Lecture 10 - Planning under Uncertainty (II)

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Readings: Poole & Mackworth (2nd ed.)Chapter 9.5

Agents as Processes

Agents carry out actions:

- forever infinite horizon
- until some stopping criteria is met indefinite horizon
- finite and fixed number of steps finite horizon

Decision-theoretic Planning

What should an agent do when

- it gets rewards (and punishments) and tries to maximize its rewards received
- actions can be noisy; the outcome of an action can't be fully predicted
- there is a model that specifies the probabilistic outcome of actions
- the world is fully observable: the current state is always fully in evidence

for the various planning horizons?

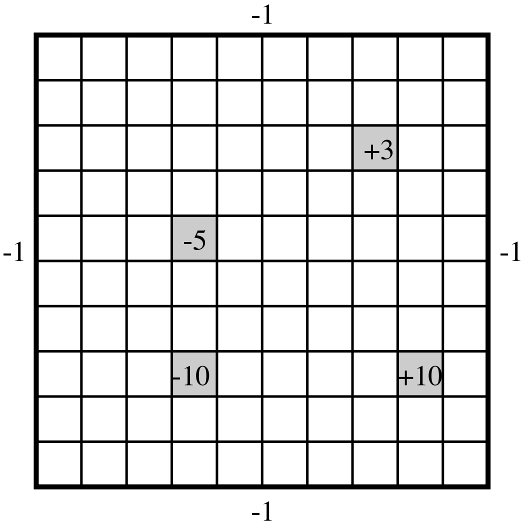
World State

- The world state is the information such that if you knew the world state, no information about the past is relevant to the future. Markovian assumption.
- Let S_i, A_i be the state,action at time i

$$P(S_{t+1}|S_0, A_0, \ldots, S_t, A_t) = P(S_{t+1}|S_t, A_t)$$

  $P(s'|s,a)$ is the probability that the agent will be in state s' immediately after doing action a in state s.
- The dynamics is stationary if the distribution is the same for each time point.

Example: Simple Grid World



Grid World Model

- Actions: up, down, left, right.
- 100 states corresponding to the positions of the robot.
- Robot goes in the commanded direction with probability 0.7, and one of the other directions with probability 0.1.
- If it crashes into an outside wall, it remains in its current position and has a reward of -1.
- Four special rewarding states; the agent gets the reward when leaving the state.

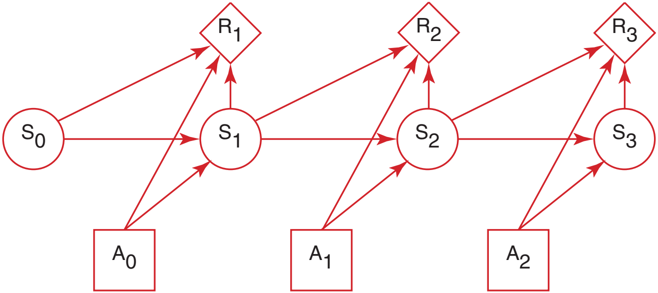
Planning Horizons

The planning horizon is how far ahead the planner looks to make a decision.

- The robot gets flung to one of the corners at random after leaving a positive ($+10$ or $+3$) reward state.
  - the process never halts
  - infinite horizon
- The robot gets $+10$ or $+3$ entering the state, then it stays there getting no reward. These are absorbing states.
  - The robot will eventually reach the absorbing state.
  - indefinite horizon

Decision Processes

- A Markov decision process augments a Markov chain with actions and values (information arcs not shown).



Markov Decision Processes

For an MDP you specify:

- set S of states.
- set A of actions.
- $P(S_{t+1}|S_t, A_t)$ specifies the dynamics.
- $R(S_t, A_t, S_{t+1})$ specifies the reward. The agent gets a reward at each time step (rather than just a final reward). $R(s, a, s')$ is the expected reward received when the agent is in state s, does action a and ends up in state s'.

