Agents as Processes

Agents carry out actions:
- forever **infinite horizon**
- until some stopping criteria is met **indefinite horizon**
- finite and fixed number of steps **finite horizon**
Decision-theoretic Planning

What should an agent do when

- it gets rewards (and punishments) and tries to maximize its rewards received
- actions can be noisy; the outcome of an action can’t be fully predicted
- there is a model that specifies the probabilistic outcome of actions
- the world is fully observable for the various planning horizons?
The world state is the information such that if you knew the world state, no information about the past is relevant to the future. **Markovian assumption**.

Let $S_i, A_i$ be the state, action at time $i$

$$P(S_{t+1}|S_0, A_0, \ldots, S_t, A_t) = P(S_{t+1}|S_t, A_t)$$

$P(s'|s, a)$ is the probability that the agent will be in state $s'$ immediately after doing action $a$ in state $s$.

The dynamics is **stationary** if the distribution is the same for each time point.
Example: Simple Grid World

[Grid diagram with numbers +10, -10, -5, -1, -1, +3, -10, +10, -5]
Grid World Model

- Actions: up, down, left, right.
- 100 states corresponding to the positions of the robot.
- Robot goes in the commanded direction with probability 0.7, and one of the other directions with probability 0.1.
- If it crashes into an outside wall, it remains in its current position and has a reward of $-1$.
- Four special rewarding states; the agent gets the reward when leaving.
Planning Horizons

The planning horizon is how far ahead the planner looks to make a decision.

- The robot gets flung to one of the corners at random after leaving a positive (+10 or +3) reward state.
  - the process never halts
  - **infinite horizon**

- The robot gets +10 or +3 entering the state, then it stays there getting no reward. These are **absorbing states**.
  - The robot will eventually reach the absorbing state.
  - **indefinite horizon**
A **Markov decision process** augments a Markov chain with actions and values (information arcs not shown).
Markov Decision Processes

For an MDP you specify:

- set $S$ of states.
- set $A$ of actions.
- $P(S_{t+1}|S_t, A_t)$ specifies the dynamics.
- $R(S_t, A_t, S_{t+1})$ specifies the reward. The agent gets a reward at each time step (rather than just a final reward). $R(s, a, s')$ is the expected reward received when the agent is in state $s$, does action $a$ and ends up in state $s'$. 
Information Availability

What information is available when the agent decides what to do?

- **fully-observable MDP** the agent gets to observe $S_t$ when deciding on action $A_t$.

- **partially-observable MDP** (POMDP) the agent has some noisy sensor of the state. It needs to remember its sensing and acting history. It can do this by maintaining a sufficiently complex *belief state*. 
Suppose the agent receives the sequence of rewards $r_1, r_2, r_3, r_4, \ldots$. What value should be assigned?

- **Total reward**: $V = \sum_{i=1}^{\infty} r_i$
- **Average reward**: $V = \lim_{n \to \infty} \frac{(r_1 + \cdots + r_n)}{n}$
- **Discounted reward**: $V = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$

\(\gamma\) is the **discount factor** $0 \leq \gamma \leq 1$. 
A stationary policy is a function:

$$\pi : S \rightarrow A$$

Given a state $s$, $\pi(s)$ specifies what action the agent who is following $\pi$ will do.

An optimal policy is one with maximum expected discounted reward.

For a fully-observable MDP with stationary dynamics and rewards with infinite or indefinite horizon, there is always an optimal stationary policy.
Value of a Policy

- $Q^\pi(s, a)$, where $a$ is an action and $s$ is a state, is the expected value of doing $a$ in state $s$, then following policy $\pi$.
- $V^\pi(s)$, where $s$ is a state, is the expected value of following policy $\pi$ in state $s$.
- $Q^\pi$ and $V^\pi$ can be defined mutually recursively:
  \[
  Q^\pi(s, a) = \sum_{s'} P(s'|a, s) \left( r(s, a, s') + \gamma V^\pi(s') \right)
  \]
  \[
  V^\pi(s) = Q^\pi(s, \pi(s))
  \]
Value of the Optimal Policy

