Formal Concepts, Galois Connections, and the Categorical Structure of Cognitive Explanation

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Outline

What is a theory?

What is a theory?
Name a psychological theory.

A celestial example

	size			distance from sun		moon	
	small	medium	large	near	far	yes	no
Mercury	×			×			×
Venus	×			×			×
Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
Saturn			×		×	×	
Uranus		×			×	×	
Neptune		×			×	×	
Pluto	×				×	×	

Thagard's Principles of Explanatory Coherence

1. Symmetry
2. Analogy
3. Contradiction
4. Competition

SACCADE

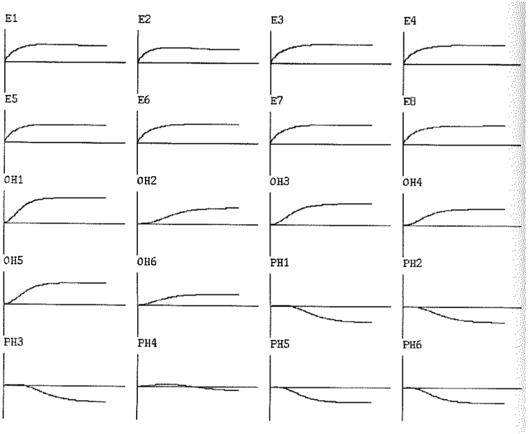
5. Acceptability
6. Data Priority
7. Explanation

Taking up all the oxygen in the room

Table I. *Input propositions for Lavoisier (1862) example*

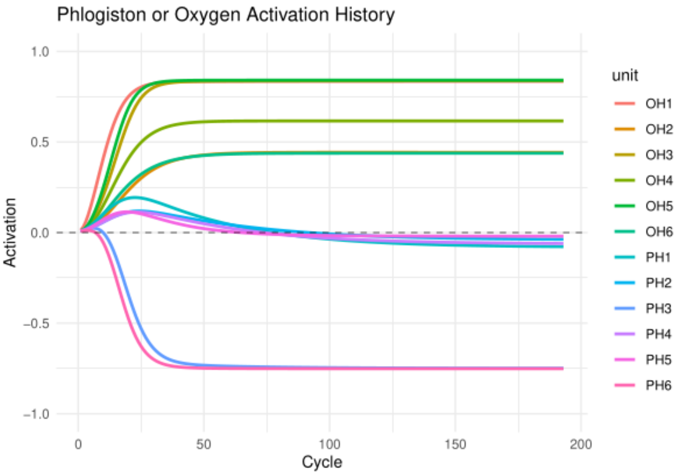
Evidence	
(proposition 'E1	"In combustion, heat and light are given off.")
(proposition 'E2	"Inflammability is transmittable from one body to another.")
(proposition 'E3	"Combustion only occurs in the presence of pure air.")
(proposition 'E4	"Increase in weight of a burned body is exactly equal to weight of air absorbed.")
(proposition 'E5	"Metals undergo calcination.")
(proposition 'E6	"In calcination, bodies increase weight.")
(proposition 'E7	"In calcination, volume of air diminishes.")
(proposition 'E8	"In reduction, effervescence appears.")
Oxygen hypotheses	
(proposition 'OH1	"Pure air contains oxygen principle.")
(proposition 'OH2	"Pure air contains matter of fire and heat.")
(proposition 'OH3	"In combustion, oxygen from the air combines with the burning body.")
(proposition 'OH4	"Oxygen has weight.")
(proposition 'OH5	"In calcination, metals add oxygen to become calxes.")
(proposition 'OH6	"In reduction, oxygen is given off.")
Phlogiston hypotheses	
(proposition 'PH1	"Combustible bodies contain phlogiston.")
(proposition 'PH2	"Combustible bodies contain matter of heat.")
(proposition 'PH3	"In combustion, phlogiston is given off.")
(proposition 'PH4	"Phlogiston can pass from one body to another.")
(proposition 'PH5	"Metals contain phlogiston.")
(proposition 'PH6	"In calcination, phlogiston is given off.")

And the network said ...



```
# ECHO Model in R
# Ported from Paul Thagard's MacLisp implementation
echo_env <- new.env()
echo_env$min_activation <- -1.0
echo_env$max_activation <- 1.0
create_unit <- function(name, activation = echo_env$default_activation) {
  unit <- list(
    name = name,
    activation = activation,
    new_activation = activation,
  )
weight_of_link_between <- function(unit1_name, unit2_name) {
  exp_unit <- get_unit(exp)
  exp_unit$explains <- unique(c(exp_unit$explains, explanandum))
}
# Run Lavoisier example
h2 <- example_lavoisier()
```

And the network said ... (ECHO, Echo, echo ...)