Information Availability

What information is available when the agent decides what to do?

- fully-observable MDP the agent gets to observe S_t when deciding on action A_t.
- partially-observable MDP (POMDP) the agent has some noisy sensor of the state. It needs to remember its sensing and acting history. It can do this by maintaining a sufficiently complex belief state.

Rewards and Values

Suppose the agent receives the sequence of rewards $r_1, r_2, r_3, r_4, \ldots$. What value should be assigned?

- total reward $V = \sum_{i=1}^{\infty} r_i$
- average reward $V = \lim_{n \to \infty} (r_1 + \cdots + r_n)/n$
- discounted reward $V = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$
  γ is the discount factor $0 \le \gamma \le 1$.

Policies

- A stationary policy is a function:

  $$\pi : S \to A$$

  Given a state s, $\pi(s)$ specifies what action the agent who is following π will do.
- An optimal policy is one with maximum expected discounted reward.
- For a fully-observable MDP with stationary dynamics and rewards with infinite or indefinite horizon, there is always an optimal stationary policy.

Value of a Policy

- $Q^{\pi}(s, a)$, where a is an action and s is a state, is the expected value of doing a in state s, then following policy π.
- $V^{\pi}(s)$, where s is a state, is the expected value of following policy π in state s.
- Q^{π} and V^{π} can be defined mutually recursively:

$$
\begin{aligned}
Q^{\pi}(s, a) &= \sum_{s'} P(s'|a, s)\left(r(s, a, s') + \gamma V^{\pi}(s')\right) \\
V^{\pi}(s) &= Q^{\pi}(s, \pi(s))
\end{aligned}
$$

Value of the Optimal Policy

- $Q^*(s, a)$, where a is an action and s is a state, is the expected value of doing a in state s, then following the optimal policy.
- $\pi^*(s)$ is the optimal action to take in state s
- $V^*(s)$, where s is a state, is the expected value of following the optimal policy in state s.
- Q^* and V^* can be defined mutually recursively:

$$
\begin{aligned}
Q^*(s, a) &= \sum_{s'} P(s'|a, s) \left(r(s, a, s') + \gamma V^*(s') \right) \\
V^*(s) &= \max_a Q^*(s, a) \\
\pi^*(s) &= argmax_a Q^*(s, a)
\end{aligned}
$$

Value Iteration

- The *t-step lookahead value function*, V^t is the expected value with t steps to go
- Idea: Given an estimate of the t-step lookahead value function, determine the $t+1$-step lookahead value function.

Value Iteration

- Set V^0 arbitrarily, $t = 1$
- Compute Q^t, V^t from V^{t-1}.

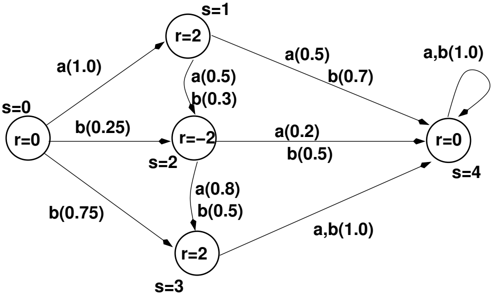
$$Q^t(s,a) = \left[R(s) + \gamma \sum_{s'} Pr(s'|s,a) V^{t-1}(s') \right]$$

$$V^t(s) = \max_a Q^t(s,a)$$

- The policy with t stages to go is simply the actions that maximizes this
  $\pi^t(s) = \arg\max_a \left[R(s) + \gamma \sum_{s'} Pr(s'|s,a) V^{t-1}(s') \right]$
- This is dynamic programming
- This converges exponentially fast (in t) to the optimal value function.
- Convergence when $||V^t(s) - V^{t-1}(s)|| < \epsilon \frac{(1-\gamma)}{\gamma}$ ensures V^t is within ϵ of optimal ($||X|| = max\{|x|, x \in X\}$)