- \( Q^*(s, a) \), where \( a \) is an action and \( s \) is a state, is the expected value of doing \( a \) in state \( s \), then following the optimal policy.
- \( \pi^*(s) \) is the optimal action to take in state \( s \)
- \( V^*(s) \), where \( s \) is a state, is the expected value of following the optimal policy in state \( s \).
- \( Q^* \) and \( V^* \) can be defined mutually recursively:
  \[
  Q^*(s, a) = \sum_{s'} P(s'|a, s) \left( r(s, a, s') + \gamma V^*(s') \right)
  \]
  \[
  V^*(s) = \max_a Q^*(s, a)
  \]
  \[
  \pi^*(s) = \arg\max_a Q^*(s, a)
  \]
Value Iteration

- The $t$-step lookahead value function, $V^t$ is the expected value with $t$ steps to go.
- Idea: Given an estimate of the $t$-step lookahead value function, determine the $t+1$-step lookahead value function.
Value Iteration

- Set $V^0$ arbitrarily, $t = 1$
- Compute $Q^t$, $V^t$ from $V^{t-1}$.

$$Q^t(s, a) = \left [ R(s) + \gamma \sum_{s'} Pr(s'|s, a)V^{t-1}(s') \right ]$$

$$V^t(s) = \max_a Q^t(s, a)$$

- The policy with $t$ stages to go is simply the actions that maximizes this

$$\pi^t(s) = \arg \max_a \left [ R(s) + \gamma \sum_{s'} Pr(s'|s, a)V^{t-1}(s') \right ]$$

- This is dynamic programming
- This converges exponentially fast (in $t$) to the optimal value function.
- Convergence when $||V^t(s) - V^{t-1}(s)|| < \epsilon \frac{(1-\gamma)}{\gamma}$ ensures $V^t$ is within $\epsilon$ of optimal ($||X|| = \max\{|x|, x \in X\}$)
Value Iteration: Simple Example
This same graph, represented as a decision network, would have the following factors, where the \((row, col) = (i, j)\) entry in each probability table is \(P(S' = j|S = i, A)\)

\[
P(S'|S, A = a) = \begin{bmatrix}
0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.5 & 0.0 & 0.5 \\
0.0 & 0.0 & 0.0 & 0.8 & 0.2 \\
0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 1.0
\end{bmatrix}
\]

\[
P(S'|S, A = b) = \begin{bmatrix}
0.0 & 0.0 & 0.25 & 0.75 & 0.0 \\
0.0 & 0.0 & 0.3 & 0.0 & 0.7 \\
0.0 & 0.0 & 0.0 & 0.5 & 0.5 \\
0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 1.0
\end{bmatrix}
\]

\[
R(S) = \begin{bmatrix}
0 \\
2 \\
-2 \\
2 \\
0
\end{bmatrix}
\]
first iteration, using $\gamma = 0.9$

\[
V^0(s') = R(s')
\]
\[
Q^1(s, a) = R(s) + \gamma \sum_{s'} P(s'|s, a) V^0(s')
\]

\[
= \begin{bmatrix}
1.8 & 1.1 & -0.56 & 2.0 & 0 \\
0.9 & 1.46 & -1.1 & 2.0 & 0 
\end{bmatrix}
\]

\[
V^1(s) = \max_a(Q^1(s, a))
\]

\[
= \begin{bmatrix}
1.8 & 1.46 & -0.56 & 2.0 & 0 
\end{bmatrix}
\]

\[
\pi^1(s) = [a \ b \ a \ a \ a]
\]
Value Iteration: Simple Example

second iteration

\[ Q^2(s, a) = R(s) + \gamma \sum_{s'} P(s'|s, a)V^1(s') \]

