THE CALCULUS OF NEO-PEIRCEAN RELATIONS

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Abstract. The calculus of relations was introduced by De Morgan and Peirce during the second half of the 19th century. Later developments on quantification theory by Frege and Peirce himself, paved the way to what is known today as first-order logic, causing the calculus of relations to be long forgotten. This was until 1941, when Tarski raised the question on the existence of a complete axiomatisation for it. This question found only negative answers: there is no finite axiomatisation for the calculus of relations and many of its fragments, as shown later by several no-go theorems. In this paper we show that – by moving from traditional syntax (cartesian) to a diagrammatic one (monoidal) – it is possible to have complete axiomatisations for the full calculus. The no-go theorems are circumvented by the fact that our calculus, named the calculus of neo-Peircean relations, is more expressive than the calculus of relations and, actually, as expressive as first-order logic. The axioms are obtained by combining two well known categorical structures: cartesian and linear bicategories.

1. Introduction

The modern understanding of first order logic (FOL) is the result of an evolution with contributions from many philosophers and mathematicians. Amongst these, particularly relevant for our exposition is the calculus of relations (CR) by Charles S. Peirce [Pei97]. Peirce, inspired by De Morgan [Mor60], proposed a relational analogue of Boole's algebra [Boo47]: a rigorous mathematical language for combining relations with operations governed by algebraic laws.

With the rise of first order logic, Peirce's calculus was forgotten until Tarski, who in [Tar41] recognised its algebraic flavour. In the introduction to [TG88], written shortly before his death, Tarski put much emphasis on two key features of CR: (a) its lack of quantifiers and (b) its sole deduction rule of substituting equals by equals. The calculus, however, comes with two great shortcomings: (c) it is strictly less expressive than FOL [LÖW15] and (d) it is *not* axiomatisable [Mon64].

Key words and phrases: calculus of relations, string diagrams, deep inference.

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Despite these limitations, CR had —and continues to have— a great impact in computer science, e.g., in the theory of databases [Cod83] and in the semantics of programming languages [Pra76, HJ86, Las98, BDM96, LG22]. Indeed, the lack of quantifiers avoids the usual burden of bindings, scopes and capture-avoid substitutions (see [GP02, Pit13, PE88, Hof99, FPT99, GMS+23] for some theories developed to address specifically the issue of bindings). This feature, together with purely equational proofs, makes CR particularly suitable for proof assistants [Pou13, Pou16, KN12].

Less influential in computer science, there are two others quantifiers-free alternatives to FOL that are worth mentioning: first, *predicate functor logic* (PFL) [Qui71] that was thought by Quine as the first order logic analogue of combinatory logic [CFC+58] for the λ-calculus; second, Peirce's *existential graphs* (EGs) [Rob73] and, in particular, its fragment named *system* β. In this system FOL formulas are *diagrams* and the deduction system consists of rules for their manipulation. Peirce's work on EGs remained unpublished during his lifetime.

Diagrams have been used as formal entities since the dawn of computer science, e.g. in the Böhm-Jacopini theorem [BJ79]. More recently, the spatial nature of mobile computations led Milner to move from the traditional term-based syntax of process calculi to bigraphs [Mil09]. Similarly, the impossibility of copying quantum information and, more generally, the paradigm-shift of treating data as a physical resource (see e.g. [OLEI19, GKO+16]), has led to the use [BE15, BHP+19, BSZ15, CD11, FSR16, FS20, GJ16, MCG18, PZ21, BGK+22] of *string diagrams* [JS91, Sel10] as syntax. String diagrams, formally arrows of a freely generated symmetric (strict) monoidal category, combine the rigour of traditional terms with a visual and intuitive graphical representation. Like traditional terms, they can be equipped with a compositional semantics.

In this paper, we introduce the calculus of *neo-Peircean relations*, a string diagrammatic account of FOL that has several key features:

(1) Its diagrammatic syntax is closely related to Peirce's EGs, but it can also be given through a context free grammar equipped with an elementary type system;
(2) It is quantifier-free and, differently than FOL, its compositional semantics can be given by few simple rules: see (3.4);
(3) Terms and predicates are not treated as separate syntactic and semantic entities;
(4) Its sole deduction rule is substituting equals by equals, like CR, but differently, it features a complete axiomatisation;
(5) The axioms are those of well-known algebraic structures, also occurring in different fields such as linear algebra [BSZ17] or quantum foundations [CD11];
(6) It allows for compositional encodings of FOL, CR and PFL;
(7) String diagrams disambiguate interesting corner cases where traditional FOL encounters difficulties. One perk is that we allow empty models —forbidden in classical treatments— leading to (slightly) more general Gödel completeness;
(8) The corner case of empty models coincides with *propositional* models and in that case our axiomatisation simplifies to the deep inference Calculus of Structures [Brü03, Gug07].

By returning to the algebraic roots of logic we preserve CR's benefits (a) and (b) while overcoming its limitations (c) and (d).

Cartesian syntax. To ease the reader into this work, we show how traditional terms appear as string diagrams. Consider a signature Σ consisting of a unary symbol f and two binary symbols g and h. The term $h(\,g(f(x_3), f(x_3)), x_1\,)$ corresponds to the string diagram on the

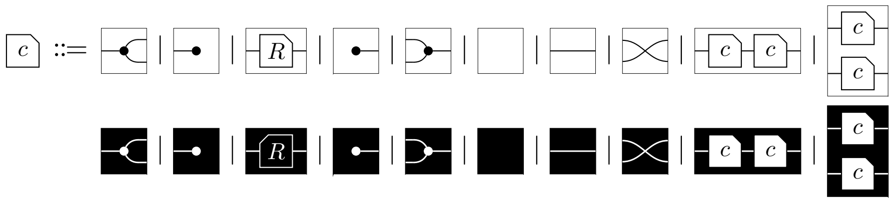
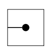
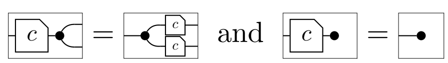


FIGURE 1. Diagrammatic syntax of NPR_Σ

left below.

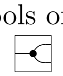
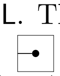
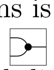
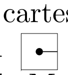
A difference wrt traditional syntax tree is the explicit treatment of copying and discarding. The discharger  informs us that the variable x_2 does not appear in the term; the copier  makes clear that the variable x_3 is shared by two sub-terms. The string diagram on the represents the same term if one admits the equations



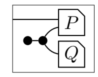
(Nat)

Fox [Fox76] showed that (Nat) together with axioms asserting that copier and discard form a *comonoid* ((◀°-as), (◀°-un), (◀°-co) in Fig. 3) force the monoidal category of string diagrams to be *cartesian* ($\otimes$ is the categorical product): actually, it is the *free* cartesian category on Σ.

Functorial semantics. The work of Lawvere [Law63] illustrates the deep connection of syntax with semantics, explaining why cartesian syntax is so well-suited to functional structures, but also hinting at its limitations when denoting other structures, e.g. relations. Given an algebraic theory $\mathbb{T}$ in the universal algebraic sense, i.e., a signature Σ with a set of equations E, one can freely generate a cartesian category $\mathbf{L}_\mathbb{T}$. *Models* –in the standard algebraic sense– are in one-to-one correspondence with cartesian functors $\mathcal{M}$ from $\mathbf{L}_\mathbb{T}$ to $\mathbf{Set}$, the category of sets and functions. More generally, models of the theory in any cartesian category $\mathbf{C}$ are cartesian functors $\mathcal{M}\colon \mathbf{L}_\mathbb{T} \to \mathbf{C}$. By taking $\mathbf{C}$ to be $\mathbf{Rel}^\circ$, the category of sets and relations, one could wish to use the same approach for relational theories but any such attempt fails immediately since the cartesian product of sets is not the categorical product in $\mathbf{Rel}^\circ$.

Cartesian bicategories. An evolution of Lawvere's approach for relational structures is developed in [BPS17, BSS18, See20]. Departing from cartesian syntax, it uses string diagrams generated by the *first* row of the grammar in Fig. 1, where R is taken from a monoidal signature Σ – a set of symbols equipped with both an arity and also a *coarity* – and can be thought of as akin to relation symbols of FOL. The diagrams are subject to the laws of cartesian bicategories [CW87] in Fig. 3:  and  form a comonoid, but the category of diagrams is not cartesian since the equations in (Nat) hold laxly ((◀°-nat), (!°-nat)). The diagrams  and  form a *monoid* ((▶°-as), (▶°-un), (▶°-co)) and are right *adjoint* to copier and discard. Monoids and comonoids together satisfy *special Frobenius* equations ((S°),(F°)). The category of diagrams $\mathbf{CB}_\Sigma$ is the free cartesian bicategory generated by Σ and, like in Lawvere's functorial semantics, models are morphisms of cartesian bicategories $\mathcal{M}\colon \mathbf{CB}_\Sigma \to \mathbf{Rel}^\circ$. Importantly, the laws of cartesian bicategories provide a complete axiomatisation for $\mathbf{Rel}^\circ$, meaning that c, d in $\mathbf{CB}_\Sigma$ are provably equal with the laws of cartesian bicategories iff $\mathcal{M}(c) = \mathcal{M}(d)$ for all models $\mathcal{M}$.

The (co)monoid structures allow one to express existential quantification: for instance, the FOL formula $\exists x_2.P(x_1, x_2) \wedge Q(x_2)$ is depicted as the diagram on the right. The expressive power of $\mathbf{CB}_\Sigma$ is, however, limited to the existential-conjunctive fragment of FOL.



Cocartesian bicategories. To express the universal-disjunctive fragment, we consider the category $\mathbf{CB}_\Sigma$ of string diagrams generated by the *second* row of the grammar in Fig. 1 and subject to the laws of cocartesian bicategories in Fig. 4: those of cartesian bicategories but with the reversed order $\geq$. The diagrams of $\mathbf{CB}_\Sigma$ are photographic negative of those in $\mathbf{CB}_\Sigma$. To explain this change of colour, note that sets and relations form *another* category: $\mathbf{Rel}^\bullet$. Composition $⨟$ in $\mathbf{Rel}^\bullet$ is the De Morgan dual of the usual relational composition:

$$R ⨟ S \stackrel{\text{def}}{=} \{(x, z) \mid \exists y.(x, y) \in R \wedge (y, z) \in S\}$$

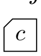
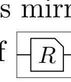
but

$$R ⨟ S \stackrel{\text{def}}{=} \{(x, z) \mid \forall y.(x, y) \in R \vee (y, z) \in S\}.$$

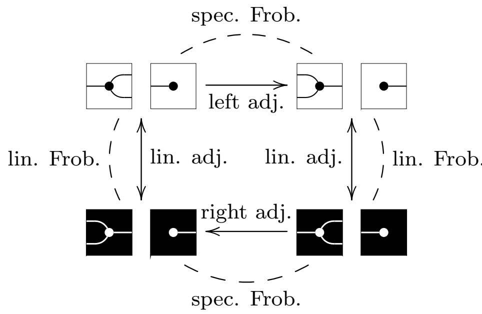
While $\mathbf{Rel}^\circ$ is a cartesian bicategory, $\mathbf{Rel}^\bullet$ is *cocartesian*. Interestingly, the "black" composition $⨟$ was used in Peirce's approach [Pei83] to relational algebra.

Just as $\mathbf{CB}_\Sigma$ is complete with respect to $\mathbf{Rel}^\circ$, dually, $\mathbf{CB}_\Sigma$ is complete wrt $\mathbf{Rel}^\bullet$. The former accounts for the existential-conjunctive fragment of FOL; the latter for its universal-disjunctive fragment. This raises a natural question:

How do white and black structures combine into a complete account of first-order logic?

Linear bicategories. Although $\mathbf{Rel}^\circ$ and $\mathbf{Rel}^\bullet$ have the same objects and arrows, there are two different compositions ($⨟$ and $⨟$). The appropriate categorical structures to deal with these situations are *linear bicategories* introduced in [CKS00] as a horizontal categorification of linearly distributive categories [dP91, CS97b]. The laws of linear bicategories are in Fig. 5: the key law is *linearly distributivity* of $⨟$ over $⨟$ ((δ_l), (δ_r)), that was already known to hold for relations since the work of Peirce [Pei83]. Another crucial property observed by Peirce is that for any $R \subseteq X \times Y$, the relation $R^\perp \subseteq Y \times X \stackrel{\text{def}}{=} \{(y, x) \mid (x, y) \notin R\}$ is its *linear adjoint*. This operation has an intuitive graphical depiction: given , take its mirror image  and then its photographic negative . For instance, the linear adjoint of  is .

First order bicategories. The final step is to characterise how cartesian, cocartesian and linear bicategories combine: *(i)* white and black (co)monoids are linear adjoints that *(ii)* satisfy a "linear" version of the Frobenius law. We dub the result *first order bicategories*. We shall see that this is a complete axiomatisation for first order logic, yet all of the algebraic machinery is compactly summarised in the following picture, named "the Tao of Logic".

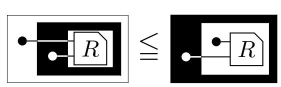


Functorial semantics for first order theories. In the spirit of functorial semantics, we take the free first order bicategory $\mathbf{FOB}_{\mathbb{T}}$ generated by a theory $\mathbb{T}$ and observe that models of $\mathbb{T}$ in a first order bicategory $\mathbf{C}$ are morphisms $\mathcal{M}\colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$. Taking $\mathbf{C} = \mathbf{Rel}$, the first order bicategory of sets and relations, these are models in the sense of FOL with one notable exception: in FOL models with the empty domain are forbidden. As we shall wee, theories with empty models are exactly the propositional theories.

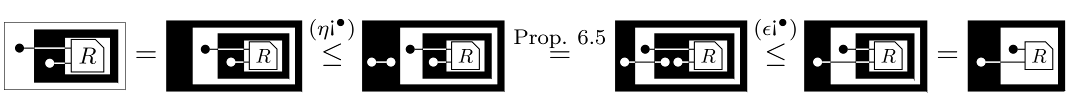
Completeness. We prove that the laws of first order bicategories provide a complete axiomatisation for first order logic. The proof is a diagrammatic adaptation of Henkin's proof [Hen49] of Gödel's completeness theorem. However, in order to properly consider models with an empty domain, we make a slight additional step to go beyond Gödel completeness.

A taste of diagrammatic logic. Before we introduce the calculus of neo-Peircean relations, we start with a short worked example to give the reader a taste of using the calculus to prove a non-trivial result of first order logic. Doing so lets us illustrate the methodology of proof within the calculus, which is sometimes referred to as diagrammatic reasoning or string diagram surgery.

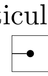
Let R be a symbol with arity 2 and coarity 0. The two diagrams below correspond to FOL formulas $\exists x.\, \forall y.\, R(x,y)$ and $\forall y.\, \exists x.\, R(x,y)$: see § 9 for a dictionary of translating between FOL and diagrams.



It is well-known that $\exists x.\, \forall y.\, R(x,y) \models \forall y.\, \exists x.\, R(x,y)$, i.e. in any model, if the first formula evaluates to true then so does the second. Within our calculus, this statement is expressed as the above inequality. This can be proved by mean of the axiomatisation we introduce in this work:



(1.1)

The central step relies on the particularly good behaviour of *maps*, intuitively those relations that are functional. In particular  is an example. The details are not important at this stage.

Synopsis. We begin by recalling CR in § 2. The calculus of neo-Peircean relations is introduced in § 3, together with the statement of our main result (Theorem 3.4). We recall (co)cartesian and linear bicategories in § 4 and § 5. The categorical structures most important for our work are first-order bicategories, introduced in § 6. In § 7 we consider first order theories, the diagrammatic version of the deduction theorem (Theorem 7.13) and some subtle differences with FOL that play an important role in the proof of completeness in § 8. In § 9 we discuss the relationship of NPR_Σ with FOL, while in § 10 with CR, PFL and EGs. In § 11, we give a closer look to the work of Peirce, illustrating how fo-bicategories provide solid categorical foundations to most of his intuitions. This paper is an extended version of [BGHS24]: it additionally contains proofs and detailed comparisons with the work of Quine and Peirce.