Value Iteration: Simple Example

State space graph (NOT a DBN):



Value Iteration: Simple Example

This same graph, represented as a decision network, would have the following factors, where the $(row, col) = (i, j)$ entry in each probability table is $P(S' = j|S = i, A)$

$$P(S'|S, A = a) = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.8 & 0.2 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$$P(S'|S, A = b) = \begin{bmatrix} 0.0 & 0.0 & 0.25 & 0.75 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.0 & 0.7 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \quad R(S) = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

Value Iteration: Simple Example

first iteration, using $\gamma = 0.9$

$$
\begin{aligned}
V^0(s') &= R(s') \\
Q^1(s,a) &= R(s) + \gamma \sum_{s'} P(s'|s,a) V^0(s') \\
&= \begin{bmatrix} 1.8 & 1.1 & -0.56 & 2.0 & 0 \\ 0.9 & 1.46 & -1.1 & 2.0 & 0 \end{bmatrix} \\
V^1(s) &= max_a(Q^1(s,a)) \\
&= \begin{bmatrix} 1.8 & 1.46 & -0.56 & 2.0 & 0 \end{bmatrix} \\
\pi^1(s) &= \begin{bmatrix} a & b & a & a & a \end{bmatrix}
\end{aligned}
$$

Value Iteration: Simple Example

second iteration

$$Q^2(s,a) = R(s) + \gamma \sum_{s'} P(s'|s,a) V^1(s')$$

$$= \begin{bmatrix} 1.31 & 1.75 & -0.56 & 2.0 & 0 \\ 1.22 & 1.85 & -1.1 & 2.0 & 0 \end{bmatrix}$$

$$V^2(s) = max_a(Q^2(s,a))$$

$$= \begin{bmatrix} 1.31 & 1.84 & -0.56 & 2.0 & 0 \end{bmatrix}$$

$$\pi^2(s) = \begin{bmatrix} a & b & a & a & a \end{bmatrix}$$

on convergence, optimal value function is

$$V^*(s) = \begin{bmatrix} 1.66 & 1.85 & -0.56 & 2.0 & 0 \end{bmatrix}$$

policy is

$$\pi^*(s) = \begin{bmatrix} a & b & a & a & a \end{bmatrix}$$

Asynchronous Value Iteration

- You don't need to sweep through all the states, but can update the value function for each state individually.
- This converges to the optimal value function, if each state and action is visited infinitely often in the limit.
- You can either store $V[s]$ or $Q[s, a]$.

Asynchronous VI: storing $V[s]$

- Repeat forever:
  - Select state s;
  - $V[s] \leftarrow \max_a \sum_{s'} P(s'|s,a)\,(R(s,a,s') + \gamma V[s']);$

Asynchronous VI: storing $Q[s, a]$

- Repeat forever:
  - Select state s, action a;
  - $Q[s, a] \leftarrow \sum_{s'} P(s'|s, a) \left(R(s, a, s') + \gamma \max_{a'} Q[s', a'] \right);$

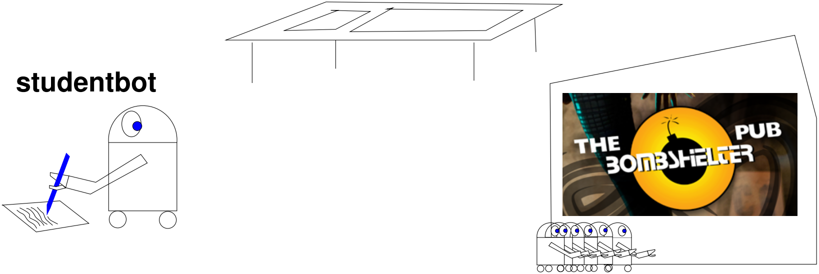
Markov Decision Processes: Factored State