\[
= \begin{bmatrix}
1.31 & 1.75 & -0.56 & 2.0 & 0 \\
1.22 & 1.85 & -1.1 & 2.0 & 0
\end{bmatrix}
\]

\[ V^2(s) = \max_a (Q^2(s, a)) \]

\[
= \begin{bmatrix}
1.31 & 1.84 & -0.56 & 2.0 & 0 \\
\end{bmatrix}
\]

\[ \pi^2(s) = [a \ b \ a \ a \ a] \]

on convergence, optimal value function is

\[ V^*(s) = [1.66 \ 1.85 \ -0.56 \ 2.0 \ 0] \]

policy is

\[ \pi^*(s) = [a \ b \ a \ a \ a] \]
Asynchronous Value Iteration

- You don’t need to sweep through all the states, but can update the value functions for each state individually.
- This converges to the optimal value functions, if each state and action is visited infinitely often in the limit.
- You can either store $V[s]$ or $Q[s, a]$. 
Asynchronous VI: storing $V[s]$

- Repeat forever:
  - Select state $s$;
  - $V[s] \leftarrow \max_a \sum_{s'} P(s'|s,a) (R(s,a,s') + \gamma V[s'])$;
Asynchronous VI: storing $Q[s, a]$

- Repeat forever:
  - Select state $s$, action $a$;
  - $Q[s, a] \leftarrow \sum_{s'} P(s'|s, a) \left( R(s, a, s') + \gamma \max_{a'} Q[s', a'] \right)$;
Markov Decision Processes: Factored State

- Represent \( S = \{X_1, X_2, \ldots, X_n\} \)
- for each \( X_i \), and each action \( a \in A \), we have \( P(X_i'|S, A) \)
- Reward \( R(X_1, X_2, \ldots, X_N) \) may be additive:
  \[
  R(X_1, X_2, \ldots, X_N) = \sum_i R(X_i)
  \]
- Value iteration proceeds as usual but can do one variable at a time (e.g. variable elimination)
Example: studentbot

state variables (3x2x4x2=48 states):

- **tired**: studentbot is tired (no/a bit/very)
- **passtest**: studentbot passes test (no/yes)
- **knows**: studentbot’s state of knowledge (nothing/a bit/a lot/everything)
- **goodtime**: studentbot has a good time (no/yes)
studentbot actions:

- **study**: studentbot’s knowledge increases, studentbot gets tired
- **sleep**: studentbot gets less tired
- **party**: studentbot has a good time if he’s not tired, but gets tired and loses knowledge
- **take test**: studentbot takes a test (can take test anytime)
Example: studentbot

studentbot

studentbot rewards:

- +20 if studentbot passes the test
- +2 if studentbot has a good time

basic tradeoff: short term vs. long-term rewards
State-based:

\[ P(s'|s, a) = [48 \times 48] \]

\[ R(s) = [48 \times 1] \]
As a dynamic decision network:
Studentbot Policy

tired

knows

nothing

study

take_test

tired

knows

very

a_bit

party

nothing

a_bit

take_test

a_lot

everything

a_bit

sleep

a_lot

everything
A POMDP is like an MDP, but some variables are not observed. It is a tuple $\langle S, A, T, R, O, \Omega \rangle$

- $S$: finite set of unobservable states
- $A$: finite set of agent actions
- $T: S \times A \rightarrow S$ transition function
- $R: S \times A \rightarrow \mathcal{R}$ reward function
- $O$: set of observations
- $\Omega: S \times A \rightarrow O$ observation function
e.g. 1-D Tiger problem

\[ P(s|b,a)V(b)? \]
Value Functions and Conditional Plans

\[ V^{k+1}(b) = \max_a R^a(b) + \gamma \sum_o Pr(o|b, a)V^k(b^a_o) \]

\( V(b) \) can be represented with a piecewise linear function over the belief space - pieces are called \( \alpha \) vectors.
e.g. Tiger problem, after zero iterations