2. The Calculus of Binary Relations

The calculus of binary relations, in the original presentation given by Peirce [Pei83], features two forms of relational compositions $\mathbin{;}$ and $\mathbin{\bullet\!\!;}$, defined for all relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ as

$$R \mathbin{;} S \stackrel{\text{def}}{=} \{(x,z) \mid \exists y{\in}Y \,.\, (x,y) \in R \wedge (y,z) \in S\} \subseteq X \times Z \text{ and}$$
$$R \mathbin{\bullet\!\!;} S \stackrel{\text{def}}{=} \{(x,z) \mid \forall y{\in}Y \,.\, (x,y) \in R \vee (y,z) \in S\} \subseteq X \times Z \tag{2.1}$$

with units the equality and the difference relations respectively, defined for all sets X as

$$id^{\circ}_X \stackrel{\text{def}}{=} \{(x,y) \,|\, x = y\} \subseteq X \times X \text{ and } id^{\bullet}_X \stackrel{\text{def}}{=} \{(x,y) \,|\, x \neq y\} \subseteq X \times X. \tag{2.2}$$

Beyond the usual union $\cup$, intersection $\cap$, and their units $\bot$ and $\top$, the calculus also features two unary operations $(\cdot)^{\dagger}$ and $\overline{(\cdot)}$ denoting the opposite and the complement:

$$R^{\dagger} \stackrel{\text{def}}{=} \{(y,x) \mid (x,y) \in R\} \text{ and } \overline{R} \stackrel{\text{def}}{=} \{(x,y) \mid (x,y) \notin R\}. \tag{2.3}$$

In summary, its syntax is given by the following context free grammar

$$\begin{array}{rclllllllllll} E & ::= & R & \mid & id^{\circ} & \mid & E \mathbin{;} E & \mid & id^{\bullet} & \mid & E \mathbin{\bullet\!\!;} E & \mid \\ & & E^{\dagger} & \mid & \top & \mid & E \cap E & \mid & \bot & \mid & E \cup E & \mid & \overline{E} \end{array} \tag{CR_Σ}$$

where R is taken from a given set Σ of generating symbols. The semantics is defined wrt a *relational interpretation* $\mathcal{I}$, that is, a set X together with a binary relation $\rho(R) \subseteq X \times X$ for each $R \in \Sigma$. The following inductive definition assigns a binary relation to each expression.

$$\begin{array}{llll} \langle id^{\circ} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^{\circ}_X & \langle E_1 \mathbin{;} E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \mathbin{;} \langle E_2 \rangle_{\mathcal{I}} & \langle \top \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} X \times X & \langle E_1 \cap E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cap \langle E_2 \rangle_{\mathcal{I}} \\ \langle id^{\bullet} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^{\bullet}_X & \langle E_1 \mathbin{\bullet\!\!;} E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \mathbin{\bullet\!\!;} \langle E_2 \rangle_{\mathcal{I}} & \langle \bot \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \varnothing & \langle E_1 \cup E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cup \langle E_2 \rangle_{\mathcal{I}} \\ & \langle R \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \rho(R) & \langle E^{\dagger} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E \rangle_{\mathcal{I}}^{\dagger} & \langle \overline{E} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \overline{\langle E \rangle_{\mathcal{I}}} \end{array} \tag{2.4}$$

Two expressions E_1, E_2 are said to be *equivalent*, written $E_1 \equiv_{\mathsf{CR}} E_2$, if and only if $\langle E_1 \rangle_{\mathcal{I}} = \langle E_2 \rangle_{\mathcal{I}}$, for all interpretations $\mathcal{I}$. Inclusion, denoted by $\leq_{\mathsf{CR}}$, is defined analogously by replacing $=$ with $\subseteq$. For instance, the following inclusions hold, witnessing the fact that $\mathbin{;}$ *linearly distributes* over $\mathbin{\bullet\!\!;}$.

$$R \mathbin{;} (S \mathbin{\bullet\!\!;} T) \leq_{\mathsf{CR}} (R \mathbin{;} S) \mathbin{\bullet\!\!;} T \qquad (R \mathbin{\bullet\!\!;} S) \mathbin{;} T \leq_{\mathsf{CR}} R \mathbin{\bullet\!\!;} (S \mathbin{;} T) \tag{2.5}$$

Along with the boolean laws, in 'Note B' [Pei83] Peirce states (2.5) and stresses its importance. However, since $R \mathbin{\bullet\!\!;} S \equiv_{\mathsf{CR}} \overline{\overline{R} \mathbin{;} \overline{S}}$ and $id^{\bullet} \equiv_{\mathsf{CR}} \overline{id^{\circ}}$, both $\mathbin{\bullet\!\!;}$ and $id^{\bullet}$ are often considered redundant, for instance ommited by Tarski [Tar41] and in much of the later work.

Tarski asked whether $\equiv_{\mathsf{CR}}$ can be axiomatised, i.e., is there a basic set of laws from which one can prove all the valid equivalences? Unfortunately, there is no finite complete axiomatisations for the whole calculus [Mon64] nor for several fragments, e.g., [HM00, Red64, FS90, BFS20, Pou18].

Our work returns to the same problem, but from a radically different perspective: we see the calculus of relations as a sub-calculus of a more general system for arbitrary (i.e. not merely binary) relations. The latter is strictly more expressive than CR_Σ – actually it is as expressive as first order logic (FOL)– but allows for an elementary complete axiomatisation based on the interaction of two influential algebraic structures: that of linear bicategories and cartesian bicategories.

$$
\begin{array}{c}
id^\circ_0 \colon 0 \to 0 \qquad id^\circ_1 \colon 1 \to 1 \qquad \sigma^\circ_{1,1} \colon 2 \to 2 \qquad \blacktriangleleft^\circ_1 \colon 1 \to 2 \qquad !^\circ_1 \colon 1 \to 0 \qquad \blacktriangleright^\circ_1 \colon 2 \to 1 \qquad ¡^\circ_1 \colon 0 \to 1 \\
\dfrac{ar(R) = n \quad coar(R) = n}{R^\circ \colon n \to m} \qquad \dfrac{ar(R) = n \quad coar(R) = m}{R^\bullet \colon m \to n} \qquad \dfrac{c \colon n_1 \to m_1 \quad d \colon n_2 \to m_2}{c \otimes d \colon n_1 + n_2 \to m_1 + m_2} \qquad \dfrac{c \colon n \to m \quad d \colon m \to o}{c \fatsemi d \colon n \to o}
\end{array}
$$

$$
\begin{array}{lll}
id^\circ_0 = id^\circ_0 & \sigma^\circ_{0,0} = id^\circ_0 & \sigma^\circ_{1,0} = \sigma^\circ_{0,1} = id^\circ_1 \\
id^\circ_{1+n} = id^\circ_1 \otimes id^\circ_n & \sigma^\circ_{1,1+1} = (\sigma^\circ_{1,n} \otimes id^\circ_1) \fatsemi (id^\circ_n \otimes \sigma^\circ_{1,1}) & \sigma^\circ_{m+1,n} = (id^\circ_1 \otimes \sigma^\circ_{m,n}) \fatsemi (\sigma^\circ_{1,n} \otimes id^\circ_m) \\
\blacktriangleleft^\circ_0 = id^\circ_0 & \blacktriangleright^\circ_0 = id^\circ_0 & !^\circ_0 = id^\circ_0 \qquad ¡^\circ_0 = id^\circ_0 \\
\blacktriangleleft^\circ_{1+n} = (\blacktriangleleft^\circ_1 \otimes \blacktriangleleft^\circ_n) \fatsemi (id^\circ_1 \otimes \sigma^\circ_{1,n} \otimes id^\circ_n) & \blacktriangleright^\circ_{1+n} = (id^\circ_1 \otimes \sigma^\circ_{1,n} \otimes id^\circ_n) \fatsemi (\blacktriangleright^\circ_1 \otimes \blacktriangleright^\circ_n) & !^\circ_{1+n} = !^\circ_1 \otimes !^\circ_n \qquad ¡^\circ_{n+1} = ¡^\circ_1 \otimes ¡^\circ_n
\end{array}
$$

$$
\begin{array}{c}
a \fatsemi (b \fatsemi c) = (a \fatsemi b) \fatsemi c \qquad id^\circ_n \fatsemi c = c = c \fatsemi id^\circ_m \qquad (a \otimes b) \fatsemi (c \otimes d) = (a \fatsemi c) \otimes (b \fatsemi d) \\
a \otimes (b \otimes c) = (a \otimes b) \otimes c \quad id^\circ_0 \otimes c = c = c \otimes id^\circ_0 \quad (c \otimes id^\circ_o) \fatsemi \sigma^\circ_{m,o} = \sigma^\circ_{n,o} \fatsemi (id^\circ_o \otimes c) \quad \sigma^\circ_{1,1} \fatsemi \sigma^\circ_{1,1} = id^\circ_2
\end{array}
$$

TABLE 1. Typing rules (top); inductive definitions of syntactic sugar (middle); structural congruence (bottom)

3. Neo-Peircean Relations

Here we introduce the calculus of *neo-Peircean relations* (NPR_Σ).

The first step is to move from binary relations $R \subseteq X \times X$ to relations $R \subseteq X^n \times X^m$ where, for any $n \in \mathbb{N}$, X^n denotes the set of row vectors $(x_1, \dots, x_n)$ with all $x_i \in X$. In particular, X^0 is the one element set $\mathbb{1} \stackrel{\text{def}}{=} \{\star\}$. Considering these kinds of relations allows us to identify two novel fundamental constants: the *copier* $\blacktriangleleft^\circ_X \subseteq X \times X^2$, which is the diagonal function $\langle id^\circ_X, id^\circ_X \rangle \colon X \to X \times X$ considered as a relation, and the *discharger* $!^\circ_X \subseteq X \times \mathbb{1}$ which is, similarly, the unique function from X to $\mathbb{1}$. By combining them with opposite and complement we obtain, in total, 8 basic relations.

$$
\begin{array}{ll}
\blacktriangleleft^\circ_X \stackrel{\text{def}}{=} \{(x,\ (y,z)) \mid x = y \wedge x = z\} \subseteq X \times (X \times X) & !^\circ_X \stackrel{\text{def}}{=} \{(x, \star) \mid x \in X\} \subseteq X \times \mathbb{1} \\
\blacktriangleright^\circ_X \stackrel{\text{def}}{=} \{((y,z),\ x) \mid x = y \wedge x = z\} \subseteq (X \times X) \times X & ¡^\circ_X \stackrel{\text{def}}{=} \{(\star, x) \mid x \in X\} \subseteq \mathbb{1} \times X \\
\hline
\blacktriangleleft^\bullet_X \stackrel{\text{def}}{=} \{(x,\ (y,z)) \mid x \neq y \vee x \neq z\} \subseteq X \times (X \times X) & !^\bullet_X \stackrel{\text{def}}{=} \varnothing \subseteq X \times \mathbb{1} \\
\blacktriangleright^\bullet_X \stackrel{\text{def}}{=} \{((y,z),\ x) \mid x \neq y \vee x \neq z\} \subseteq (X \times X) \times X & ¡^\bullet_X \stackrel{\text{def}}{=} \varnothing \subseteq \mathbb{1} \times X
\end{array}
\tag{3.1}
$$

Together with the identities id°_X and $id^\bullet_X$ and the compositions $\fatsemi$ and $\boldsymbol{\fatsemi}$ from (2.2), there are black and white *symmetries*:

$$
\begin{array}{rcl}
\sigma^\circ_{X,Y} & \stackrel{\text{def}}{=} & \{(\ (x_1, y_1), (y_2, x_2)\ ) \mid x_1 = x_2 \wedge y_1 = y_2\} \subseteq (X \times Y) \times (Y \times X) \\
\sigma^\bullet_{X,Y} & \stackrel{\text{def}}{=} & \{(\ (x_1, y_1), (y_2, x_2)\ ) \mid x_1 \neq x_2 \vee y_1 \neq y_2\} \subseteq (X \times Y) \times (Y \times X)\ .
\end{array}
\tag{3.2}
$$

The calculus does *not* feature the boolean operators nor the opposite and the complement: these can be derived using the above structure and two *monoidal products* $\otimes$ and $\boldsymbol{\otimes}$, defined for $R \subseteq X \times Y$ and $S \subseteq V \times W$ as

$$
\begin{array}{rcl}
R \otimes S & \stackrel{\text{def}}{=} & \{(\,(x,v),(y,w)\,) \mid (x,y) \in R \wedge (v,w) \in S\} \subseteq (X \times V) \times (Y \times W) \\
R \mathbin{\boldsymbol{\otimes}} S & \stackrel{\text{def}}{=} & \{(\,(x,v),(y,w)\,) \mid (x,y) \in R \vee (v,w) \in S\} \subseteq (X \times V) \times (Y \times W)\ .
\end{array}
\tag{3.3}
$$

Syntax. Terms are defined by the following context free grammar

$$
\begin{array}{rcl}
c & ::= & \blacktriangleleft^{\circ}_{1} \mid !^{\circ}_{1} \mid R^{\circ} \mid ¡^{\circ}_{1} \mid \blacktriangleright^{\circ}_{1} \mid id^{\circ}_{0} \mid id^{\circ}_{1} \mid \sigma^{\circ}_{1,1} \mid c \mathbin{\circ;} c \mid c \otimes c \mid \\
 & & \blacktriangleleft^{\bullet}_{1} \mid !^{\bullet}_{1} \mid R^{\bullet} \mid ¡^{\bullet}_{1} \mid \blacktriangleright^{\bullet}_{1} \mid id^{\bullet}_{0} \mid id^{\bullet}_{1} \mid \sigma^{\bullet}_{1,1} \mid c \mathbin{\bullet;} c \mid c \mathbin{\boldsymbol{\otimes}} c
\end{array}
\tag{NPR_Σ}
$$

where R, like in CR_Σ, belongs to a fixed set Σ of *generators*. Differently than in CR_Σ, each $R \in \Sigma$ comes with two natural numbers: arity $ar(R)$ and coarity $coar(R)$. The tuple $(\Sigma, ar, coar)$, hereafter referred simply as Σ, is a *monoidal signature*. Intuitively, every $R \in \Sigma$ represents some relation $R \subseteq X^{ar(R)} \times X^{coar(R)}$.

In the first row of (NPR_Σ) there are eight constants and two operations: white composition ($\mathbin{\circ;}$) and white monoidal product ($\otimes$). These, together with identities (id°_{0} and id°_{1}) and symmetry ($\sigma^{\circ}_{1,1}$) are typical of symmetric monoidal categories. Apart from R°, the constants are the copier ($\blacktriangleleft^{\circ}_{1}$), discharger ($!^{\circ}_{1}$) and their opposite cocopier ($\blacktriangleright^{\circ}_{1}$) and codischarger ($¡^{\circ}_{1}$). The second row contains the "black" versions of the same constants and operations. Note that our syntax does not have variables, no quantifiers, nor the usual associated meta-operations like capture-avoiding substitution.

We shall refer to the terms generated by the first row as the *white fragment*, while to those of second row as the *black fragment*. Sometimes, we use the gray colour to be agnostic wrt white or black. The rules in top of Table 1 assigns to each term at most one type $n \to m$. We consider only those terms that can be typed. For all $n, m \in \mathbb{N}$, $id^{\circ}_{n} \colon n \to n$, $\sigma^{\circ}_{n,m} \colon n+m \to m+n$, $\blacktriangleleft^{\circ}_{n} \colon n \to n+n$, $\blacktriangleright^{\circ}_{n} \colon n+n \to n$, $!^{\circ}_{n} \colon n \to 0$ and $¡^{\circ}_{n} \colon 0 \to n$ are as in middle of Table 1.

Semantics. As for CR_Σ, the semantics of NPR_Σ needs an interpretation $\mathcal{I} = (X, \rho)$: a set X, the *semantic domain*, and $\rho(R) \subseteq X^{ar(R)} \times X^{coar(R)}$ for each $R \in \Sigma$. The semantics of terms is defined inductively as follows.

$$
\begin{array}{llll}
\mathcal{I}^\sharp(\blacktriangleleft^{\circ}_{1}) \stackrel{\text{def}}{=} \blacktriangleleft^{\circ}_{X} & \mathcal{I}^\sharp(!^{\circ}_{1}) \stackrel{\text{def}}{=} !^{\circ}_{X} & \mathcal{I}^\sharp(\blacktriangleright^{\circ}_{1}) \stackrel{\text{def}}{=} \blacktriangleright^{\circ}_{X} & \mathcal{I}^\sharp(¡^{\circ}_{1}) \stackrel{\text{def}}{=} ¡^{\circ}_{X} \\
\mathcal{I}^\sharp(id^{\circ}_{0}) \stackrel{\text{def}}{=} id^{\circ}_{\mathbb{1}} & \mathcal{I}^\sharp(id^{\circ}_{1}) \stackrel{\text{def}}{=} id^{\circ}_{X} & \mathcal{I}^\sharp(\sigma^{\circ}_{1,1}) \stackrel{\text{def}}{=} \sigma^{\circ}_{X,X} & \mathcal{I}^\sharp(R^{\circ}) \stackrel{\text{def}}{=} \rho(R) \\
\mathcal{I}^\sharp(c ; d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) ; \mathcal{I}^\sharp(d) & \mathcal{I}^\sharp(c \otimes d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \otimes \mathcal{I}^\sharp(d) & & \mathcal{I}^\sharp(R^{\bullet}) \stackrel{\text{def}}{=} \overline{\rho(R)}^{\dagger}
\end{array}
\tag{3.4}
$$

The constants and operations appearing on the right-hand-side of the above equations are amongst those defined in (2.1), (2.2), (2.3) (3.1), (3.2) and (3.3). A simple inductive argument confirms that $\mathcal{I}^\sharp$ maps terms c of type $n \to m$ to relations $R \subseteq X^n \times X^m$.

Remark 3.1. In particular, $id^{\circ}_{0} \colon 0 \to 0$ is sent to $id^{\circ}_{\mathbb{1}} \subseteq \mathbb{1} \times \mathbb{1}$, since $X^0 = \mathbb{1}$ independently of X. Note that there are only two relations on the singleton set $\mathbb{1} = \{\star\}$: the relation $\{(\star,\star)\} \subseteq \mathbb{1} \times \mathbb{1}$ and the empty relation $\varnothing \subseteq \mathbb{1} \times \mathbb{1}$. These are $id^{\circ}_{\mathbb{1}}$ and $id^{\bullet}_{\mathbb{1}}$ since

$$
\begin{array}{rclcl}
id^{\circ}_{\mathbb{1}} & \stackrel{(2.2)}{=} & \{(x,y) \in \mathbb{1} \times \mathbb{1} \mid x = y\} & = & \{(\star,\star)\} \quad \text{and} \\
id^{\bullet}_{\mathbb{1}} & \stackrel{(2.2)}{=} & \{(x,y) \in \mathbb{1} \times \mathbb{1} \mid x \neq y\} & = & \varnothing.
\end{array}
$$

It is worth emphasising that $id^{\circ}_{\mathbb{1}}$ and $id^{\bullet}_{\mathbb{1}}$ provide *truth* and *falsity* independently of the chosen domain of interpretation.

Example 3.2. Take Σ with two symbols R and S with arity and coarity 1. From Table 1, the two terms below have type $1 \to 1$.

$$!^{\circ}_{1} \fatsemi ¡^{\circ}_{1} \qquad \blacktriangleleft^{\circ}_{1} \fatsemi ((R^{\circ} \otimes S^{\circ}) \fatsemi \blacktriangleright^{\circ}_{1}) \tag{3.5}$$

For any interpretation $\mathcal{I} = (X, \rho)$, the semantics of the leftmost term, $\mathcal{I}^{\sharp}(!^{\circ}_{1} \fatsemi ¡^{\circ}_{1})$, is the top relation $X \times X$, denoted in CR_{Σ} by $\top$:

$$\begin{aligned}
\mathcal{I}^{\sharp}(!^{\circ}_{1} \fatsemi ¡^{\circ}_{1}) &= !^{\circ}_{X} \fatsemi ¡^{\circ}_{X} && (3.4)\\
&= \{(x, \star) \mid x \in X\} \fatsemi \{(\star, x) \mid x \in X\} && (3.1)\\
&= \{(x, y) \mid x, y \in X\} && (2.1)\\
&= X \times X &&\\
&= \langle \top \rangle_{\mathcal{I}}. && (2.4)
\end{aligned}$$

Similarly, $\mathcal{I}^{\sharp}(\blacktriangleleft^{\circ}_{1} \fatsemi ((R^{\circ} \otimes S^{\circ}) \fatsemi \blacktriangleright^{\circ}_{1}) = \rho(R) \cap \rho(S)$ which is denoted in CR_{Σ} by $R \cap S$.