- Represent $S = \{X_1, X_2, \ldots, X_n\}$
- X_i are random variables
- for each X_i, and each action $a \in A$, we have $P(X_i'|S, A)$
- Reward $R(X_1, X_2, \ldots, X_N)$ may be additive:

$$R(X_1, X_2, \ldots, X_N) = \sum_i R(X_i)$$

- Value iteration proceeds as usual but can do one variable at a time (e.g. variable elimination)

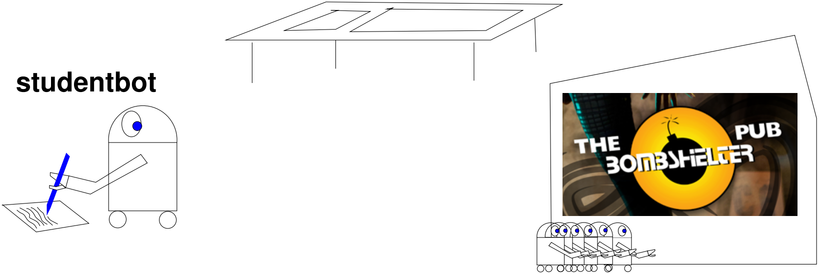
Example: studentbot



state variables (3x2x4x2=48 states):

- tired : studentbot is tired (no/a bit/very)
- passtest : studentbot passes test (no/yes)
- knows : studentbot's state of knowledge (nothing/a bit/a lot/everything)
- goodtime : studentbot has a good time (no/yes)

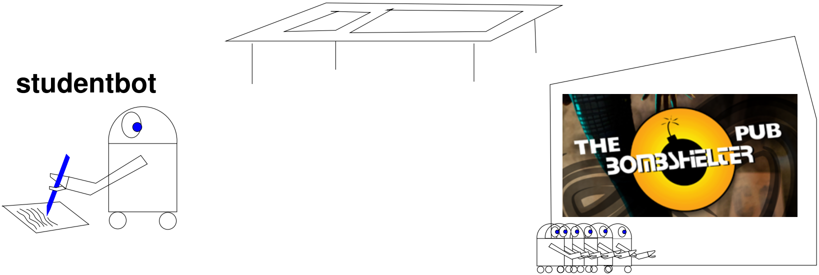
Example: studentbot



studentbot actions:

- study : studentbot's knowledge increases, studentbot gets tired
- sleep : studentbot gets less tired
- party : studentbot has a good time if he's not tired, but gets tired and loses knowledge
- take test : studentbot takes a test (can take test anytime)

Example: studentbot



studentbot rewards:

- +20 if studentbot passes the test
- +2 if studentbot has a good time

basic tradeoff: short term vs. long-term rewards

Studentbot

State-based:

$$P(s'|s, a) = [48 \times 48]$$

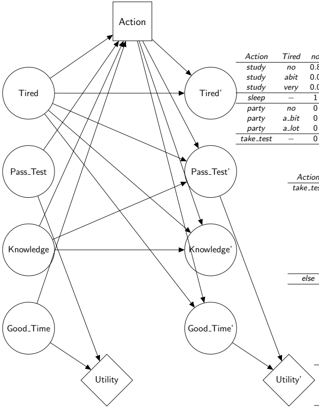
$$R(s) = [48 \times 1]$$

Studentbot

As a dynamic decision network:

			Knowledge′			
Action	*Tired*	*Knowledge*	*nothing*	*a_bit*	*a_lot*	*everything*
study	*no*	*nothing*	0.5	0.5	0	0
	no	*a_bit*	0	0.5	0.5	0
	no	*a_lot*	0.0	0.0	0.5	0.5
	no	*everything*	0		0	1.0
	a_bit	*nothing*	0.5	0.5	0	0
	a_bit	*a_bit*	0	0.5	0.5	0.0
	a_bit	*a_lot*	0	0	0.5	0.5
	a_bit	*everything*	0	0	0	1.0
	very	*nothing*	1.0	0	0	0
	very	*a_bit*	0	1.0	0.0	0.0
	very	*a_lot*	0	0	1.0	0.0
	very	*everything*	0	0	0	1.0
party	–	*nothing*	1.0	0.0	0	0
	no	*a_bit*	0.5	0.5	0.0	0.0
	no	*a_lot*	0.0	0.5	0.5	0.0
	no	*everything*	0	0	0.5	0.5
sleep	–	*nothing*	1.0	0.0	0	0
	–	*a_bit*	0.0	1.0	0.0	0.0
	–	*a_lot*	0.0	0.0	1.0	0.0
	–	*everything*	0	0.0	0.0	1.0
take_test	–	–	1.0	0.0	0	0

