \frac{1}{b} \right)^{n-1} \] extend to infinity

\[ = b^k \left( \frac{b}{1-b} \right)^2 \] derivative of the geometric series
The definition of searching is \textbf{symmetric}: find path from start nodes to goal node or from goal node to start nodes.

\textbf{Forward branching factor:} number of arcs out of a node.

\textbf{Backward branching factor:} number of arcs into a node.

Search complexity is $b^n$. Should use forward search if forward branching factor is less than backward branching factor, and vice versa.

Note: sometimes when graph is dynamically constructed, you may not be able to construct the backwards graph.
You can search backward from the goal and forward from the start **simultaneously**.

This wins as $2b^{k/2} \ll b^k$. This **can result** in an exponential saving in time and space.

The main problem is making sure the **frontiers meet**.

This is often used with one **breadth-first** method that builds a set of locations that can lead to the goal. In the other direction another method can be used to find a path to these interesting locations.
Island Driven Search

- **Idea:** find a set of islands between \( s \) and \( g \).

\[ s \rightarrow i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_{m-1} \rightarrow g \]

There are \( m \) smaller problems rather than 1 big problem.

- This can win as \( mb^{k/m} \ll b^k \).
- The problem is to identify the islands that the path must pass through. It is difficult to guarantee optimality.
- You can solve the subproblems using islands \( \Rightarrow \) hierarchy of abstractions.
Dynamic Programming

- Start from goal and work backwards
- Compute the cost-to-goal at each node recursively
- e.g. Dijkstra’s algorithm
- Cost from $n \rightarrow$ goal is
  Cost from $m \rightarrow$ goal + cost from $n$ to $m$
- $dist(n)$ is cost-to-goal from node $n$, and $cost(n, m)$ is cost to go from $n$ to $m$
  
  $$
  dist(n) = \begin{cases} 
  0 & \text{if } n \text{ is goal} \\
  \min_m (cost(n, m) + dist(m)) & \text{otherwise}
  \end{cases}
  $$

- $dist(n)$ is a value function over nodes
- $policy(n)$ is best $m$ for each $n$, so best path is
  
  $$
  path(n, goal) = \arg \min_m (cost(n, m) + dist(m))
  $$

- problem: space needed to store entire graph
Discounted Dynamic Programming

- assume goal has a reward, \( R(n) \)
- arcs still have costs
- Rewards far in the future are less valuable
- “a bird in the hand is worth two in the bush”
- Discount factor \( \beta < 1 \)
- maximize rewards

\[
\text{dist}(n) = \begin{cases} 
R(n) & \text{if } n \text{ is goal} \\
\max_m (\beta \text{dist}(m) - \text{cost}(n, m)) & \text{otherwise}
\end{cases}
\]
Minimax Search

- for competitive, two-person, zero-sum games (tic-tac-toe)
- try to find the best option for you ("O")
- assume competitor ("X") will take the worst option for you
- label each node here with expected reward (-1, 0 or +1)

TURN from top of tree

X

O

X

O

+1 −1 −1 −1 0 −1 0 −1 0 0

FROM TOP OF TREE

IGNORE

TURN from top of tree

X

O

X

O

+1 −1 −1 −1 0 −1 0 −1 0 0

FROM TOP OF TREE

IGNORE

TURN from top of tree

X

O

X

O

+1 −1 −1 −1 0 −1 0 −1 0 0

FROM TOP OF TREE

IGNORE

TURN from top of tree

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X

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+1 −1 −1 −1 0 −1 0 −1 0 0

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FROM TOP OF TREE

IGNORE

TURN from top of tree

X

O

X

O

+1 −1 −1 −1 0 −1 0 −1 0 0

FROM TOP OF TREE

IGNORE

TURN from top of tree
Next:

- Constraints (Poole & Mackworth chapter 4)