We leave to the reader to check that the following two terms

$$!^{\bullet}_{1} \fatsemi^{\bullet} ¡^{\bullet}_{1} \qquad \blacktriangleleft^{\bullet}_{1} \fatsemi^{\bullet} ((R^{\circ} \otimes^{\bullet} S^{\circ}) \fatsemi^{\bullet} \blacktriangleright^{\bullet}_{1})$$

denote, instead, the bottom relation $\varnothing$ and $\rho(R) \cap \rho(S)$ corresponding to the CR_{Σ} expressions $\bot$ and $R \cup S$. In §10.1, we will illustrate in detail an encoding of the whole CR_{Σ} into NPR_{Σ}.

Remark 3.3. We will see in §9 that NPR_{Σ} is as expressive as FOL. We draw the reader's attention to the simplicity of the inductive definition of semantics compared to the traditional FOL approach where variables and quantifiers make the definition more involved. Moreover, in traditional accounts, the domain of an interpretation is required to be a *non-empty* set. In our calculus this is unnecessary and it is *not* a mere technicality: in § 7 we shall see that empty models capture the propositional calculus.

Semantic equivalence. Like in CR_{Σ}, *semantic equivalece* plays a key role. For all terms $c, d \colon n \to m$, we write

$$c \equiv d \text{ iff, for all interpretations } \mathcal{I}, \mathcal{I}^{\sharp}(c) = \mathcal{I}^{\sharp}(d). \tag{$\equiv$}$$

Semantic inclusion, written $\leqq$, is defined analogously replacing $=$ with $\subseteq$.

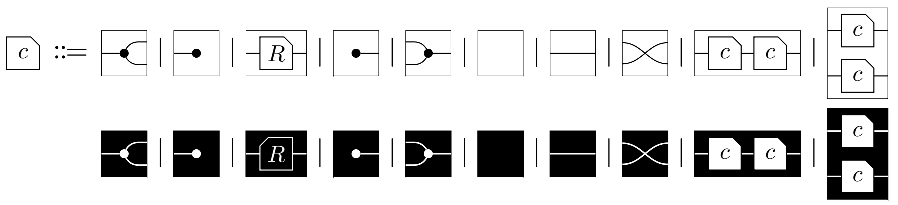
By definition $\equiv$ and $\leqq$ only relate terms of the same type. Throughout the paper, we will encounter several relations amongst terms of the same type. To avoid any confusion with the relations denoted by the terms, we call them *well-typed relations* and use symbols $\mathbb{I}$ rather than the usual R, S, T. In the following, we write $c \mathbb{I} d$ for $(c, d) \in \mathbb{I}$ and $\mathtt{pc}(\mathbb{I})$ for the smallest precongruence (wrt $\otimes$, $\otimes^{\bullet}$, $\fatsemi$ and $\fatsemi^{\bullet}$,) generated by $\mathbb{I}$, i.e., the relation inductively generated as

$$\begin{array}{ccc}
\dfrac{c \mathbb{I} d}{c \,\mathtt{pc}(\mathbb{I})\, d}\ (id) & \dfrac{-}{c \,\mathtt{pc}(\mathbb{I})\, c}\ (r) & \dfrac{a \,\mathtt{pc}(\mathbb{I})\, b \quad b \,\mathtt{pc}(\mathbb{I})\, c}{a \,\mathtt{pc}(\mathbb{I})\, c}\ (t) \\[2ex]
\dfrac{c_1 \,\mathtt{pc}(\mathbb{I})\, c_2 \quad d_1 \,\mathtt{pc}(\mathbb{I})\, d_2}{c_1 \fatsemi d_1 \,\mathtt{pc}(\mathbb{I})\, c_2 \fatsemi d_2}\ (\fatsemi) & & \dfrac{c_1 \,\mathtt{pc}(\mathbb{I})\, c_2 \quad d_1 \,\mathtt{pc}(\mathbb{I})\, d_2}{c_1 \otimes d_1 \,\mathtt{pc}(\mathbb{I})\, c_2 \otimes d_2}\ (\otimes)
\end{array} \tag{3.6}$$

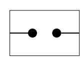
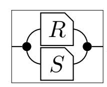
Similarly, we will use $\mathtt{c}(\mathbb{I})$ for the smallest congruence generated by $\mathbb{I}$, namely the well typed relation inductively generated by adding the symmetric closure to the rules above:

$$\dfrac{a \,\mathtt{c}(\mathbb{I})\, b}{b \,\mathtt{c}(\mathbb{I})\, a}\ (s)$$

Diagrams. Terms of NPR_Σ enjoy an elegant diagrammatic representation inspired by string diagrams [JS91, Sel10]. Actually, diagrams take centre stage in our presentation. A term $c \colon n \to m$ is drawn as a diagram with n ports on the left and m ports on the right; $\fatsemi$ is depicted as horizontal composition while $\otimes$ by vertically "stacking" diagrams. The two compositions $\fatsemi$ and $\pmb{\fatsemi}$ and two monoidal products $\otimes$ and $\pmb{\otimes}$ are distinguished with different colours. All constants in the white fragment have white background, mutatis mutandis for the black fragment: for instance id°_1 and $id^\bullet_1$ are drawn and . Indeed, the diagrammatic version of (NPR_Σ) is the grammar in Fig.1, reported below.



For instance, the two terms in (3.5), $!^\circ_1 \fatsemi ¡^\circ_1$ and $\blacktriangleleft^\circ_1 \fatsemi ((R^\circ \otimes S^\circ) \fatsemi \blacktriangleright^\circ_1)$ are drawn as

 and .

Note that one diagram may correspond to more than one term: for instance the diagram on the right above does not only represent the term $\blacktriangleleft^\circ_1 \fatsemi ((R^\circ \otimes S^\circ) \fatsemi \blacktriangleright^\circ_1)$, but also $(\blacktriangleleft^\circ_1 \fatsemi (R^\circ \otimes S^\circ)) \fatsemi \blacktriangleright^\circ_1$. Indeed, it is clear that traditional term-based syntax carries more information than the diagrammatic one (e.g. associativity). From the point of view of the semantics, however, this bureaucracy is irrelevant and is conveniently discarded by the diagrammatic notation.

To formally show this, we recall that diagrams capture only the axioms of symmetric monoidal categories [JS91, Sel10], illustrated in the bottom of Figure 1; let $\mathbb{SMC}$ be the well-typed relation obtained by substituting a, b, c, d in Fig. 1 with terms of the appropriate type; we call *structural congruence*, written $\approx$, the well-typed congruence generated by $\mathbb{SMC}$,

$$\approx \stackrel{\text{def}}{=} \mathsf{c}(\mathbb{SMC}), \tag{$\approx$}$$

and we observe that $\approx \subseteq \equiv$.

We consider diagram as first class mathematical objects. It is possible to perform calculations with either terms or diagrams, but we usually prefer working with diagrams because of the lesser bureaucratic overhead and the compelling visual intuitions afforded by the notation. Moreover, as we will illustrate in the following sections, diagrams allow for a visual representation of opposite $(\cdot^\dagger)$ and complement $(\bar{\cdot})$ operations: the *mirror image* of a diagram –obtained by swapping left and right– provides the opposite, while the *photographic negative* –obtained by swapping white and black– provides the complement.

Nevertheless, for typographical reasons, it is sometimes convenient to use terms, and thus, the reader should become familiar with both notations. To ease this transition, in the early stages of the paper, we will occasionally present key notions in both notations. Additionally, Table 6 in Appendix B provides a summary of the diagrammatic conventions, which the reader may find useful.

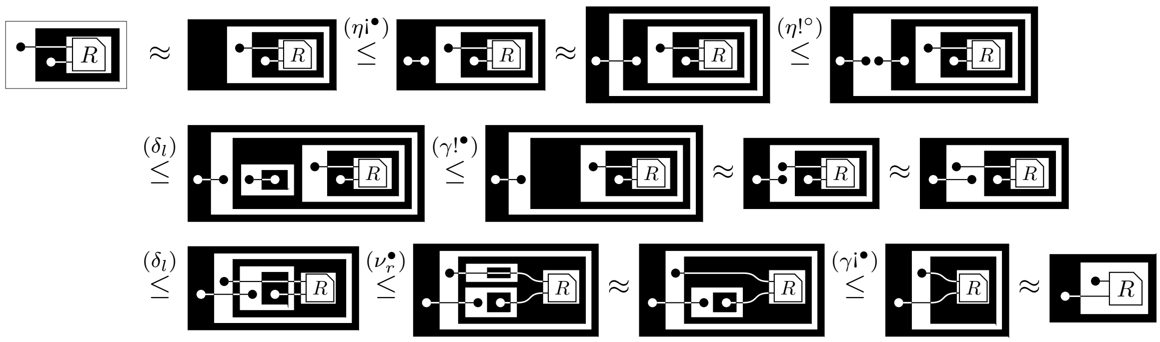


FIGURE 2. Completely axiomatic proof of (1.1).

Axioms. Figure 10 in Appendix A illustrates a complete system of axioms for the semantic inclusion $\leqq$. Let $\mathbb{FOB}$ be the well-typed relation obtained by substituting a, b, c, d in Fig. 10 with terms of the appropriate type and and call *syntactic inclusion*, written $\lesssim$, the precongruence generated by $\mathbb{FOB}$ and $\approx$. In symbols,

$$\lesssim \stackrel{\text{def}}{=} \mathtt{pc}(\mathbb{FOB} \cup \approx). \tag{$\lesssim$}$$

We will also write $\cong \stackrel{\text{def}}{=} \lesssim \cap \gtrsim$ for the *syntactic equivalence*. Our main result is:

Theorem 3.4. *For all terms $c, d \colon n \to m$, $c \lesssim d$ iff $c \leqq d$.*

The axiomatisation is far from minimal and is redundant in several respects. We chose the more verbose presentation in order to emphasise both the underlying categorical structures and the various dualities that we will highlight in the next sections. We confined the complete axiomatisation to the appendix because the axioms in Figure 10 appear also in Figures 3, 4, 5, 6 in form of diagrams. This allows a more principled, staged presentation and place each axiom in its proper context, highlighting their provenance from one of the categorical structures involved.

Proofs as diagrams rewrites. Proofs in NPR_Σ are rather different from those of traditional proof systems: since the only inference rules are those in (3.6), any proof of $c \lesssim d$ consists of a sequence of applications of axioms. As an example consider the proof in Fig. 2 which is the same of (1.1) from the Introduction, but only using axioms. Note that, when applying axioms, we are in fact performing diagram rewriting: an instance of the left hand side of an axiom is found within a larger diagram and replaced with the right hand side. Since such rewrites can happen anywhere, there is a close connection between proofs in NPR_Σ and the work on *deep inference* [HSW21, Brü03, Gug07] – see Example 7.10.

The theory of neo-Peircean relations is both rich and elegant, built upon well-established concepts from category theory. For this reason, we now move away from a traditional, term-based exposition and adopt the more abstract framework of category theory. In the next two sections, we provide the necessary categorical background –covering cartesian and linear bicategories –before introducing first-order bicategories in §6. We will then return to NPR_Σ in §6.1, where we explain how it gives rise to a first-order bicategory, specifically the freely generated one: any equivalence that holds in NPR_Σ also holds in arbitrary fo-bicategories.

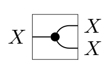
We hope that this section, which deliberately avoids categorical language, has been enjoyable even for readers who are not experts in category theory and that it has sparked their curiosity to continue reading.

4. (Co)Cartesian Bicategories

Although the term bicategory might seem ominous, the beasts considered in this paper are actually quite simple. We consider *poset enriched symmetric monoidal categories*: every homset carries a partial order $\leq$, and composition $\fatsemi$ and monoidal product $\otimes$ are monotone. That is, if $a \leq b$ and $c \leq d$ then $a \fatsemi c \leq b \fatsemi d$ and $a \otimes c \leq b \otimes d$. A *poset enriched symmetric monoidal functor* is a (strong, and usually strict) symmetric monoidal functor that preserves the order $\leq$. The notion of *adjoint arrows*, which will play a key role, amounts to the following: for $c \colon X \to Y$ and $d \colon Y \to X$, c is *left adjoint* to d, or d is *right adjoint* to c, written $d \vdash c$, if $id^{\circ}_X \leq c \fatsemi d$ and $d \fatsemi c \leq id^{\circ}_Y$.

For a symmetric monoidal bicategory $(\mathbf{C}, \otimes, I)$, we will write $\mathbf{C}^{\mathsf{op}}$ for the bicategory having the same objects as $\mathbf{C}$ but homsets $\mathbf{C}^{\mathsf{op}}[X,Y] \stackrel{\text{def}}{=} \mathbf{C}[Y,X]$: ordering, identities and monoidal product are defined as in $\mathbf{C}$, while composition $c \fatsemi d$ in $\mathbf{C}^{\mathsf{op}}$ is $d \fatsemi c$ in $\mathbf{C}$. Similarly, we will write $\mathbf{C}^{\mathsf{co}}$ to denote the bicategory having the same objects and arrows of $\mathbf{C}$ but equipped with the reversed ordering $\geq$. Composition, identities and monoidal product are defined as in $\mathbf{C}$. In this paper, we will often tacitly use the facts that, by definition, both $(\mathbf{C}^{\mathsf{op}})^{\mathsf{op}}$ and $(\mathbf{C}^{\mathsf{co}})^{\mathsf{co}}$ are $\mathbf{C}$ and that $(\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ is $(\mathbf{C}^{\mathsf{op}})^{\mathsf{co}}$.

All monoidal categories considered throughout this paper are tacitly assumed to be strict [ML78], i.e. $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $I \otimes X = X = X \otimes I$ for all objects X, Y, Z. This is harmless: strictification [ML78] allows to transform any monoidal category into a strict one, enabling the sound use of string diagrams that will be exploited in this and the next sections. These are like the diagrams of NPR_{Σ} in §3 but are interpreted as arrows of the categorical structures of interest and wires are labeled by objects of such categories. For instance, the diagrams



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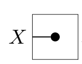

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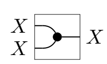
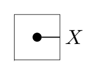


and


represent arrows

$$\blacktriangleleft^{\circ}_X \colon X \to X \otimes X, \quad !^{\circ}_X \colon X \to I, \quad \blacktriangleright^{\circ}_X \colon X \otimes X \to X \quad \text{and} \quad ¡^{\circ}_X \colon I \to X$$

of cartesian bicategories, introduced below.

Definition 4.1. A *cartesian bicategory* $(\mathbf{C}, \otimes, I, \blacktriangleleft^{\circ}, !^{\circ}, \blacktriangleright^{\circ}, ¡^{\circ})$, shorthand $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$, is a poset enriched symmetric monoidal category $(\mathbf{C}, \otimes, I)$ and, for every object X in $\mathbf{C}$, arrows $\blacktriangleleft^{\circ}_X \colon X \to X \otimes X$, $!^{\circ}_X \colon X \to I$, $\blacktriangleright^{\circ}_X \colon X \otimes X \to X$, $¡^{\circ}_X \colon I \to X$ such that

(1) $(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ is a comonoid and $(\blacktriangleright^{\circ}_X, ¡^{\circ}_X)$ a monoid, i.e., the followings hold:

$$\begin{array}{rcl@{\qquad}rcl}
\blacktriangleleft^{\circ}_X \fatsemi (id^{\circ}_X \otimes \blacktriangleleft^{\circ}_X) & \stackrel{(\blacktriangleleft^{\circ}\text{-as})}{=} & \blacktriangleleft^{\circ}_X \fatsemi (\blacktriangleleft^{\circ}_X \otimes id^{\circ}_X) & (id^{\circ}_X \otimes \blacktriangleright^{\circ}_X) \fatsemi \blacktriangleright^{\circ}_X & \stackrel{(\blacktriangleright^{\circ}\text{-as})}{=} & (\blacktriangleright^{\circ}_X \otimes id^{\circ}_X) \fatsemi \blacktriangleright^{\circ}_X \\
\blacktriangleleft^{\circ}_X \fatsemi (id^{\circ}_X \otimes !^{\circ}_X) & \stackrel{(\blacktriangleleft^{\circ}\text{-un})}{=} & id^{\circ}_X & (id^{\circ}_X \otimes ¡^{\circ}_X) \fatsemi \blacktriangleright^{\circ}_X & \stackrel{(\blacktriangleright^{\circ}\text{-un})}{=} & id^{\circ}_X \\
\blacktriangleleft^{\circ}_X \fatsemi \sigma^{\circ}_{X,X} & \stackrel{(\blacktriangleleft^{\circ}\text{-co})}{=} & \blacktriangleleft^{\circ}_X & \sigma^{\circ}_{X,X} \fatsemi \blacktriangleright^{\circ}_X & \stackrel{(\blacktriangleright^{\circ}\text{-co})}{=} & \blacktriangleright^{\circ}_X
\end{array}$$

(2) the comonoid $(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ is lax natural, i.e., for all arrows $c \colon X \to Y$ the followings hold:

$$c \fatsemi \blacktriangleleft^{\circ}_X \stackrel{(\blacktriangleleft^{\circ}\text{-nat})}{\leq} \blacktriangleleft^{\circ}_Y \fatsemi (c \otimes c) \qquad c \fatsemi !^{\circ}_X \stackrel{(!^{\circ}\text{-nat})}{\leq} !^{\circ}_Y$$