		Good_Time′	
Action	*Tired*	*no*	*yes*
party	*no*	0.0	1.0
party	*yes*	1.0	0.0
other	–	1.0	0.0

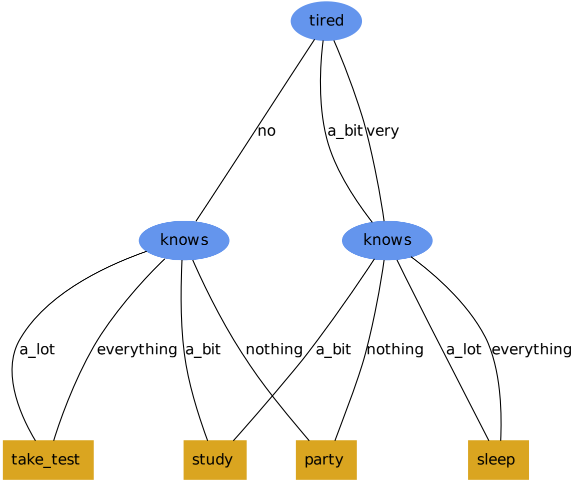


		Tired′		
Action	*Tired*	*no*	*a_bit*	*very*
study	*no*	0.8	0.2	0.0
study	*abit*	0.0	0.8	0.2
study	*very*	0.0	0.0	1.0
sleep	–	1	0	0
party	*no*	0	1	0
party	*a_bit*	0	0	1
party	*a_lot*	0	0	1
take_test	–	0	0	1

			PassTest′	
Action	*Tired*	*Knowledge*	*no*	*yes*
take_test	*no*	*nothing*	1	0
	no	*a_bit*	0.9	0.1
	no	*a_lot*	0.7	0.3
	no	*everything*	0.1	0.9
	a_bit	*nothing*	1	0
	a_bit	*a_bit*	0.9	0.1
	a_bit	*a_lot*	0.7	0.3
	a_bit	*everything*	0.1	0.9
	very	–	1	0
else	–	–	1	0

Good_Time	*Pass_Test*	*U(Good_Time, Pass_Test)*
yes	*yes*	22
yes	*no*	2
no	*yes*	20
no	*no*	0

Studentbot Policy



Partially Observable Markov Decision Processes (POMDPs)