- $p=0$: Tiger: left
  - Value: 0
  - $a=\text{listen}$
  - $a=\text{open right}$

- $p=1$: Tiger: right
  - Value: 0
  - $a=\text{listen}$
  - $a=\text{open left}$

Value range: -10 to 2
e.g. Tiger problem, after one iteration

all generated $\alpha$ vectors

optimal value function
1. Generate belief samples to make belief set belief set $\mathcal{B}$

2. compute forward-propagated belief states

$$b_o^a(s') = \sum_{s \in S} T(s'|a, s) \Omega(o|s', a)b(s) \quad \forall b \in \mathcal{B}$$
1. start with one alpha vector: $\alpha_0 = R(s, a)$

2. repeat until converged:
   2.1 for each belief sample, $b$:
   $$\Gamma^a_b = R(s, a) + \sum_{s' \in S} \sum_{o \in O} T(s'|a, s)\Omega(o|s', a) \arg\max_{\alpha_j} \alpha_j(s') \cdot b^a_o(s') \quad \forall \ a \in A, b \in B$$

   2.2 Maximize over actions at each $b$:
   $$\alpha^\dagger = \bigcup_{b \in B} \{ \arg\max \Gamma^a_b \cdot b_j \}$$
Policy: maps beliefs states into actions $\pi(b(s)) \rightarrow a$

Two ways to compute a policy

1. Backwards search
   - Dynamic programming (Variable Elimination)
   - in MDP:
     $$Q_t(s, a) = R(s, a) + \gamma \sum_{s'} Pr(s'|s, a) \max_{a'} Q_{t-1}(s', a')$$
   - in POMDP: $Q_t(b(s), a)$
   - Point-based backups make this efficient

2. Forwards search: Monte Carlo Tree Search (MCTS)
   - Expand the search tree
   - Expand more deeply in promising directions
   - Ensure exploration using e.g. UCB
MCTS

Selection → Expansion → Simulation → Backpropagation

Select node to visit based on tree policy.

A new node is added to the tree upon selection.

Run trial simulation based on a default policy (usually random) from the newly created node until terminal node is reached.

Sampled statistics from the simulated trial is propagated back up from the child nodes to the ancestor nodes.
procedure GetValue(b(s))

for each action-observation pair a, o:

$\tilde{b}_o^a(s') \leftarrow$ propagate the full belief state forwards

for each action and observation (using stochastic simulation)

if $\tilde{b}_o^a(s')$ not at a leaf:

evaluate recursively by further growing the tree:

$V_o^a \leftarrow$ GetValue($\tilde{b}_o^a(s')$)

else:

create a new leaf for a, o

do a series of single-belief point rollouts

(e.g. propagate a single belief forward stochastically gathering reward until termination condition is met),

use the total returned value as $V_o^a$.

return $R(b(s)) + \max_a \{ \gamma \sum_o P(o|b(s), a) \sum_{s'} V_o^a \tilde{b}_o^a(s') \}$
e.g. Tiger problem, two steps expanded
Partialy Observable Markov Decision Process:

**A: smile, yell, sympathise, do_nothing**

<table>
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<th>sad</th>
<th>angry</th>
<th>neutral</th>
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**E**:  

- **E**:  
- **E’**:  

**R(E)**:  

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**O**:  

- **O**:  
- **O’**:  

**E’**:  

- **E’**:  
- **E**:  

**O’**:  

- **O’**:  
- **O**:  

**A: do_nothing**
Partially Observable Markov Decision Process:

### Action Set ($A$)

- smile
- yell
- sympathise
- do nothing

### Observation Set ($O$)

- happy
- sad
- angry
- neutral

### State Transition ($R$)

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### Reward ($R(E)$)

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POMDPs for Emotional Interaction

Partially Observable Markov Decision Process:

A: smile, yell, sympathise, do_nothing

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Reinforcement Learning Poole & Mackworth (2nd ed.) Chapter 12.1, 12.3-12.9

Deep Reinforcement Learning