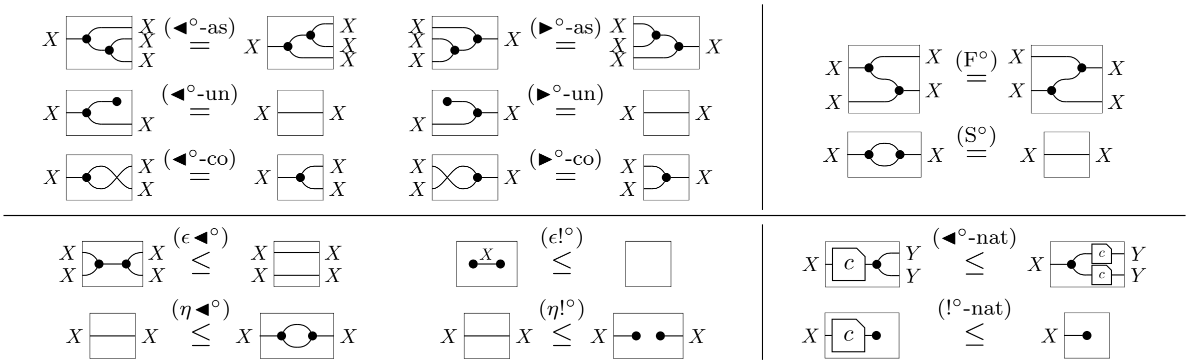


FIGURE 3. Axioms of cartesian bicategories

(3) $(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ are left adjoints to $(\blacktriangleright^{\circ}_X, ¡^{\circ}_X)$, i.e.:

$$\blacktriangleright^{\circ}_X \fatsemi \blacktriangleleft^{\circ}_X \overset{(\epsilon\blacktriangleleft^{\circ})}{\leq} (id^{\circ}_X \otimes id^{\circ}_X) \qquad id^{\circ}_X \overset{(\eta\blacktriangleleft^{\circ})}{\leq} \blacktriangleleft^{\circ}_X \fatsemi \blacktriangleright^{\circ}_X \qquad id^{\circ}_X \overset{(\eta!^{\circ})}{\leq} !^{\circ}_X \fatsemi ¡^{\circ}_X \qquad ¡^{\circ}_X \fatsemi !^{\circ}_X \overset{(\epsilon!^{\circ})}{\leq} id^{\circ}_0$$

(4) $(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ and $(\blacktriangleright^{\circ}_X, ¡^{\circ}_X)$ form special Frobenius algebras, i.e.:

$$(\blacktriangleleft^{\circ}_X \otimes id^{\circ}_X) \fatsemi (id^{\circ}_X \otimes \blacktriangleright^{\circ}_X) \overset{(F^{\circ})}{=} (id^{\circ}_X \otimes \blacktriangleleft^{\circ}_X) \fatsemi (\blacktriangleright^{\circ}_X \otimes id^{\circ}_X) \qquad \blacktriangleleft^{\circ}_X \fatsemi \blacktriangleright^{\circ}_X \overset{(S^{\circ})}{=} id^{\circ}_X$$

(5) $(\blacktriangleleft^{\circ}_X, !^{\circ}_X)$ and $(\blacktriangleright^{\circ}_X, ¡^{\circ}_X)$ satisfy the usual coherence conditions:

$$\begin{array}{llll} \blacktriangleleft^{\circ}_I = id^{\circ}_I & \blacktriangleleft^{\circ}_{X\otimes Y} = (\blacktriangleleft^{\circ}_X \otimes \blacktriangleleft^{\circ}_Y) \fatsemi (id^{\circ}_X \otimes \sigma^{\circ}_{X,Y} \otimes id^{\circ}_Y) & !^{\circ}_I = id^{\circ}_I & !^{\circ}_{X\otimes Y} = !^{\circ}_X \otimes !^{\circ}_Y \\ \blacktriangleright^{\circ}_I = id^{\circ}_I & \blacktriangleright^{\circ}_{X\otimes Y} = (id^{\circ}_X \otimes \sigma^{\circ}_{X,Y} \otimes id^{\circ}_Y) \fatsemi (\blacktriangleright^{\circ}_X \otimes \blacktriangleright^{\circ}_Y) & ¡^{\circ}_I = id^{\circ}_I & ¡^{\circ}_{X\otimes Y} = ¡^{\circ}_X \otimes ¡^{\circ}_Y \end{array}$$

$\mathbf{C}$ is a *cocartesian bicategory* if $\mathbf{C}^{\mathbf{co}}$ is a cartesian bicategory. A *morphism of (co)cartesian bicategories* is a poset enriched strong symmetric monoidal functor preserving monoids and comonoids.

Remark 4.2. In the original presentation of [CW87], the structures in Definition 4.1 are named cartesian bicategories *of relations*. Here we have chosen to go for brevity and just call them cartesian bicategories.

Fig. 3 illustrates the axioms of cartesian bicategories by means of diagrams: the axioms on the top-left corner are those of comonoids and monoids; on the bottom-left corner, there are the axioms of adjointness; bottom-right illustrates lax naturality and top-right the special Frobenius. The axioms of coherence –point (5) in Definition 4.1– that we purposely do not display in diagrams, corresponds to the inductive definitions of $\blacktriangleleft^{\circ}_n$, $\blacktriangleright^{\circ}_n$, $!^{\circ}_n$ and $¡^{\circ}_n$ in Tab. 1: $\otimes \mapsto +$,$I \mapsto 0$, $X \mapsto 1$ and $Y \mapsto n$.

The archetypal example of a cartesian bicategory is $(\mathbf{Rel}^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$. $\mathbf{Rel}^{\circ}$ the bicategory of sets and relations ordered by inclusion $\subseteq$ with white composition $\fatsemi$ and identities id° defined as in (2.1) and (2.2). The monoidal product on objects is the cartesian product of sets with unit I the singleton set $\mathbb{1}$. On arrows, $\otimes$ is defined as in (3.3). It is immediate to check that, for every set X, the arrows $\blacktriangleleft^{\circ}_X$, $!^{\circ}_X$ defined in (3.1) form a comonoid in $\mathbf{Rel}^{\circ}$, while $\blacktriangleright^{\circ}_X$, $¡^{\circ}_X$ a monoid. Simple computations also proves all the (in)equalities in Fig. 3.

Lax naturality of the comonoid $(\blacktriangleright^{\circ}_X, ¡^{\circ}_X)$ is the most interesting to show: since for any relation $R \subseteq X \times Y$

$$R \fatsemi \blacktriangleleft^{\circ}_Y = \{(x,(y,y)) \mid (x,y) \in R\} \ \subseteq\ \{(x,(y,z)) \mid (x,y) \in R \wedge (x,z) \in R\} = \blacktriangleleft^{\circ}_X \fatsemi (R \otimes R)$$

and

$$R \,⨾\, !^{\circ}_{Y} = \{(x,\star) \mid \exists y \in X \,.\, (x,y) \in R\} \;\subseteq\; \{(x,\star) \mid x \in X\} = !^{\circ}_{X},$$

the axioms ($\blacktriangleleft^{\circ}$-nat) and ($!^{\circ}$-nat) hold in $\mathbf{Rel}^{\circ}$. The reversed inclusions are also interesting to consider: $R ⨾ \blacktriangleleft^{\circ}_{Y} \supseteq \blacktriangleleft^{\circ}_{X} ⨾ (R \otimes R)$ holds iff the relation R is single valued namely, for all $x \in X$, there is *at most* one $y \in Y$ such that $(x,y) \in R$, while $R ⨾ !^{\circ}_{Y} \supseteq !^{\circ}_{X}$ iff R is total i.e., for all $x \in X$, there is *at least* one $y \in Y$ such that $(x,y) \in R$. That is, the two inequalities in Definition 4.1.(2) are equalities iff the relation R is a *function*. This justifies the following:

Definition 4.3. Let $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ be a cartesian bicategory. An arrow $c \colon X \to Y$ is a *map* if

$$c ⨾ \blacktriangleleft^{\circ}_{Y} \geq \blacktriangleleft^{\circ}_{X} ⨾ (c \otimes c) \qquad \text{and} \qquad c ⨾ !^{\circ}_{Y} \;\geq\; !^{\circ}_{X}.$$

In diagrams,

The category of maps of $\mathbf{C}$, denoted by $\mathtt{Map}(\mathbf{C})$, is the subcategory of $\mathbf{C}$ having as objects those of $\mathbf{C}$ and as arrows only the maps.

Proposition 4.4. *$\mathtt{Map}(\mathbf{C})$ is a cartesian category: the final object is I and the product is $\otimes$.*

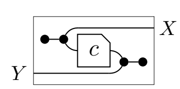
Proof. See Theorem 1.6 in [CW87]. □

The cartesianity of $\mathtt{Map}(\mathbf{C})$ provide several properties of maps that will be useful later.

Lemma 4.5. *In a cartesian bicategory $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ the following holds:*
(1) *For all objects X, $id^{\circ}_{X} \colon X \to X$, $\blacktriangleleft^{\circ}_{X} \colon X \to X \otimes X$ and $!^{\circ}_{X} \colon X \to I$ are maps;*
(2) *For maps a and b properly typed, $a ⨾ b$ and $a \otimes b$ are maps;*
(3) *If $a \colon I \to I$ is a map, then $a = id^{\circ}_{I}$;*
(4) *If $a \colon I \to X \otimes Y$ is a map, then there are maps $c \colon I \to X, d \colon I \to Y$ such that $a = c \otimes d$.*

Proof. See Theorem 1.6 in [CW87]. □

Given a cartesian bicategory $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$, one can take $\mathbf{C}^{\mathsf{op}}$, swap monoids and comonoids and thus, obtain a cartesian bicategory $(\mathbf{C}^{\mathsf{op}}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ})$. Most importantly, there is an identity on objects isomorphism $(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ defined for all arrows $c \colon X \to Y$ as

$c^{\dagger} \stackrel{\text{def}}{=}$  (4.1)

Note that in § 2, we used the same symbol $(\cdot)^{\dagger}$ to denote the converse relation. This is no accident: in $\mathbf{Rel}^{\circ}$, $R^{\dagger}$ as in (4.1) is exactly $\{(y,x) \mid (x,y) \in R\}$.

Proposition 4.6. *$(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{op}$ is an isomorphism of cartesian bicategories, namely the laws in the first three rows of Table 2.(a) hold.*

Proof. See Theorem 2.4 in [CW87]. □

Lemma 4.7. *Let $\mathcal{F} \colon \mathbf{C_1} \to \mathbf{C_2}$ be a morphism of cartesian bicategories. Then, for all $c \colon X \to Y$, $\mathcal{F}(c)^{\dagger} = \mathcal{F}(c^{\dagger})$.*

Proof. See Remark 2.9 in [CW87]. □

The following result generalises the well-known fact that a relation R is a function if and only if it is left adjoint to $R^{\dagger}$.

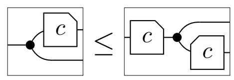
Proposition 4.8. *In a cartesian bicategory, an arrow $c\colon X \to Y$ is a map iff $c^\dagger \vdash c$, namely*

$$id_X^\circ \leq c \,;\, c^\dagger \qquad \textit{and} \qquad c^\dagger \,;\, c \leq id_Y^\circ .$$

Proof. See Lemma 2.5 in [CW87]. □

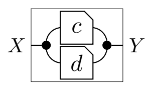
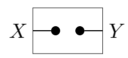
Hereafter, we write c (mirrored box) for $\boxed{c}^\dagger$ and we call it the *mirror image* of $\boxed{c}$. With this notation, we can nicely express the following result that we will often use in our diagrammatic proofs.

Lemma 4.9. *In a cartesian bicategory, the following holds for every arrow $c\colon X \to Y$.*



Proof. See e.g. [BPS17]. □

For all objects X, Y and arrows $c, d\colon X \to Y$, one can define $c \sqcap d$ and $\top$ as follows.

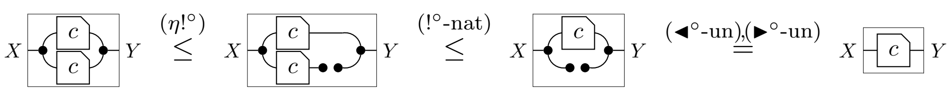
$c \sqcap d \stackrel{\text{def}}{=}$  $\qquad \top \stackrel{\text{def}}{=}$  (4.2)

We have already seen in Example 3.2 that these diagram, when interpreted in $\mathbf{Rel}^\circ$, denote respectively intersection and top. It is easy to show that in any cartesian bicategory $\mathbf{C}$, $\sqcap$ and $\top$ form a *meet-semilattice with top*, namely $\sqcap$ is associative, commutative, idempotent and has $\top$ as unit.

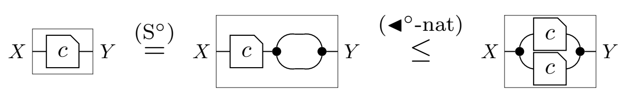
Lemma 4.10. *For all arrows $c, d, e\colon X \to Y$ of a cartesian bicategory, the followings hold.*

$$(c \sqcap d) \sqcap e = c \sqcap (d \sqcap e) \qquad c \sqcap \top = c \qquad c \sqcap d = d \sqcap c \qquad c \sqcap c = c$$

Proof. The first three equalities — associativity, unitality and commutativity — follow directly from (◀°-as), (◀°-un), (◀°-co), as well as (▶°-as), (▶°-un), (▶°-co). The last equality, idempotency, is proved diagrammatically below, where we show the two inclusions separately:



and



□

Remark 4.11. Note that, however, $\mathbf{C}$ is usually *not* enriched over meet-semilattices since $;$ distributes only laxly over $\sqcap$. Indeed, in $\mathbf{Rel}^\circ$,

$$R \,;\, (S \cap T) \subseteq (R \,;\, S) \cap (R \,;\, T)$$

holds for all (properly typed) relations R, S, T, but the reverse does not.

Let us now turn to *co*cartesian bicategories. Our main example is $(\mathbf{Rel}^\bullet, \blacktriangleleft^\bullet, \blacktriangleright^\bullet)$. $\mathbf{Rel}^\bullet$ is the bicategory of sets and relations ordered by $\subseteq$ with composition $\bullet\!;$, identities $id^\bullet$ and $\otimes$ defined as in (2.1), (2.2) and (3.3). Comonoids $(\blacktriangleleft^\bullet_X, !^\bullet_X)$ and monoids $(\blacktriangleright^\bullet_X, ¡^\bullet_X)$ are those of (3.1). To see that $\mathbf{Rel}^\bullet$ is a cocartesian bicategory, observe that the complement $\overline{(\cdot)}$ is a poset-enriched symmetric monoidal isomorphism $\overline{(\cdot)}\colon (\mathbf{Rel}^\circ)^{\mathsf{co}} \to \mathbf{Rel}^\bullet$ preserving (co)monoids.

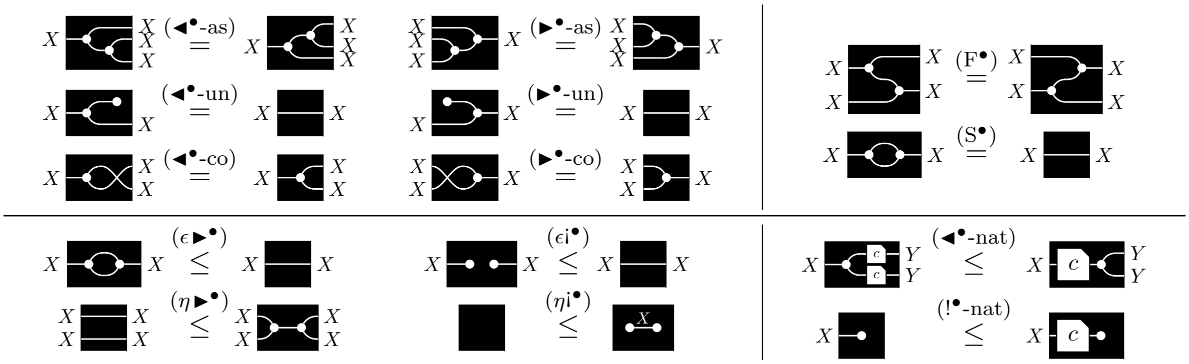
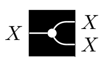
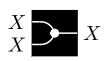
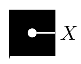
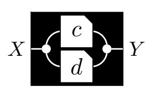
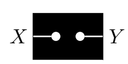


FIGURE 4. Axioms of cocartesian bicategories

We draw arrows of cocartesian bicategories in black: $◀^\bullet_X$, $!^\bullet_X$, $▶^\bullet_X$ and $¡^\bullet_X$ are drawn , ,  and . Following this convention, the axioms of cocartesian bicategories are in Fig. 4; they can also be obtained from Fig. 3 by inverting both the colours and the order.

It is not surprising that in a cocartesian bicategory $\mathbf{C}$, every homset $\mathbf{C}[X, Y]$ carries a join semi-lattice with bottom, where $c \sqcup d$ and $\bot$ are defined for all arrows $c, d \colon X \to Y$ as follows.

$$c \sqcup d \stackrel{\text{def}}{=}$$  $$\bot \stackrel{\text{def}}{=}$$  (4.3)

Lemma 4.12. *For all arrows $c, d, e \colon X \to Y$ of a cocartesian bicategory, the followings hold.*

$$(c \sqcup d) \sqcup e = c \sqcup (d \sqcup e) \qquad c \sqcup \bot = c \qquad c \sqcup d = d \sqcup c \qquad c \sqcup c = c$$

Proof. The proof is analogous to that of Lemma 4.10, but relies on the axioms in Figure 4. □

5. LINEAR BICATEGORIES

We have seen that $\mathbf{Rel}^\circ$ forms a cartesian, and $\mathbf{Rel}^\bullet$ a cocartesian bicategory. Categorically, they are remarkably similar — as evidenced by the isomorphism $\overline{(\cdot)}$ — but from a logical viewpoint they represent two complimentary parts of FOL: $\mathbf{Rel}^\circ$ the existential conjunctive fragment, and $\mathbf{Rel}^\bullet$ the universal disjunctive fragment. To discover the full story, we must merge them into one entity and study the algebraic interactions between them. However, the coexistence of two different compositions $\fatsemi$ and $\boldsymbol{;}$ brings us out of the realm of ordinary categories. The solution is linear bicategories [CKS00]. Here $\fatsemi$ linearly distributes over $\boldsymbol{;}$, as in Pierce's calculus. To keep our development easier, we stick to the poset enriched case and rely on diagrams, using white and black to distinguish $\fatsemi$ and $\boldsymbol{;}$.