A POMDP is like an MDP, but

some variables are not observed. It is

a tuple $\langle S, A, T, R, O, \Omega \rangle$

S: finite set of unobservable states

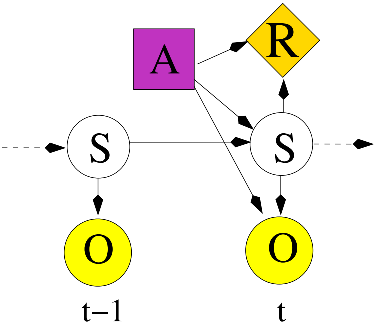
A: finite set of agent actions

$T : S \times A \rightarrow S$ transition function

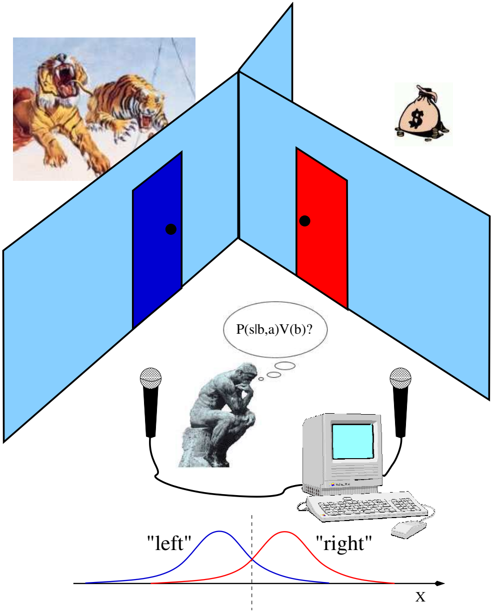
$R : S \times A \rightarrow \mathcal{R}$ reward function

O: set of observations

$\Omega : S \times A \rightarrow O$ observation function



e.g. 1-D Tiger problem



The material after this is optional

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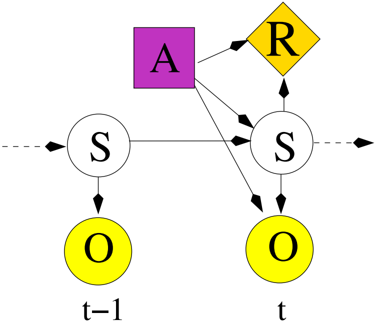
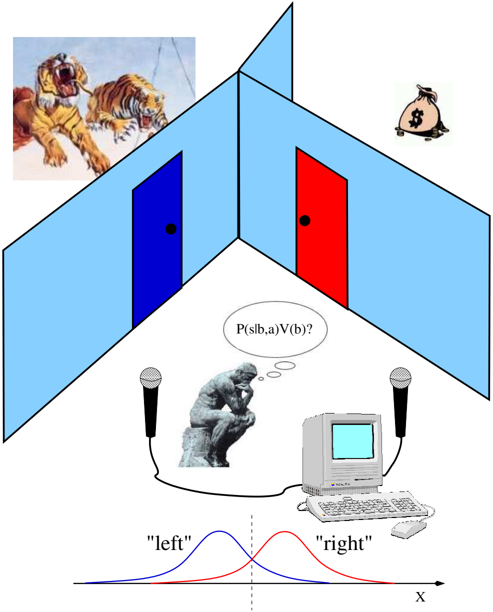
$\Omega : S \times A \rightarrow O$ observation function



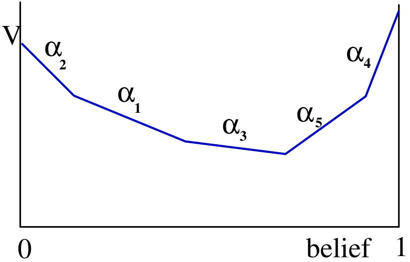
e.g. 1-D Tiger problem

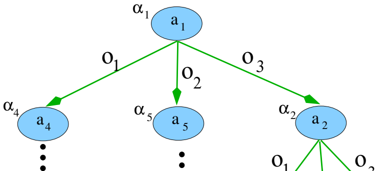


Value Functions and Conditional Plans

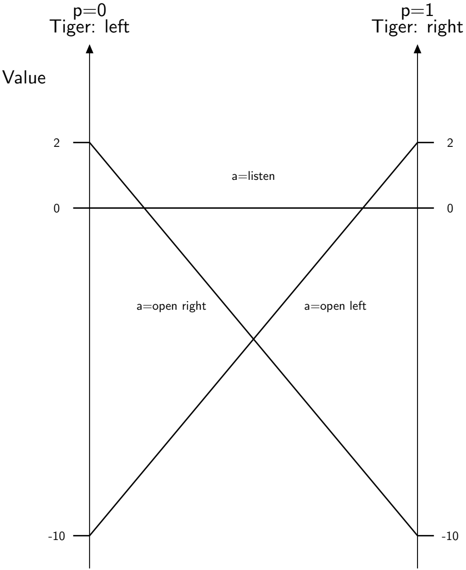
$$V^{k+1}(b) = \max_a R^a(b) + \gamma \sum_o Pr(o|b,a) V^k(b_o^a)$$