- A set of objects (we'll denote with G by convention).
- A set of attributes (we'll denote with M by convention).
- And a relation (R) between these two sets.
- It is often convenient to present this as a matrix (as we did for the planets).

Formal CONCEPTS

- Sets of objects that all share the same attributes.
- Sets of attributes that are all possesed by the same objects.
- Both with nothing left out.
- That is, for $A \subseteq G$ and $B \subseteq M$ if we define the following (’) operations thusly:

$$A' = \{m \in M \mid \forall g \in A \; gRm\} \tag{1}$$

$$B' = \{g \in G \mid \forall m \in B \; gRm\} \tag{2}$$

$$\text{then, } A'' = A\text{, defines a concept.} \tag{3}$$

Definition (Galois Connection)

$a \leq F(b) \iff G(a) \geq b$

Where $a \subseteq A$, and $b \subseteq B$ with $F : B \rightarrow A$, and $G : A \rightarrow B$ bijective and monotonic functions.

Note that this works for our concepts.

A celestial example

	size			distance from sun		moon	
	small	medium	large	near	far	yes	no
Mercury	×			×			×
Venus	×			×			×
Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
Saturn			×		×	×	
Uranus		×			×	×	
Neptune		×			×	×	
Pluto	×				×	×	

Is the combination of Earth and Mars a concept?

A celestial example

	size			distance from sun		moon	
	small	medium	large	near	far	yes	no
Mercury	×			×			×
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Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
Saturn			×		×	×	
Uranus		×			×	×	
Neptune		×			×	×	
Pluto	×				×	×	

Is the combination of Earth and Mars a concept?

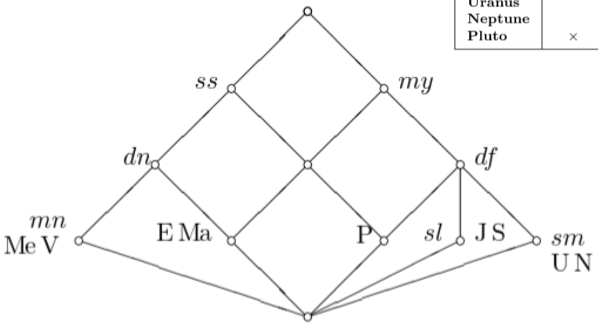
$$A = \{Earth,\ Mars\} \tag{4}$$

$$A' = \{small,\ near,\ moon\} \tag{5}$$

$$A'' = \{Earth,\ Mars\} \tag{6}$$

Jacob's Ladder

	size			distance from sun		moon	
	small	medium	large	near	far	yes	no
Mercury	×			×			×
Venus	×			×			×
Earth	×			×		×	
Mars	×			×		×	
Jupiter			×		×	×	
Saturn			×		×	×	
Uranus		×			×	×	
Neptune		×			×	×	
Pluto	×				×	×	



From ECHO to FCA: The Lavoisier Context

```
[1] "Lavoisier vs Phlogiston Explanatory Context:"
```

	OH1	OH2	OH3	OH4	OH5	OH6	PH1	PH2	PH3	PH4	PH5	PH6
E1	1	1	1	0	0	0	1	1	1	0	0	0
E2	0	0	0	0	0	0	1	0	1	1	0	0
E3	1	0	1	0	0	0	0	0	0	0	0	0
E4	1	0	1	1	0	0	0	0	0	0	0	0
E5	1	0	0	0	1	0	0	0	0	0	1	1
E6	1	0	0	1	1	0	0	0	0	0	0	0
E7	1	0	0	0	1	0	0	0	0	0	0	0
E8	1	0	0	0	0	1	0	0	0	0	0	0

Evidence E1-E8 as objects, hypotheses OH1-OH6 (Oxygen) and PH1-PH6 (Phlogiston) as attributes.

A value of 1 means the hypothesis explains the evidence.