Definition 5.1. A *linear bicategory* $(\mathbf{C}, \fatsemi, id^\circ, \boldsymbol{;}, id^\bullet)$ consists of two poset enriched categories $(\mathbf{C}, \fatsemi, id^\circ)$ and $(\mathbf{C}, \boldsymbol{;}, id^\bullet)$ with the same objects, arrows and orderings but possibly different identities and compositions such that $\fatsemi$ linearly distributes over $\boldsymbol{;}$, i.e., the following hold.

$$a \fatsemi (b \boldsymbol{;} c) \stackrel{(\delta_l)}{\leq} (a \fatsemi b) \boldsymbol{;} c \qquad (a \boldsymbol{;} b) \fatsemi c \stackrel{(\delta_r)}{\leq} a \boldsymbol{;} (b \fatsemi c)$$

Definition 5.2. A *symmetric monoidal linear bicategory* $(\mathbf{C}, \mathbin{\mathring{;}}, id^{\circ}, \mathbin{\overset{\bullet}{;}}, id^{\bullet}, \otimes, \sigma^{\circ}, \boldsymbol{\otimes}, \sigma^{\bullet}, I)$, shortly $(\mathbf{C}, \otimes, \boldsymbol{\otimes}, I)$, consists of a linear bicategory $(\mathbf{C}, \mathbin{\mathring{;}}, id^{\circ}, \mathbin{\overset{\bullet}{;}}, id^{\bullet})$ and two poset enriched symmetric monoidal categories $(\mathbf{C}, \otimes, I)$ and $(\mathbf{C}, \boldsymbol{\otimes}, I)$ such that $\otimes$ and $\boldsymbol{\otimes}$ agree on objects, i.e., $X \otimes Y = X \boldsymbol{\otimes} Y$, share the same unit I and

(1) there are linear strengths for $(\otimes, \boldsymbol{\otimes})$, i.e.,:

$$
\begin{array}{lclclcl}
(a \mathbin{\overset{\bullet}{;}} b) \otimes (c \mathbin{\overset{\bullet}{;}} d) & \overset{(\nu_l^{\circ})}{\leq} & (a \otimes c) \mathbin{\overset{\bullet}{;}} (b \boldsymbol{\otimes} d) & & (a \boldsymbol{\otimes} c) \mathbin{\mathring{;}} (b \otimes d) & \overset{(\nu_l^{\bullet})}{\leq} & (a \mathbin{\mathring{;}} b) \boldsymbol{\otimes} (c \mathbin{\mathring{;}} d) \\
(a \mathbin{\overset{\bullet}{;}} b) \otimes (c \mathbin{\overset{\bullet}{;}} d) & \overset{(\nu_r^{\circ})}{\leq} & (a \boldsymbol{\otimes} c) \mathbin{\overset{\bullet}{;}} (b \otimes d) & & (a \otimes c) \mathbin{\mathring{;}} (b \boldsymbol{\otimes} d) & \overset{(\nu_r^{\bullet})}{\leq} & (a \mathbin{\mathring{;}} b) \boldsymbol{\otimes} (c \mathbin{\mathring{;}} d)
\end{array}
$$

(2) the black tensor $\boldsymbol{\otimes}$ preserves id° colaxly and $\otimes$ preserves $id^{\bullet}$ laxly, i.e.,;

$$
id^{\circ}_{X \otimes Y} \overset{(\boldsymbol{\otimes}^{\circ})}{\leq} id^{\circ}_X \boldsymbol{\otimes} id^{\circ}_Y \qquad id^{\bullet}_X \otimes id^{\bullet}_Y \overset{(\otimes^{\bullet})}{\leq} id^{\bullet}_{X \boldsymbol{\otimes} Y}
$$

A *morphism of symmetric monoidal linear bicategories* $\mathcal{F} \colon (\mathbf{C_1}, \otimes, \boldsymbol{\otimes}, I) \to (\mathbf{C_2}, \otimes, \boldsymbol{\otimes}, I)$ consists of two poset enriched symmetric monoidal functors $\mathcal{F}^{\circ} \colon (\mathbf{C_1}, \otimes, I) \to (\mathbf{C_2}, \otimes, I)$ and $\mathcal{F}^{\bullet} \colon (\mathbf{C_1}, \boldsymbol{\otimes}, I) \to (\mathbf{C_2}, \boldsymbol{\otimes}, I)$ that agree on objects and arrows, namely $\mathcal{F}^{\circ}(X) = \mathcal{F}^{\bullet}(X)$ and $\mathcal{F}^{\circ}(c) = \mathcal{F}^{\bullet}(c)$ for all objects X and arrows c.

Remark 5.3. In the literature $\mathbin{\mathring{;}}$, id°, $\mathbin{\overset{\bullet}{;}}$ and $id^{\bullet}$ are written with the linear logic notation $\otimes$, $\top$, $\oplus$ and $\perp$. Modulo this, the traditional notion of linear bicategory (Definition 2.1 in [CKS00]) coincides with the one in Definition 5.1 whenever the 2-structure is collapsed to a poset. Monoidal products on linear bicategories are not much studied although the axioms in Definition 5.2.(1) already appeared in [Nae]. They are the linear strengths of the pair $(\otimes, \boldsymbol{\otimes})$ seen as a linear functor (Definition 2.4 in [CKS00]), a notion of morphism that crucially differs from ours on the fact that the $\mathcal{F}^{\circ}$ and $\mathcal{F}^{\bullet}$ may not coincide on arrows. Instead the inequalities $(\otimes^{\bullet})$ and $(\boldsymbol{\otimes}^{\circ})$ are, to the best of our knowledge, novel. Beyond being natural, they are crucial for Lemma 5.4 below.

Fig. 5 illustrates the diagrams corresponding to the axioms of Definition 5.1 in the top-left corner and Definition 5.2 in the bottom.

All linear bicategories in this paper are symmetric monoidal. We therefore omit the adjective *symmetric monoidal* and refer to them simply as linear bicategories. For a linear bicategory $(\mathbf{C}, \otimes, \boldsymbol{\otimes}, I)$, we will often refer to $(\mathbf{C}, \otimes, I)$ as the *white structure*, shorthand $\mathbf{C}^{\circ}$, and to $(\mathbf{C}, \boldsymbol{\otimes}, I)$ as the *black structure*, $\mathbf{C}^{\bullet}$. Note that a morphism $\mathcal{F}$ is a mapping of objects and arrows that preserves the ordering, the white and black structures; thus we write $\mathcal{F}$ for both $\mathcal{F}^{\circ}$ and $\mathcal{F}^{\bullet}$.

If $(\mathbf{C}, \otimes, \boldsymbol{\otimes}, I)$ is linear bicategory then $(\mathbf{C}^{\mathsf{op}}, \otimes, \boldsymbol{\otimes}, I)$ is a linear bicategory. Similarly $(\mathbf{C}^{\mathsf{co}}, \boldsymbol{\otimes}, \otimes, I)$, the bicategory obtained from $\mathbf{C}$ by reversing the ordering and swapping the white and the black structure, is a linear bicategory.

Our main example is the linear bicategory $(\mathbf{Rel}, \otimes, \boldsymbol{\otimes}, \mathbb{1})$ of sets and relations ordered by $\subseteq$. The white structure is the symmetric monoidal category $(\mathbf{Rel}^{\circ}, \otimes, \mathbb{1})$, introduced in the previous section and the black structure is $(\mathbf{Rel}^{\bullet}, \boldsymbol{\otimes}, \mathbb{1})$. Observe that the two have the same objects, arrows and ordering. The white and black monoidal products $\otimes$ and $\boldsymbol{\otimes}$ agree on objects and are the cartesian product of sets. As common unit object, they have the singleton set $\mathbb{1}$. We already observed in (2.5) that the white composition $\mathbin{\mathring{;}}$ distributes over $\mathbin{\overset{\bullet}{;}}$ and thus (δ_l) and (δ_r) hold. By using the definitions in (2.1), (2.2) and (3.3), the reader can easily check also the inequalities in Definition 5.2.(1) and (2).

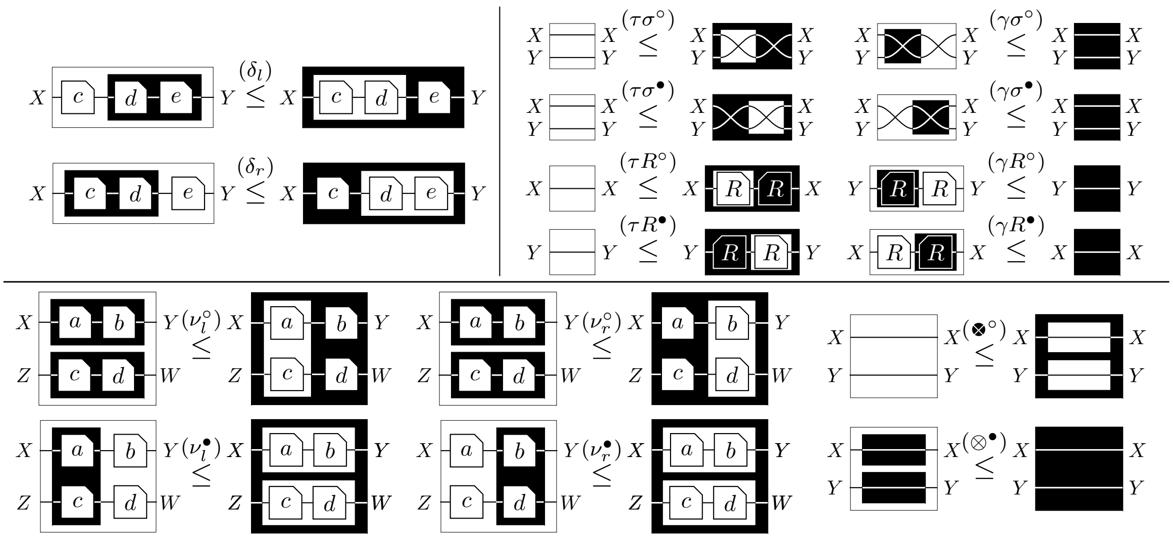


FIGURE 5. Axioms of closed symmetric monoidal linear bicategories

Lemma 5.4. *Let $(\mathbf{C}, \otimes, \boldsymbol{\otimes}, I)$ be a linear bicategory. For all arrows a, b, c the following hold:*

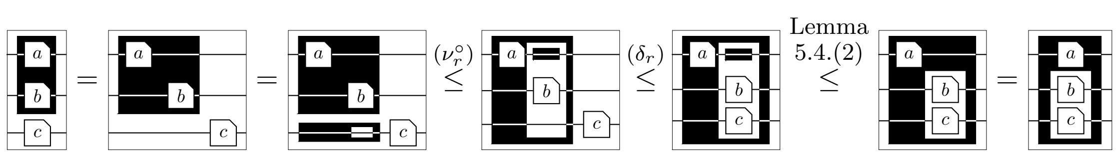
$$(1)\ id_I^{\bullet} \leq id_I^{\circ} \qquad (2)\ a \otimes b \leq a \boldsymbol{\otimes} b \qquad (3)\ (a \boldsymbol{\otimes} b) \otimes c \leq a \boldsymbol{\otimes} (b \otimes c).$$

Proof. The proof of (1) is on the left and (2) on the right:

$$\begin{aligned} id_I^{\bullet} &= id_I^{\bullet} \fatsemi id_I^{\circ} \\ &= id_I^{\bullet} \fatsemi (id_I^{\bullet} \mathbin{\boldsymbol{;}} id_I^{\circ}) \\ &\leq (id_I^{\bullet} \fatsemi id_I^{\bullet}) \mathbin{\boldsymbol{;}} id_I^{\circ} && (\delta_l) \\ &= (id_I^{\bullet} \otimes id_I^{\bullet}) \mathbin{\boldsymbol{;}} id_I^{\circ} && (\text{SMC}) \\ &\leq (id_I^{\bullet} \boldsymbol{\otimes} id_I^{\bullet}) \mathbin{\boldsymbol{;}} id_I^{\circ} && (\otimes^{\bullet}) \\ &= id_I^{\circ} \end{aligned}$$

$$\begin{aligned} a \otimes b &= (a \mathbin{\boldsymbol{;}} id^{\bullet}) \otimes (b \mathbin{\boldsymbol{;}} id^{\bullet}) \\ &\leq (a \boldsymbol{\otimes} b) \mathbin{\boldsymbol{;}} (id^{\bullet} \otimes id^{\bullet}) && (\nu_r^{\circ}) \\ &\leq (a \boldsymbol{\otimes} b) \mathbin{\boldsymbol{;}} (id^{\bullet} \boldsymbol{\otimes} id^{\bullet}) && (\otimes^{\bullet}) \\ &= a \boldsymbol{\otimes} b \end{aligned}$$

The proof of (3) is given diagrammatically as follows:



□

Remark 5.5. As $\otimes$ linearly distributes over $\boldsymbol{\otimes}$, it may seem that symmetric monoidal linear bicategories of Definition 5.2 are linearly distributive [dP91, CS97b]. Moreover (1), (2) of Lemma 5.4 may suggest that they are mix categories [CS97a]. This is not the case: functoriality of $\otimes$ over $\mathbin{\boldsymbol{;}}$ and of $\boldsymbol{\otimes}$ over $\fatsemi$ fails in general.

5.1. **Closed linear bicategories.** In § 4, we recalled adjoints of arrows in bicategories; in linear bicategories one can define *linear* adjoints. For $a \colon X \to Y$ and $b \colon Y \to X$, a is *left linear adjoint* to b, or b is *right linear adjoint* to a, written $b \Vdash a$, if $id^{\circ}_X \leq a \overset{\bullet}{;} b$ and $b \overset{\circ}{;} a \leq id^{\bullet}_Y$.

Next we discuss some properties of right linear adjoints. Those of left adjoints are analogous but they do not feature in our exposition since in the categories of interest — in next section — left and right linear adjoint coincide. As expected, linear adjoints are unique.

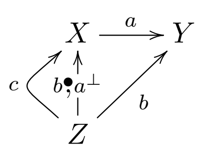
Lemma 5.6. *If $b \Vdash a$ and $c \Vdash a$, then $b = c$.*

Proof. By the following two derivations.

$$\begin{aligned} b &= b \overset{\circ}{;} id^{\circ}_X \\ &\leq b \overset{\circ}{;} (a \overset{\bullet}{;} c) && (c \Vdash a) \\ &\leq (b \overset{\circ}{;} a) \overset{\bullet}{;} c && (\delta_l) \\ &\leq id^{\bullet}_Y \overset{\bullet}{;} c && (b \Vdash a) \\ &= c \end{aligned} \qquad \Bigg| \qquad \begin{aligned} c &= c \overset{\circ}{;} id^{\circ}_X \\ &\leq c \overset{\circ}{;} (a \overset{\bullet}{;} b) && (b \Vdash a) \\ &\leq (c \overset{\circ}{;} a) \overset{\bullet}{;} b && (\delta_l) \\ &\leq id^{\bullet}_Y \overset{\bullet}{;} b && (c \Vdash a) \\ &= b \end{aligned}$$

□

By virtue of the above result we can write $a^{\perp} \colon Y \to X$ for *the* right linear adjoint of $a \colon X \to Y$. With this notation one can write the *left residual* of $b \colon Z \to Y$ by $a \colon X \to Y$ as $b \overset{\bullet}{;} a^{\perp} \colon Z \to X$. The left residual is the greatest arrow $Z \to X$ making the diagram below commute laxly in $\mathbf{C}^{\circ}$, namely if $c \overset{\circ}{;} a \leq b$ then $c \leq b \overset{\bullet}{;} a^{\perp}$.



When a and b have the same type and c is the identity, the principle of residuation provides an alternative characterisation of the order on the arrows.

Lemma 5.7 (Residuation)**.** *$a \leq b$ iff $id^{\circ}_X \leq b \overset{\bullet}{;} a^{\perp}$.*

Proof. In the leftmost derivation we prove $a \leq b \Rightarrow id^{\circ}_X \leq b \overset{\bullet}{;} a^{\perp}$ and in the rightmost $a \leq b \Leftarrow id^{\circ}_X \leq b \overset{\bullet}{;} a^{\perp}$.

$$\begin{aligned} id^{\circ}_X &\leq a \overset{\bullet}{;} a^{\perp} && (a^{\perp} \Vdash a) \\ &\leq b \overset{\bullet}{;} a^{\perp} && (a \leq b) \end{aligned} \qquad \Bigg| \qquad \begin{aligned} a &= id^{\circ}_X \overset{\circ}{;} a \\ &\leq (b \overset{\bullet}{;} a^{\perp}) \overset{\circ}{;} a && (id^{\circ}_X \leq b \overset{\bullet}{;} a^{\perp}) \\ &\leq b \overset{\bullet}{;} (a^{\perp} \overset{\circ}{;} a) && (\delta_r) \\ &\leq b \overset{\bullet}{;} id^{\bullet}_Y && (a^{\perp} \Vdash a) \\ &= b \end{aligned}$$

□

Definition 5.8. A linear bicategory $(\mathbf{C}, \otimes, \boldsymbol{\otimes}, I)$ is said to be *closed* if every $a \colon X \to Y$ has both a left and a right linear adjoint and the white symmetry is both left and right linear adjoint to the black symmetry, i.e., the following inequalities hold.

$$id^{\circ}_{n+m} \overset{(\tau\sigma^{\circ})}{\leq} \sigma^{\circ}_{n,m} \overset{\bullet}{;} \sigma^{\bullet}_{m,n} \quad \sigma^{\bullet}_{n,m} \overset{\circ}{;} \sigma^{\circ}_{m,n} \overset{(\gamma\sigma^{\circ})}{\leq} id^{\bullet}_{n+m} \quad id^{\circ}_{n+m} \overset{(\tau\sigma^{\bullet})}{\leq} \sigma^{\bullet}_{n,m} \overset{\bullet}{;} \sigma^{\circ}_{m,n} \quad \sigma^{\circ}_{n,m} \overset{\circ}{;} \sigma^{\bullet}_{m,n} \overset{(\gamma\sigma^{\bullet})}{\leq} id^{\bullet}_{n+m}$$

Remark 5.9. The top-right corner of Fig. 5 contains the diagrammatic representation of the four axioms for the symmetries. The remaining four axioms are intended for NPR_Σ: for all generators $R \in \Sigma$, R° is both left and right linear adjoint to $R^\bullet$. As we will formally show in §6.1, these conditions guarantee that *all* diagrams of NPR_Σ have left and right linear adjoints and thus they give rise to a *closed* linear bicategory.