$V(b)$ can be represented with a piecewise linear function over the belief space - pieces are called α vectors

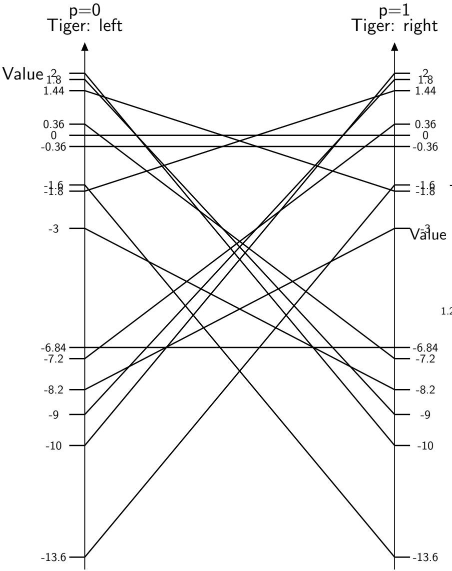




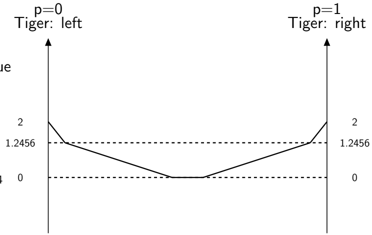
e.g. Tiger problem, after zero iterations



e.g. Tiger problem, after one iteration



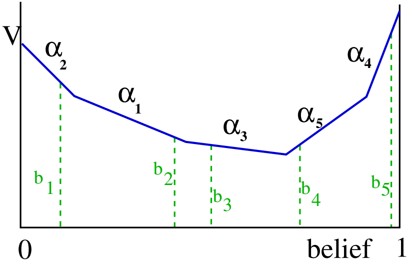
all generated α vectors



optimal value function

Point-based Value Iteration

1. Generate belief samples to make belief set belief set $\mathcal{B}$



2. compute forward-propagated belief states

$$b_o^a(s') = \sum_{s \in S} T(s'|a,s)\Omega(o|s',a)b(s) \quad \forall b \in \mathcal{B}$$

Point-Based Value Iteration II

1. start with one alpha vector: $\alpha_0 = R(s, a)$
2. repeat until converged:
   2.1 for each belief sample, b:

$$\Gamma_b^a = R(s,a) + \sum_{s' \in \mathcal{S}} \sum_{o \in \mathcal{O}} T(s'|a,s)\Omega(o|s',a) \arg\max_{\alpha_j} \alpha_j(s') \cdot b_o^a(s') \ \forall \ a \in \mathcal{A}, b \in \mathcal{B}$$

   2.2 Maximize over actions at each b:

$$\alpha^{\dagger} = \bigcup_{b \in \mathcal{B}} \{\arg\max_{\Gamma_b^a} (\Gamma_b^a \cdot b_j)\}$$