Sample Formal Concepts

```
[[1]]
({E1, E2, E3, E4, E5, E6, E7, E8}, {})

[[2]]
({E1, E2}, {PH1, PH3})

[[3]]
({E2}, {PH1, PH3, PH4})

[[4]]
({E1, E3, E4, E5, E6, E7, E8}, {OH1})

[[5]]
({E8}, {OH1, OH6})
```

Each concept shows:

- **Extent**: Set of evidence explained together
- **Intent**: Set of hypotheses that explain exactly that evidence

Implications in the Lavoisier Context

```
[1] "Key implications discovered:"
Implication set with 3 implications.
Rule 1: {OH1, OH4, OH6} -> {OH2, OH3, OH5, PH1, PH2, PH3, PH4, PH5, PH6}
Rule 2: {OH1, OH4, OH5, PH5, PH6} -> {OH2, OH3, OH6, PH1, PH2, PH3, PH4}
Rule 3: {OH1, OH3, OH6} -> {OH2, OH4, OH5, PH1, PH2, PH3, PH4, PH5, PH6}
```

Implications reveal logical dependencies: if evidence supports certain hypotheses, what other hypotheses must also be supported?

The Lavoisier Concept Lattice

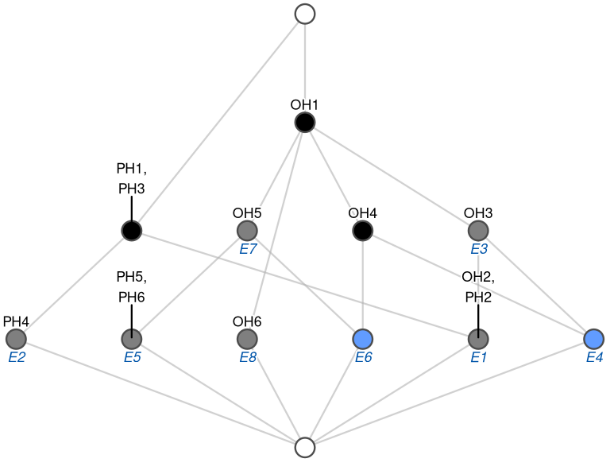


Figure: "The lattice structure reveals the hierarchical organization of explanatory relationships."

Theorem (Data Processing Inequality)

For random variables X, Y, and Z that form a Markov chain $X \to Y \to Z$, we have:

$$I(X; Z) \leq I(X; Y)$$

where $I(X; Y)$ denotes the mutual information between X and Y.

In our context: Information cannot be gained by processing data through additional theoretical layers.

Definition (Chu Space)

A Chu space over a set K is a triple (A, r, X) where:

- A and X are sets (called *points* and *states* respectively)
- $r : A \times X \to K$ is a function (the *evaluation map*)

The Chu space defines a duality via:

$$a^{\perp} = \{x \in X \mid r(a, x) = 0\}$$

$$x^{\perp} = \{a \in A \mid r(a, x) = 0\}$$

Formal contexts are Chu spaces over $\{0, 1\}$ where objects are points, attributes are states, and the incidence relation is the evaluation map.

Chu Spaces are *-autonomous categories

In the category of Chu spaces, every object A has a dual A^* such that:

$$A^{**} \cong A$$

The *-autonomous structure provides:

- **Duality**: Each Chu space has a canonical dual
- **Internal Hom**: $[A, B] = A^* \otimes B$
- **Self-duality**: The category is equivalent to its own opposite

This categorical framework unifies logic, geometry, and computation through the lens of duality.

Unifying Logic, Geometry, and Computation

Logic: Chu spaces model linear logic's multiplicative fragment, where $A \multimap B \equiv A^* \otimes B$

Geometry: Stone duality connects topological spaces with Boolean algebras via Chu construction

Computation: Game semantics emerges naturally—strategies are morphisms in Chu categories

Cognition: Formal concepts capture the duality between extension (objects) and intension (properties)

I probably ran out of time so come talk to me[1]

- Monoids
- Monoidal Categories
- Monoidal and Comonoidal Objects
- Hypergraphs
- Categories of Hypergraphs
- Connections to Other Areas of Cognition

[1] Because I actually don't know that much about most of what I am talking about. If you know more, or want to dive deeper into something we talked about today, I would love to work with you.

Readings

Davey, B. A., and H. A. Priestley. 2002. *Introduction to Lattices and Order*. Cambridge University Press. https://doi.org/10.1017/cbo9780511809088.

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Oberauer, Klaus, and Stephan Lewandowsky. 2019. "Addressing the Theory Crisis in Psychology." *Psychonomic Bulletin & Review* 26 (5): 1596–1618.

Thagard, Paul. 1989. "Explanatory Coherence." *Behavioral and Brain Sciences* 12 (3): 435–67. https://doi.org/10.1017/s0140525x00057046.

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