As expected, $(\mathbf{Rel}, \otimes, \circledast, \mathbb{1})$ is a a closed linear bicategory: both left and right linear adjoints of a relation $R \subseteq X \times Y$ are given by $\overline{R}^\dagger = \{(y,x) \mid (x,y) \notin R\} \subseteq Y \times X$. With this, it is easy to see that $\sigma^\bullet \Vdash \sigma^\circ \Vdash \sigma^\bullet$ in $\mathbf{Rel}$.

Observe that if a linear bicategory $(\mathbf{C}, \otimes, \circledast, I)$ is closed, then also $(\mathbf{C}^{\mathsf{op}}, \otimes, \circledast, I)$ and $(\mathbf{C}^{\mathsf{co}}, \circledast, \otimes, I)$ are closed. The assignment $a \mapsto a^\perp$ gives rise to an identity on objects functor $(\cdot)^\perp \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$.

Proposition 5.10. *$(\cdot)^\perp \colon \mathbf{C} \to (\mathbf{C}^{co})^{op}$ is a morphism of linear bicategories, i.e., the laws in the first two columns of Table 2.(b) hold.*

Proof. The laws for $;$, id°, $\bullet$ and $id^\bullet$ are well-known, e.g. from [CKS00]. For convenience of the reader, we report anyway their proofs in Appendix C. The remaining cases are illustrated below where a and b range over arbitrary morpshims $a \colon X_1 \to Y_1$ and $b \colon X_2 \to Y_2$.

- $(a \otimes b)^\perp = a^\perp \circledast b^\perp$. The following two derivations

$$\begin{array}{lr} id^\circ_{X_1 \otimes X_2} & \\ = id^\circ_{X_1} \otimes id^\circ_{X_2} & \\ \leq (a \bullet a^\perp) \otimes (b \bullet b^\perp) & (a^\perp \Vdash a\ ,\ b^\perp \Vdash b) \\ \leq (a \otimes b) \bullet (a^\perp \circledast b^\perp) & (\nu_r^\bullet) \end{array} \qquad \begin{array}{lr} (a^\perp \circledast b^\perp) ; (a \otimes b) & \\ \leq (a^\perp ; a) \circledast (b^\perp ; b) & (\nu_l^\bullet) \\ \leq id^\bullet_{Y_1} \circledast id^\bullet_{Y_2} & (a^\perp \Vdash a\ ,\ b^\perp \Vdash b) \\ = id^\bullet_{Y_1 \circledast Y_2} & \end{array}$$

  show that $(a^\perp \circledast b^\perp) \Vdash (a \otimes b)$. Thus, by Lemma 5.6, $(a \otimes b)^\perp = b^\perp \circledast a^\perp$.
- $(a \circledast b)^\perp = a^\perp \otimes b^\perp$. The following two derivations

$$\begin{array}{lr} id^\circ_{X_1 \otimes X_2} & \\ = id^\circ_{X_1} \otimes id^\circ_{X_2} & \\ \leq (a \bullet a^\perp) \otimes (b \bullet b^\perp) & (a^\perp \Vdash a\ ,\ b^\perp \Vdash b) \\ \leq (a \circledast b) \bullet (a^\perp \otimes b^\perp) & (\nu_r^\circ) \end{array} \qquad \begin{array}{lr} (a^\perp \otimes b^\perp) ; (a \circledast b) & \\ \leq (a^\perp ; a) \circledast (b^\perp ; b) & (\nu_l^\bullet) \\ \leq id^\bullet_{Y_1} \circledast id^\bullet_{Y_2} & (a^\perp \Vdash a\ ,\ b^\perp \Vdash b) \\ = id^\bullet_{Y_1 \circledast Y_2} & \end{array}$$

  show that $(a^\perp \otimes b^\perp) \Vdash (a \circledast b)$. Thus, by Lemma 5.6, $(a \circledast b)^\perp = b^\perp \otimes a^\perp$.
- $(\sigma^\circ)^\perp = \sigma^\bullet$. By axioms $(\tau\sigma^\circ)$ and $(\gamma\sigma^\circ)$.
- $(\sigma^\bullet)^\perp = \sigma^\circ$. By axioms $(\tau\sigma^\bullet)$ and $(\gamma\sigma^\bullet)$.

□

We conclude our exposition of closed linear bicategories with the following result, stating that $(\cdot)^\perp$ commutes with any other morphism of closed linear bicategories.

Lemma 5.11. *Let $\mathcal{F} \colon \mathbf{C_1} \to \mathbf{C_2}$ be a morphism of closed linear bicategories. Then, for all $a \colon X \to Y$ in $\mathbf{C_1}$, $\mathcal{F}(a)^\perp = \mathcal{F}(a^\perp)$.*

Proof. The following two derivations

$$\begin{array}{rlr} id^\circ_X & = \mathcal{F}(id^\circ_X) & \\ & \leq \mathcal{F}(a \bullet a^\perp) & (a^\perp \Vdash a) \\ & = \mathcal{F}(a) \bullet \mathcal{F}(a^\perp) & \end{array} \qquad \begin{array}{rlr} \mathcal{F}(a^\perp) ; \mathcal{F}(a) & = \mathcal{F}(a^\perp ; a) & \\ & \leq \mathcal{F}(id^\bullet_Y) & (a^\perp \Vdash a) \\ & = id^\bullet_Y & \end{array}$$

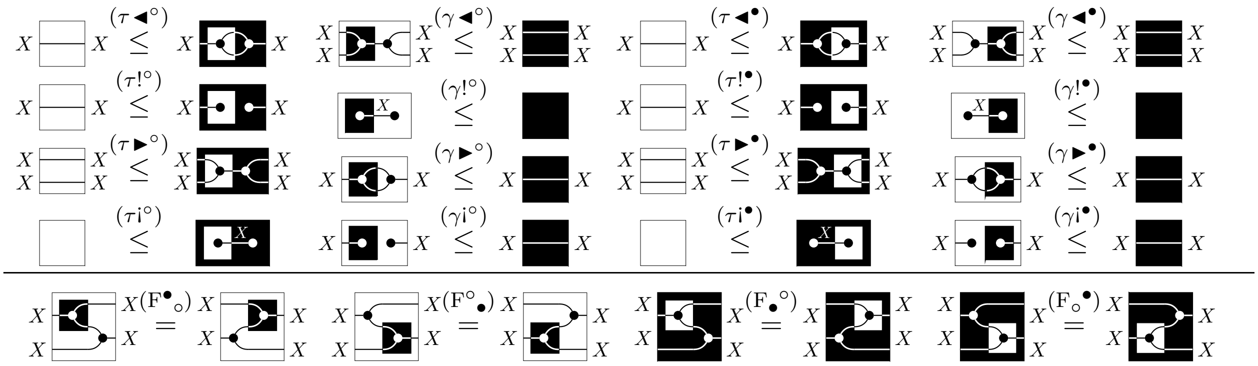


FIGURE 6. Additional axioms for fo-bicategories

show that $\mathcal{F}(a^{\perp}) \Vdash \mathcal{F}(a)$. Thus, by Lemma 5.6, we conclude that $\mathcal{F}(a)^{\perp} = \mathcal{F}(a^{\perp})$. $\square$

Hereafter, the diagram obtained from $\boxed{c}$, by taking its mirror image $\boxed{c}$ and then its photographic negative $\blacksquare_c$ will denote $\boxed{c}^{\perp}$.

6. First Order Bicategories

Here we focus on the most important and novel part of the axiomatisation. Indeed, having introduced the two main ingredients, cartesian and linear bicategories, it is time to fire up the Bunsen burner. The remit of this section is to understand how the cartesian and the linear bicategory structures interact in the context of relations. We introduce *first order bicategories* that make these interactions precise. The resulting axioms echo those of cartesian bicategories but in the linear bicategory setting: recall that in a cartesian bicategory the monoid and comonoids are adjoint and satisfy the Frobenius law. Here, the white and black (co)monoids are again related, but by *linear* adjunctions; moreover, they also satisfy appropriate "linear" counterparts of the Frobenius equations.

Definition 6.1. A *first order bicategory* $(\mathbf{C}, \otimes, \bar{\otimes}, I, \blacktriangleleft^{\circ}, !^{\circ}, \blacktriangleright^{\circ}, ¡^{\circ}, \blacktriangleleft^{\bullet}, !^{\bullet}, \blacktriangleright^{\bullet}, ¡^{\bullet})$, shorthand *fo-bicategory* $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$, consists of

- a closed linear bicategory $(\mathbf{C}, \otimes, \bar{\otimes}, I)$,
- a cartesian bicategory $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ and
- a cocartesian bicategory $(\mathbf{C}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$, such that

(1) the white comonoid $(\blacktriangleleft^{\circ}, !^{\circ})$ is left and right linear adjoint to black monoid $(\blacktriangleright^{\bullet}, ¡^{\bullet})$ and the white monoid $(\blacktriangleright^{\circ}, ¡^{\circ})$ is left and right linear adjoint to black comonoid $(\blacktriangleleft^{\bullet}, !^{\bullet})$, i.e.,

$$
\begin{array}{llll}
id^{\circ}_X \overset{(\tau\blacktriangleleft^{\circ})}{\leq} \blacktriangleleft^{\circ}_X ; \blacktriangleright^{\bullet}_X & \blacktriangleright^{\bullet}_X \mathbin{\mathring{;}} \blacktriangleleft^{\circ}_X \overset{(\gamma\blacktriangleleft^{\circ})}{\leq} id^{\bullet}_{X\bar{\otimes}X} & id^{\circ}_{X\otimes X} \overset{(\tau\blacktriangleright^{\circ})}{\leq} \blacktriangleright^{\circ}_X ; \blacktriangleleft^{\bullet}_X & \blacktriangleleft^{\bullet}_X \mathbin{\mathring{;}} \blacktriangleright^{\circ}_X \overset{(\gamma\blacktriangleright^{\circ})}{\leq} id^{\bullet}_X \\
id^{\circ}_X \overset{(\tau!^{\circ})}{\leq} !^{\circ}_X ; ¡^{\bullet}_X & ¡^{\bullet}_X \mathbin{\mathring{;}} !^{\circ}_X \overset{(\gamma!^{\circ})}{\leq} id^{\bullet}_0 & id^{\circ}_0 \overset{(\tau¡^{\circ})}{\leq} ¡^{\circ}_X ; !^{\bullet}_X & !^{\bullet}_X \mathbin{\mathring{;}} ¡^{\circ}_X \overset{(\gamma¡^{\circ})}{\leq} id^{\bullet}_X \\
id^{\circ}_X \overset{(\tau\blacktriangleleft^{\bullet})}{\leq} \blacktriangleleft^{\bullet}_X ; \blacktriangleright^{\circ}_X & \blacktriangleright^{\circ}_X \mathbin{\mathring{;}} \blacktriangleleft^{\bullet}_X \overset{(\gamma\blacktriangleleft^{\bullet})}{\leq} id^{\bullet}_{X\bar{\otimes}X} & id^{\circ}_{X\otimes X} \overset{(\tau\blacktriangleright^{\bullet})}{\leq} \blacktriangleright^{\bullet}_X ; \blacktriangleleft^{\circ}_X & \blacktriangleleft^{\circ}_X \mathbin{\mathring{;}} \blacktriangleright^{\bullet}_X \overset{(\gamma\blacktriangleright^{\bullet})}{\leq} id^{\bullet}_X \\
id^{\circ}_X \overset{(\tau!^{\bullet})}{\leq} !^{\bullet}_X ; ¡^{\circ}_X & ¡^{\circ}_X \mathbin{\mathring{;}} !^{\bullet}_X \overset{(\gamma!^{\bullet})}{\leq} id^{\bullet}_0 & id^{\circ}_0 \overset{(\tau¡^{\bullet})}{\leq} ¡^{\bullet}_X ; !^{\circ}_X & !^{\circ}_X \mathbin{\mathring{;}} ¡^{\bullet}_X \overset{(\gamma¡^{\bullet})}{\leq} id^{\bullet}_0
\end{array}
$$

(2) white and black (co)monoids satisfy the linear Frobenius laws, i.e.

$$(\blacktriangleleft^{\bullet}_X \otimes id^{\circ}_X) \fatsemi (id^{\circ}_X \otimes \blacktriangleright^{\circ}_X) \overset{(\mathrm{F}^{\bullet}{}_{\circ})}{=} (id^{\circ}_n \otimes \blacktriangleleft^{\circ}_X) \fatsemi (\blacktriangleright^{\bullet}_X \otimes id^{\circ}_X)$$

$$(\blacktriangleleft^{\circ}_X \otimes id^{\circ}_X) \fatsemi (id^{\circ}_X \otimes \blacktriangleright^{\bullet}_X) \overset{(\mathrm{F}^{\circ}{}_{\bullet})}{=} (id^{\circ}_X \otimes \blacktriangleleft^{\bullet}_X) \fatsemi (\blacktriangleright^{\circ}_X \otimes id^{\circ}_X)$$

$$(\blacktriangleleft^{\circ}_X \mathbin{\pmb{\otimes}} id^{\bullet}_X) \mathbin{\pmb{;}} (id^{\bullet}_X \mathbin{\pmb{\otimes}} \blacktriangleright^{\bullet}_X) \overset{(\mathrm{F}_{\bullet}{}^{\circ})}{=} (id^{\bullet}_X \mathbin{\pmb{\otimes}} \blacktriangleleft^{\bullet}_X) \mathbin{\pmb{;}} (\blacktriangleright^{\circ}_X \mathbin{\pmb{\otimes}} id^{\bullet}_X)$$

$$(\blacktriangleleft^{\bullet}_X \mathbin{\pmb{\otimes}} id^{\bullet}_X) \mathbin{\pmb{;}} (id^{\bullet}_X \mathbin{\pmb{\otimes}} \blacktriangleright^{\circ}_X) \overset{(\mathrm{F}_{\circ}{}^{\bullet})}{=} (id^{\bullet}_X \mathbin{\pmb{\otimes}} \blacktriangleleft^{\circ}_X) \mathbin{\pmb{;}} (\blacktriangleright^{\bullet}_X \mathbin{\pmb{\otimes}} id^{\bullet}_X)$$

A *morphism of fo-bicategories* is a morphism of linear, cartesian and cocartesian bicategories.

Remark 6.2. Fig. 6 illustrates the diagrams for the axioms of linear adjointness (top) and linear Frobenius (bottom). The latter, in particular, highlights the intuitive clarity that diagrams offer compared to term-based representations, which in particular clearly exhibit the symmetries of the axiomatisation. Regarding linear adjoints, note that — because of the symmetries involved — expressing the two statements in point (1) of the definition above requires 16 axioms. However, while the axioms governing fo-bicategories are numerous, the information conveyed by them is elegantly and concisely captured, as illustrated by "the Tao of Logic" introduced earlier.

We have seen that $(\mathbf{Rel}, \otimes, \mathbin{\pmb{\otimes}}, \mathbb{1})$ is a closed linear bicategory, $(\mathbf{Rel}^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ a cartesian bicategory and $(\mathbf{Rel}^{\bullet}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ a cocartesian bicategory. Given (3.1), it is easy to confirm linear adjointness and linear Frobenius.

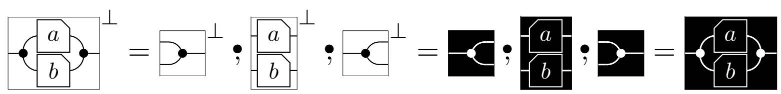
If $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ is a fo-bicategory then $(\mathbf{C}^{\mathsf{op}}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet})$ and $(\mathbf{C}^{\mathsf{co}}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ})$ are fo-bicategories: the laws of Fig. 6 are closed under mirror-reflection and photographic negative. The condition (1) in Definition 6.1 entails that the morphism of linear bicategories $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ (see Proposition 5.10) is a morphism of fo-bicategories and, similarly, the condition (2) that the morphism of cartesian bicategories $(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{op}}$ (see Proposition 4.6) is a morphism of fo-bicategories.

Proposition 6.3. *$(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{op}$ and $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{co})^{op}$ are isomorphisms of fo-bicategories, namely the laws in Table 2.(a) and (b) hold.*

The proof is illustrated in Appendix D.1. Several useful facts easily follow.

Corollary 6.4. *The laws in Table 2.(c) hold.*

Proof. $(c^{\dagger})^{\perp} = (c^{\perp})^{\dagger}$ is immediate from Proposition 6.3 and Lemma 5.11. The other laws are derived by the definitions of $\sqcap$, $\top$, $\sqcup$, $\bot$ in (4.2) and (4.3), and the laws in Tables 2.(a) and 2.(b). For instance,

$(a \sqcap b)^{\perp} =$  $= a^{\perp} \sqcup b^{\perp}.$

□

The next result about maps (Definition 4.3) plays a crucial role.