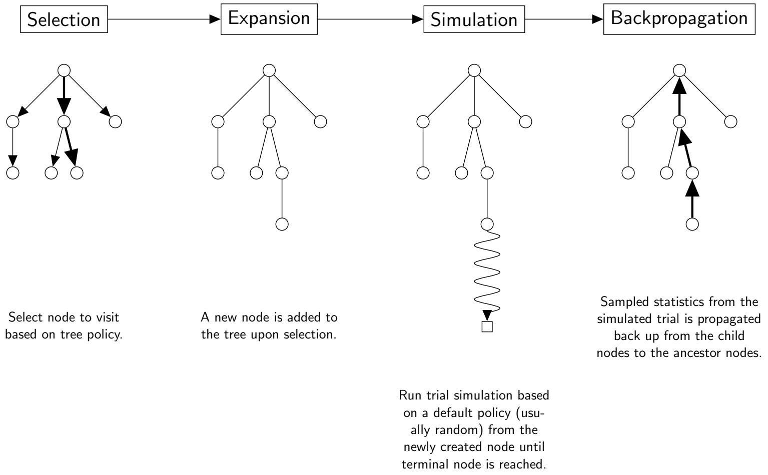
Policies

Policy: maps beliefs states into actions $\pi(b(s)) \rightarrow a$

Two ways to compute a policy

1. Backwards search
   - Dynamic programming (Variable Elimination)
   - in MDP:
     $Q_t(s,a) = R(s,a) + \gamma \sum_{s'} Pr(s'|s,a) \max_{a'} Q_{t-1}(s',a')$
   - in POMDP: $Q_t(b(s),a)$
   - Point-based backups make this efficient
2. Forwards search : Monte Carlo Tree Search (MCTS)
   - Expand the search tree
   - Expand more deeply in promising directions
   - Ensure exploration using e.g. UCB

MCTS


Selection
Expansion
Simulation
Backpropagation
Select node to visit based on tree policy.
A new node is added to the tree upon selection.
Run trial simulation based on a default policy (usually random) from the newly created node until terminal node is reached.
Sampled statistics from the simulated trial is propagated back up from the child nodes to the ancestor nodes.

Forward Monte-Carlo Search for POMDPs

procedure GetValue($b(s)$)

for each action-observation pair a, o:

$b_o^a(s') \leftarrow$ propagate the full belief state forwards
for each action and observation (using stochastic simulation)

if $b_o^a(s')$ not at a leaf:

evaluate recursively by further growing the tree:

$V_o^a \leftarrow GetValue(b_o^a(s'))$

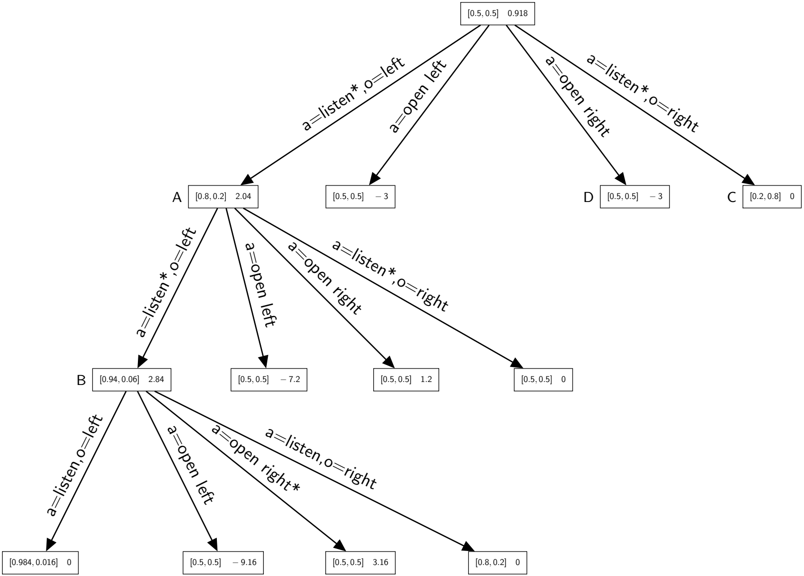
else:

create a new leaf for a, o

do a series of single-belief point rollouts
(e.g. propagate a single belief forward stochastically
gathering reward until termination condition is met),
use the total returned value as V_o^a.

return $R(b(s)) + \max_a \{\gamma \sum_o P(o|b(s), a) \sum_{s'} V_o^a b_o^a(s'))\}$

e.g. Tiger problem, two steps expanded



Next:

- Reinforcement Learning Poole & Mackworth (2nd ed.)Chapter 12.1,12.3-12.9
- Deep Reinforcement Learning