Proposition 6.5. *For all maps $f \colon X \to Y$ and arrows $c \colon Y \to Z$,*

(1) $f \fatsemi c = (f^{\dagger})^{\perp} \mathbin{\pmb{;}} c$ and (2) $c \fatsemi f^{\dagger} = c \mathbin{\pmb{;}} f^{\perp}$

and thus in particular

 (maps)

(a) Properties of $(\cdot)^{\dagger}\colon (\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}) \to (\mathbf{C}^{\mathrm{op}}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet})$			
if $c \leq d$ then $c^{\dagger} \leq d^{\dagger}$		$(c^{\dagger})^{\dagger} = c$	
$(c \mathbin{\mathring{;}} d)^{\dagger} = d^{\dagger} \mathbin{\mathring{;}} c^{\dagger}$	$(id^{\circ}_X)^{\dagger} = id^{\circ}_X$	$(\blacktriangleright^{\circ}_X)^{\dagger} = \blacktriangleleft^{\circ}_X$	$(¡^{\circ}_X)^{\dagger} = !^{\circ}_X$
$(c \otimes d)^{\dagger} = c^{\dagger} \otimes d^{\dagger}$	$(\sigma^{\circ}_{X,Y})^{\dagger} = \sigma^{\circ}_{Y,X}$	$(\blacktriangleleft^{\circ}_X)^{\dagger} = \blacktriangleright^{\circ}_X$	$(!^{\circ}_X)^{\dagger} = ¡^{\circ}_X$
$(c \mathbin{\dot{;}} d)^{\dagger} = d^{\dagger} \mathbin{\dot{;}} c^{\dagger}$	$(id^{\bullet}_X)^{\dagger} = id^{\bullet}_X$	$(\blacktriangleright^{\bullet}_X)^{\dagger} = \blacktriangleleft^{\bullet}_X$	$(¡^{\bullet}_X)^{\dagger} = !^{\bullet}_X$
$(c \mathbin{\boldsymbol{\otimes}} d)^{\dagger} = c^{\dagger} \mathbin{\boldsymbol{\otimes}} d^{\dagger}$	$(\sigma^{\bullet}_{X,Y})^{\dagger} = \sigma^{\bullet}_{Y,X}$	$(\blacktriangleleft^{\bullet}_X)^{\dagger} = \blacktriangleright^{\bullet}_X$	$(!^{\bullet}_X)^{\dagger} = ¡^{\bullet}_X$

(c) Interaction of $\cdot^{\dagger}, \cdot^{\perp}$ with $\sqcap, \sqcup$	
$(c \sqcap d)^{\dagger} = c^{\dagger} \sqcap d^{\dagger}$	$\top^{\dagger} = \top$
$(c \sqcup d)^{\dagger} = c^{\dagger} \sqcup d^{\dagger}$	$\bot^{\dagger} = \bot$
$(c \sqcap d)^{\perp} = c^{\perp} \sqcup d^{\perp}$	$\top^{\perp} = \bot$
$(c \sqcup d)^{\perp} = c^{\perp} \sqcap d^{\perp}$	$\bot^{\perp} = \top$
$(c^{\dagger})^{\perp} = (c^{\perp})^{\dagger}$	

(b) Properties of $(\cdot)^{\perp}\colon (\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}) \to ((\mathbf{C}^{\mathrm{co}})^{\mathrm{op}}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ})$			
if $c \leq d$ then $c^{\perp} \geq d^{\perp}$		$(c^{\perp})^{\perp} = c$	
$(c \mathbin{\mathring{;}} d)^{\perp} = d^{\perp} \mathbin{\dot{;}} c^{\perp}$	$(id^{\circ}_X)^{\perp} = id^{\bullet}_X$	$(\blacktriangleright^{\circ}_X)^{\perp} = \blacktriangleleft^{\bullet}_X$	$(¡^{\circ}_X)^{\perp} = !^{\bullet}_X$
$(c \otimes d)^{\perp} = c^{\perp} \mathbin{\boldsymbol{\otimes}} d^{\perp}$	$(\sigma^{\circ}_{X,Y})^{\perp} = \sigma^{\bullet}_{Y,X}$	$(\blacktriangleleft^{\circ}_X)^{\perp} = \blacktriangleright^{\bullet}_X$	$(!^{\circ}_X)^{\perp} = ¡^{\bullet}_X$
$(c \mathbin{\dot{;}} d)^{\perp} = d^{\perp} \mathbin{\mathring{;}} c^{\perp}$	$(id^{\bullet}_X)^{\perp} = id^{\circ}_X$	$(\blacktriangleright^{\bullet}_X)^{\perp} = \blacktriangleleft^{\circ}_X$	$(¡^{\bullet}_X)^{\perp} = !^{\circ}_X$
$(c \mathbin{\boldsymbol{\otimes}} d)^{\perp} = c^{\perp} \otimes d^{\perp}$	$(\sigma^{\bullet}_{X,Y})^{\perp} = \sigma^{\circ}_{Y,X}$	$(\blacktriangleleft^{\bullet}_X)^{\perp} = \blacktriangleright^{\circ}_X$	$(!^{\bullet}_X)^{\perp} = ¡^{\circ}_X$

(d) Laws of Boolean algebras	
$c \sqcap (d \sqcup e) = (c \sqcap d) \sqcup (c \sqcap e)$	
$c \sqcup (d \sqcap e) = (c \sqcup d) \sqcap (c \sqcup e)$	
$\overline{(c \sqcap d)} = \overline{c} \sqcup \overline{d}$	$\overline{\top} = \bot$
$\overline{(c \sqcup d)} = \overline{c} \sqcap \overline{d}$	$\overline{\bot} = \top$
$c \sqcap \overline{c} = \bot$	$c \sqcup \overline{c} = \top$

(e) Enrichment over join-meet semilattices		
$c \mathbin{\mathring{;}} (d \sqcup e) = (c \mathbin{\mathring{;}} d) \sqcup (c \mathbin{\mathring{;}} e)$	$(d \sqcup e) \mathbin{\mathring{;}} c = (d \mathbin{\mathring{;}} c) \sqcup (e \mathbin{\mathring{;}} c)$	$c \mathbin{\mathring{;}} \bot = \bot = \bot \mathbin{\mathring{;}} c$
$c \otimes (d \sqcup e) = (c \otimes d) \sqcup (c \otimes e)$	$(d \sqcup e) \otimes c = (d \otimes c) \sqcup (e \otimes c)$	$c \otimes \bot = \bot = \bot \otimes c$
$c \mathbin{\dot{;}} (d \sqcap e) = (c \mathbin{\dot{;}} d) \sqcap (c \mathbin{\dot{;}} e)$	$(d \sqcap e) \mathbin{\dot{;}} c = (d \mathbin{\dot{;}} c) \sqcap (e \mathbin{\dot{;}} c)$	$c \mathbin{\dot{;}} \top = \top = \top \mathbin{\dot{;}} c$
$c \mathbin{\boldsymbol{\otimes}} (d \sqcap e) = (c \mathbin{\boldsymbol{\otimes}} d) \sqcap (c \mathbin{\boldsymbol{\otimes}} e)$	$(d \sqcap e) \mathbin{\boldsymbol{\otimes}} c = (d \mathbin{\boldsymbol{\otimes}} c) \sqcap (e \mathbin{\boldsymbol{\otimes}} c)$	$c \mathbin{\boldsymbol{\otimes}} \top = \top = \top \mathbin{\boldsymbol{\otimes}} c$

Table 2. Properties of first order bicategories.

Proof. The following two derivations prove the two inclusion of (1).

$$
\begin{aligned}
f \mathbin{\mathring{;}} c &= id^{\circ}_X \mathbin{\mathring{;}} f \mathbin{\mathring{;}} c \\
&\leq ((f^{\dagger})^{\perp} \mathbin{\dot{;}} f^{\dagger}) \mathbin{\mathring{;}} f \mathbin{\mathring{;}} c && (f^{\dagger} \Vdash (f^{\dagger})^{\perp}) \\
&\leq (f^{\dagger})^{\perp} \mathbin{\dot{;}} (f^{\dagger} \mathbin{\mathring{;}} f \mathbin{\mathring{;}} c) && (\delta_r) \\
&\leq (f^{\dagger})^{\perp} \mathbin{\dot{;}} (id^{\circ}_Y \mathbin{\mathring{;}} c) && (\text{Proposition 4.8}) \\
&= (f^{\dagger})^{\perp} \mathbin{\dot{;}} c
\end{aligned}
$$

$$
\begin{aligned}
f \mathbin{\mathring{;}} c &= f \mathbin{\mathring{;}} (id^{\bullet}_X \mathbin{\dot{;}} c) \\
&\geq f \mathbin{\mathring{;}} ((f^{\dagger} \mathbin{\mathring{;}} (f^{\dagger})^{\perp}) \mathbin{\dot{;}} c) && (f^{\dagger} \Vdash (f^{\dagger})^{\perp}) \\
&\geq f \mathbin{\mathring{;}} f^{\dagger} \mathbin{\mathring{;}} ((f^{\dagger})^{\perp} \mathbin{\dot{;}} c) && (\delta_l) \\
&\geq id^{\circ}_X \mathbin{\mathring{;}} ((f^{\dagger})^{\perp} \mathbin{\dot{;}} c) && (\text{Proposition 4.8}) \\
&= (f^{\dagger})^{\perp} \mathbin{\dot{;}} c
\end{aligned}
$$

Note that $f^{\dagger} \Vdash (f^{\dagger})^{\perp}$ holds since, by Proposition 6.3, in any fo-bicategory left and right linear adjoint coincide (namely $(a^{\perp})^{\perp} = a$).

To check (2), we use Table 2.(a) and (1): $c \mathbin{\mathring{;}} f^{\dagger} = (f \mathbin{\mathring{;}} c^{\dagger})^{\dagger} = ((f^{\dagger})^{\perp} \mathbin{\dot{;}} c^{\dagger})^{\dagger} = c \mathbin{\dot{;}} f^{\perp}$. For the four equivalence, one concludes by taking as map f either $\blacktriangleleft^{\circ}$ or $!^{\circ}$. □

Recall from Remark 4.11, that cartesian bicategories are *not* enriched over $\sqcap$-semilattices, despite the fact that all homsets carry such structures. Interestingly, in a fo-bicategory, every homset carries a proper lattice, but the white structure *is* enriched over $\sqcup$ and the black structure over $\sqcap$. In **Rel**, this is the well-known fact that $R \mathbin{\mathring{;}} (S \cup T) = (R \mathbin{\mathring{;}} S) \cup (R \mathbin{\mathring{;}} T)$.

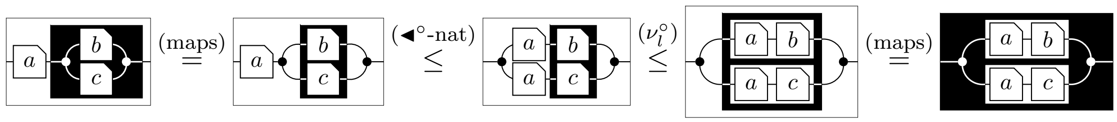
Proposition 6.6. *Let* $(\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet})$ *be a fo-bicategory. Then*

- $(\mathbf{C}, \otimes, I)$ *is monoidally enriched over* $\sqcup$*-semilattices with* $\bot$*, while*
- $(\mathbf{C}, \boldsymbol{\otimes}, I)$ *is monoidally enriched over* $\sqcap$*-semilattices with* $\top$*,*

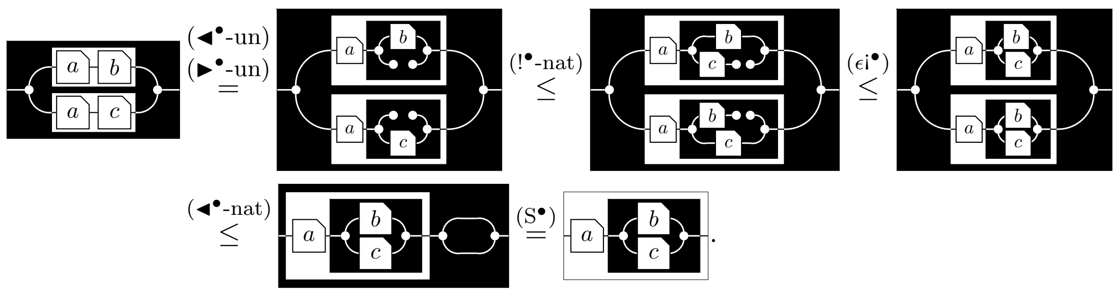
that is the laws in Table 2.(e) hold.

Proof. Below we prove that $(\mathbf{C}, \otimes, I)$ is monoidally enriched over $\sqcup$-semilattices with $\bot$. The proofs for showing that $(\mathbf{C}, \varotimes, I)$ is monoidally enriched over $\sqcap$-semilattices with $\top$ are analogous.

- For $a \fatsemi (b \sqcup c) = (a \fatsemi b) \sqcup (a \fatsemi c)$ we prove the two inclusions separately:

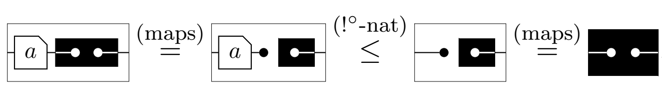


and



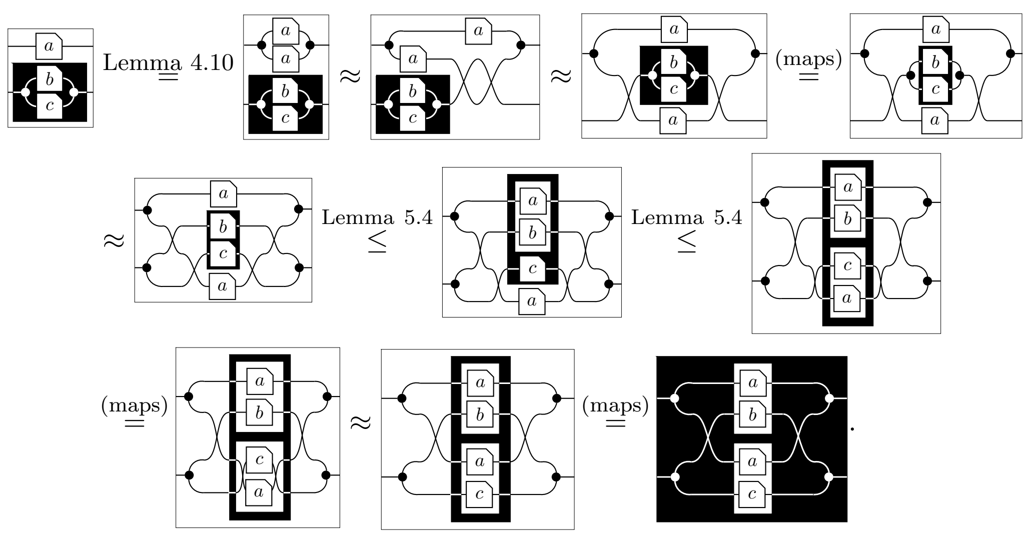
The proof for $(a \sqcup b) \fatsemi c = (a \fatsemi c) \sqcup (b \fatsemi c)$ is similar.

- We prove the left to right inclusion of $a \fatsemi \bot = \bot$:



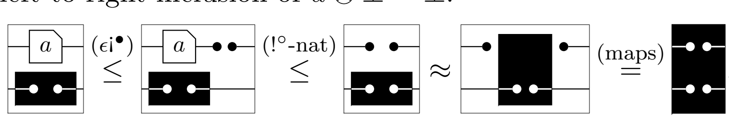
The other inclusion trivially holds. $\bot \fatsemi a = \bot$ is proved analogously.

- The right-to-left inclusion of $a \otimes (b \sqcup c) = (a \otimes b) \sqcup (a \otimes c)$ is proved by the universal property of $\sqcup$, namely: since $a \otimes b = a \otimes (b \sqcup \bot) \leq a \otimes (b \sqcup c)$ and $a \otimes c = a \otimes (\bot \sqcup c) \leq a \otimes (b \sqcup c)$, then $(a \otimes b) \sqcup (a \otimes c) \leq a \otimes (b \sqcup c)$. For the other inclusion, the following holds:



Again, $(a \sqcup b) \otimes c = (a \otimes c) \sqcup (b \otimes c)$ is proved analogously.

- We prove the left to right inclusion of $a \otimes \bot = \bot$:



The other inclusion trivially holds. $\bot \otimes a = \bot$ is proved analogously.

$\square$

For a fo-bicategory $\mathbf{C}$, we have the four isomorphisms in the diagram below, which commutes by Corollary 6.4.

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{(\cdot)^\dagger} & \mathbf{C}^{\mathsf{op}} \\ {\scriptstyle (\cdot)^\bot}\downarrow & & \downarrow{\scriptstyle (\cdot)^\bot} \\ (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}} & \xrightarrow[(\cdot)^\dagger]{} & \mathbf{C}^{\mathsf{co}} \end{array}$$

We can thus define the complement as the diagonal of the square, namely

$$\overline{(\cdot)} \stackrel{\text{def}}{=} ((\cdot)^\bot)^\dagger. \tag{6.1}$$

In diagrams, given $\boxed{c}$, its negation is $(\boxed{c}^\bot)^\dagger = \blacksquare_c^{\,\dagger} = \blacksquare_c$.

Clearly $\overline{(\cdot)} \colon \mathbf{C} \to \mathbf{C}^{\mathsf{co}}$ is an isomorphism of fo-bicategories. Moreover, it induces a Boolean algebra on each homset of $\mathbf{C}$.

Proposition 6.7. *Let $(\mathbf{C}, \blacktriangleleft^\circ, \blacktriangleright^\circ, \blacktriangleleft^\bullet, \blacktriangleright^\bullet)$ be a fo-bicategory. Then every homset of $\mathbf{C}$ is a Boolean algebra, that is the laws in Table 2.(d) hold.*

Proof. • The De Morgan laws follow immediately from the definition of $\overline{(\cdot)}$ and Corollary 6.4. We show only the following two:

$$\begin{aligned} \overline{c \sqcap d} &= ((c \sqcap d)^\bot)^\dagger && \text{(Definition of } \overline{(\cdot)}) \\ &= (c^\bot)^\dagger \sqcup (d^\bot)^\dagger && \text{(Corollary 6.4)} \\ &= \overline{c} \sqcup \overline{d} && \text{(Definition of } \overline{(\cdot)}) \end{aligned} \qquad \begin{aligned} \overline{\top} &= (\top^\bot)^\dagger && \text{(Definition of } \overline{(\cdot)}) \\ &= \bot && \text{(Corollary 6.4)} \end{aligned}$$

• The distributivity of $\sqcap$ over $\sqcup$ follows from the enrichemnt and the definition of $\sqcap$:

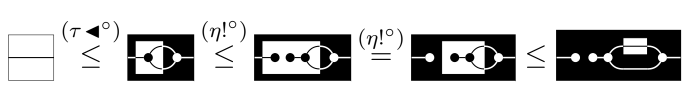
$$\begin{aligned} a \sqcap (b \sqcup c) &= \blacktriangleleft^\circ ; (a \otimes (b \sqcup c)) ; \blacktriangleright^\circ && (4.2) \\ &= \blacktriangleleft^\circ ; ((a \otimes b) \sqcup (a \otimes c)) ; \blacktriangleright^\circ && \text{(Table 2.(e))} \\ &= (\blacktriangleleft^\circ ; (a \otimes b) ; \blacktriangleright^\circ) \sqcup (\blacktriangleleft^\circ ; (a \otimes c) ; \blacktriangleright^\circ) && \text{(Table 2.(e))} \\ &= (a \sqcap b) \sqcup (a \sqcap c). && (4.2) \end{aligned}$$

Simlarly for the distributivity of $\sqcup$ over $\sqcap$.

• We are left to prove the laws of excluded middle and non-contradiction, namely $\top \leq a \sqcup \overline{a}$ and $a \sqcap \overline{a} \leq \bot$. To do that, we first prove them on the identities, namely:

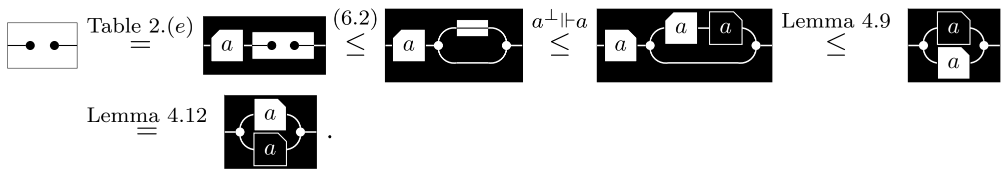
(6.2)

For the first inequality observe that the following holds:



Thus, we can conclude by residuation (Lemma 5.7). The other inequality is proved analogously.

Finally, observe that the following holds:

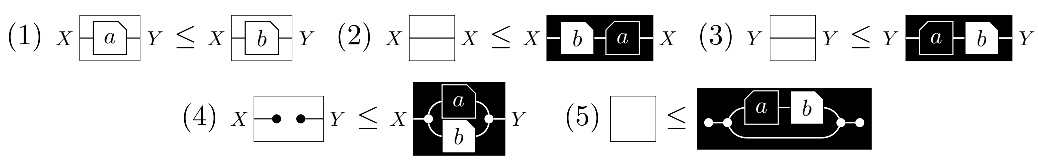


The proof of the other law is analogous.

□

We conclude this section with a result that extends Lemma 5.7 with five different possibilities to express the concept of logical entailment. It is worth emphasising that the following result stands at the core of our proofs. Once again, the diagrammatic approach proves to be an enhancement over the classical syntax. In this specific case we are looking at five (of many) different possibilities to express the ubiquitous concept of logical entailment. (1) expresses a implies b as a direct rewriting of the former into the latter. We have already seen that (2) corresponds to residuation. (3) corresponds to right residuation. (4) asserts the validity of the formula $\neg a \vee b$, thus it corresponds to the classical implication. Finally, (5) may look eccentric but it is actually a closed version of (3) that comes in handy if one has to consider closed diagrams.

Lemma 6.8. *In a fo-bicategory, the following are equivalent:*



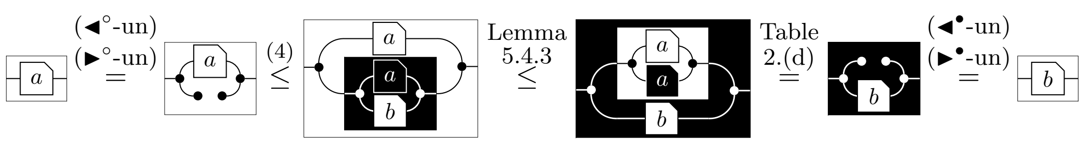
Proof. We prove that (1) is pairwise equivalent to (2), (3) and (4) and that (4) is equivalent to (5).

(1) iff (2) is Lemma 5.7.

(1) iff (3): $a \leq b$ iff $b^{\perp} \leq a^{\perp}$ by the property of $(\cdot)^{\perp}$ in Table 2.(b). By Lemma 5.7, $b^{\perp} \leq a^{\perp}$ iff $id^{\circ}_{Y} \leq a^{\perp} \bullet (b^{\perp})^{\perp}$ where $(b^{\perp})^{\perp} = b$ by the property of $(\cdot)^{\perp}$ in Table 2.(b).

(1) implies (4): $\overline{a} \sqcup b \overset{(1)}{\geq} \overline{a} \sqcup a \overset{\text{Table 2.}(d)}{=} \top$.

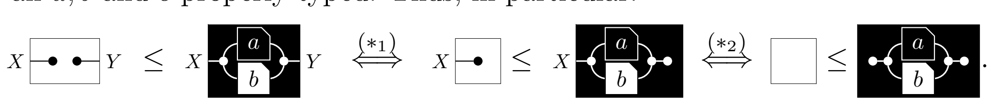
(4) implies (1):



(4) iff (5) holds by means of residuation. In particular, recall that in a fo-bicategory

$$c \leq b \bullet a^{\perp} \overset{(*_1)}{\iff} c \circ a \leq b \overset{(*_2)}{\iff} a \leq c^{\perp} \bullet b$$

for all a, b and c properly typed. Thus, in particular:



Using the compact closed structure of (co)cartesian bicategories it is immediate to show that the last diagram is equivalent to the right-hand side of (5).

□

6.1. **The freely generated first order bicategory.** We now return to NPR_Σ. Recall that $\lesssim$ is the precongruence obtained from the axioms in Figures 3, 4, 5 and 6. Its soundness (half of Theorem 3.4) is immediate since **Rel** is a fo-bicategory.

Proposition 6.9. *For all terms $c, d\colon n \to m$, if $c \lesssim d$ then $c \leqq d$.*

Proof. Let $\mathcal{I} = (X, \rho)$ be an interpretation of Σ. Recall that $\lesssim$ is defined as $\mathtt{pc}(\mathbb{FOB} \cup \approx)$. We prove by induction on the rules in (3.6), that

$$\text{if } c \lesssim d \text{ then } \mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d).$$

By definition of $\leqq$, the above statement is equivalent to the proposition.

The proof for the rules (r) and (t) is trivial. For the rule $(;)$, suppose that $c = c_1 ; c_2$ and $d = d_1 ; d_2$ with $c_1 \lesssim d_1$ and $c_2 \lesssim d_2$. Then

$$\begin{aligned}
\mathcal{I}^\sharp(c) &= \mathcal{I}^\sharp(c_1 ; c_2) \\
&= \mathcal{I}^\sharp(c_1) ; \mathcal{I}^\sharp(c_2) && (3.4) \\
&\subseteq \mathcal{I}^\sharp(d_1) ; \mathcal{I}^\sharp(d_2) && \text{(Ind. hyp.)} \\
&= \mathcal{I}^\sharp(d_1 ; d_2) && (3.4) \\
&= \mathcal{I}^\sharp(d)
\end{aligned}$$

The proof for $(\otimes)$ is analogous to the one above. The only interesting case is the rule (id): we should prove that if $(c, d) \in \mathbb{FOB}$, then $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$. However, we have already done most of the work: since all the axioms in $\mathbb{FOB} \cup \approx$ – with the only exception of the four stating $R^\bullet \Vdash R^\circ \Vdash R^\bullet$ (axioms (τR°), (γR°), $(\tau R^\bullet)$ and $(\gamma R^\bullet)$ in Figure 5) – are those of fo-bicategories and since **Rel** is a fo-bicategory, it only remains to show the soundness of those stating $R^\bullet \Vdash R^\circ \Vdash R^\bullet$. Note however that this is trivial by definition of $\mathcal{I}^\sharp(R^\bullet)$ as $\rho(R)^\perp = (\mathcal{I}^\sharp(R^\circ))^\perp$. □

Next, we show how NPR_Σ gives rise to a fo-bicategory $\mathbf{FOB}_\Sigma$. Objects are natural numbers and monoidal products $\otimes$ are defined as addition with unit object 0. Arrows from n to m are terms $c\colon n \to m$ modulo syntactic equivalence $\cong$, namely $\mathbf{FOB}_\Sigma[n, m] \stackrel{\text{def}}{=} \{[c]_\cong \mid c\colon n \to m\}$. Observe that this is well defined since $\cong$ is well-typed. Since $\cong$ is a congruence, the operations $;$ and $\otimes$ on terms are well defined on equivalence classes: $[t_1]_\cong ; [t_2]_\cong \stackrel{\text{def}}{=} [t_1 ; t_2]_\cong$ and $[t_1]_\cong \otimes [t_2]_\cong \stackrel{\text{def}}{=} [t_1 \otimes t_2]_\cong$. The partial order is given by the syntactic inclusion $\lesssim$. For all objects $n \in \mathbb{N}$, $\blacktriangleleft^\circ_n$, $!^\circ_n$, $\blacktriangleright^\circ_n$ and $¡^\circ_n$ are inductively defined as in Table 1. With this structure, one can easily prove (see Appendix D.2) the following.

Proposition 6.10. $\mathbf{FOB}_\Sigma$ *is a first order bicategory.*

A useful consequence of Proposition 6.10 is that, for any interpretation $\mathcal{I} = (X, \rho)$, the semantics $\mathcal{I}^\sharp$ gives rise to a morphism $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{Rel}$ of fo-bicategories: it is defined on objects as $n \mapsto X^n$ and on arrows by the inductive definition in (3.4). To see that it is a morphism, note that, by (3.4), all the structure of (co)cartesian bicategories and of linear bicategories is preserved (e.g. $\mathcal{I}^\sharp(\blacktriangleleft^\circ_1) = \blacktriangleleft^\circ_X$). Moreover, the ordering is preserved by Prop. 6.9. Note that, by construction,

$$\mathcal{I}^\sharp(1) = X \text{ and } \mathcal{I}^\sharp(R^\circ) = \rho(R) \text{ for all } R \in \Sigma. \tag{6.3}$$

Actually, $\mathcal{I}^\sharp$ is the unique such morphism of fo-bicategories. This is a consequence of a more general universal property: **Rel** can be replaced with an arbitrary fo-bicategory **C**. To see this, we first need to generalise the notion of interpretation.

Definition 6.11. Let Σ be a monoidal signature and **C** a first order bicategory. An *interpretation* $\mathcal{I} = (X, \rho)$ of Σ in **C** consists of an object X of **C** and an arrow $\rho(R)\colon X^n \to X^m$ for each $R \in \Sigma[n, m]$.

With this definition, we can state that $\mathbf{FOB}_\Sigma$ is *the* fo-bicategory freely generated by Σ.

Proposition 6.12. *Let Σ be a monoidal signature, **C** a first order bicategory and $\mathcal{I} = (X, \rho)$ an interpretation of Σ in **C**. There exists a unique morphism of fo-bicategories $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to$ **C** such that $\mathcal{I}^\sharp(1) = X$ and $\mathcal{I}^\sharp(R^\circ) = \rho(R)$ for all $R \in \Sigma$.*

Proof. Observe that the rules in (3.4) defining $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{Rel}$ also defines $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{C}$ for an interpretation $\mathcal{I}$ of Σ in **C** by fixing $\mathcal{I}^\sharp(R^\bullet) = (\mathcal{I}^\sharp(R^\circ))^\perp$. To prove that $\mathcal{I}^\sharp$ preserve the ordering, one can use exactly the same proof of Proposition 6.9. All the structure of (co)cartesian bicateries and linear bicategories is preserved by definition of $\mathcal{I}^\sharp$. Thus, $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{C}$ is a morphism of fo-bicategories. By definition, it also holds that $\mathcal{I}^\sharp(1) = X$ and $\mathcal{I}^\sharp(R^\circ) = \rho(R)$.

To see that it is unique, observe that a morphism $\mathcal{F}\colon \mathbf{FOB}_\Sigma \to \mathbf{C}$ should map the object 0 into I (the unit object of $\otimes$) and any other natural number n into $\mathcal{F}(1)^n$. Thus the only degree of freedom for the objects is the choice of where to map the natural number 1. Similarly, for arrows, the only degree of freedom is where to map R° and $R^\bullet$. However, the axioms in $\mathbb{FOB}$ obliges $R^\bullet$ to be mapped into the right linear adjoint of R°. Thus, by fixing $\mathcal{F}(1) = X$ and $\mathcal{F}(R^\circ) = \rho(R)$, $\mathcal{F}$ is forced to be $\mathcal{I}^\sharp$. □

We conclude this section with another useful consequence of the fact that $\mathcal{I}^\sharp\colon \mathbf{FOB}_\Sigma \to$ **Rel** is a morphism of fo-bicategories.

Lemma 6.13. *Let $\mathcal{I}$ be an interpretation of Σ in **Rel** and let $c\colon n \to m$ be an arrow in $\mathbf{FOB}_\Sigma$. Then*

$$\mathcal{I}^\sharp(c^\dagger) = (\mathcal{I}^\sharp(c))^\dagger, \qquad \mathcal{I}^\sharp(c^\perp) = (\mathcal{I}^\sharp(c))^\perp, \qquad \mathcal{I}^\sharp(\overline{c}) = \overline{(\mathcal{I}^\sharp(c))}.$$

Proof. Since $\mathcal{I}^\sharp$ is a morphism of fo-bicategories the proof for $(\cdot)^\dagger$ and $(\cdot)^\perp$ follows from Lemma 4.7 and Lemma 5.11. Negation is preserved as well, since $\overline{(\cdot)} = (\cdot^\dagger)^\perp$. □

7. Diagrammatic First-Order Theories

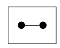
Here we take the first steps towards completeness and show that for first-order theories, fo-bicategories play an analogous role to cartesian categories in Lawvere's functorial semantics of algebraic theories.

Definition 7.1. A *first-order theory* $\mathbb{T}$ is a pair $(\Sigma, \mathbb{I})$ where Σ is a signature and $\mathbb{I}$ is a set of *axioms*: pairs (c, d) where $c, d\colon n \to m$ are in $\mathbf{FOB}_\Sigma$. A *model* of $\mathbb{T}$ is an interpretation $\mathcal{I}$ of Σ where if $(c, d) \in \mathbb{I}$, then $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$.

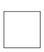
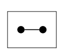
Intuitively, each pair (c, d) in the set $\mathbb{I}$ is an axiom of the theory stating that the diagram c should be included into d. As expected, models of a first-order theory are interpretations of the relational symbols in the signature Σ, such that the inequalities in $\mathbb{I}$ are preserved. Here we show a few meaningful example of first-order theories and comment on their models.

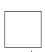
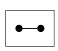
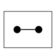
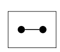
Example 7.2 (Theory of sets)**.** The simplest case is $\mathbb{T} = (\varnothing, \varnothing)$, where both the set of generating symbols and the set of inequations are empty. It is straightforward to verify that interpretations and models of this theory coincide. In particular, every possible set, also the empty set $\varnothing$, is a model of $\mathbb{T}$.

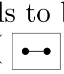
Example 7.3 (Theory of non-empty sets)**.** To discard empty sets from the models of the theory $\mathbb{T}$ defined above, it suffices to impose one single inequality. Consider the theory

$\mathbb{T}' = (\varnothing, \{($  ,  $)\})$.

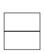
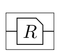
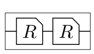
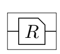
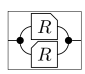
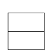
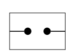
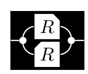
An interpretation $\mathcal{I}$ of $\mathbb{T}'$ is just a set X, since the set of generating symbols is empty. To see what is a model, it is necessary to understand the meaning of the only axiom in $\mathbb{T}'$. By the definition of $\mathcal{I}^\sharp$ in (3.4),

$\mathcal{I}^\sharp($  $) = \{(\star, \star)\}$ $\qquad$ $\mathcal{I}^\sharp($  $) = \{(\star, x) \mid x \in X\} \mathbin{;} \{(x, \star) \mid x \in X\}$

Observe that $\mathcal{I}^\sharp($  $) = \{(\star, \star)\}$ regardless of the interpretation, since X^0 is always the singleton $\mathbb{1}$. Instead, $\mathcal{I}^\sharp($  $)$ depends on the chosen domain X. In particular, if $X \neq \varnothing$, then $\mathcal{I}^\sharp($  $)$ amounts to $\{(\star, \star)\}$, otherwise if $X = \varnothing$, $\mathcal{I}^\sharp($  $) = \varnothing$.

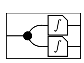
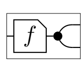
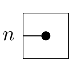
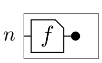
Therefore, the only inequality in $\mathbb{T}'$ forces its models to be all and only non-empty sets, i.e. all those interpretations $\mathcal{I}$ such that $\mathcal{I}^\sharp($  $) \subseteq \mathcal{I}^\sharp($  $)$.

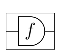
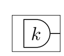
Example 7.4 (Linear orders)**.** Consider $\mathbb{T}_R = (\Sigma_R, \mathbb{I}_R)$, where $\Sigma_R = \{R \colon 1 \to 1\}$ and let $\mathbb{I}_R$ be as follows:

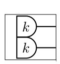
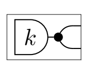
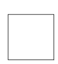
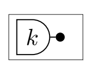
$\{($  ,  $)$, $($  ,  $)$, $($  ,  $)$, $($  ,  $)\}$.

An interpretation of $\mathbb{T}_R$ is a set X together with a relation $R \subseteq X \times X$. This is a model iff R is reflexive (i.e., $id^\circ_X \subseteq R$), transitive ($R \mathbin{;} R \subseteq R$), antisymmetric ($R \cap R^\dagger \subseteq id^\circ_X$) and total relation ($\top \subseteq R \cup R^\dagger$), thus a linear order.

Monoidal signatures Σ, differently from usual FOL alphabets, do not have function symbols. The reason is that, by adding the axioms below to $\mathbb{I}$, one forces a symbol $f \colon n \to 1 \in \Sigma$ to be a function.

n  $\leq$ n  $\qquad$ n  $\leq$ n  $\qquad (\mathbb{M}_f)$

We depict functions as n  and constants, being $0 \to 1$ functions, as  . By the definitions of $\blacktriangleleft^\circ_0$ and $!^\circ_0$ in Table 1, the axioms $(\mathbb{M}_f)$ for constants become the following.

 $\leq$  $\qquad$  $\leq$  $\qquad (\mathbb{M}_k)$

The axioms of a theory together with $\lesssim$ form a deduction system. Formally, the *deduction relation* induced by $\mathbb{T} = (\Sigma, \mathbb{I})$ is the closure (see (3.6)) of $\lesssim \cup\, \mathbb{I}$, i.e.

$$\lesssim_{\mathbb{T}} \stackrel{\text{def}}{=} \mathtt{pc}(\lesssim \cup\, \mathbb{I}). \qquad (\lesssim_{\mathbb{T}})$$

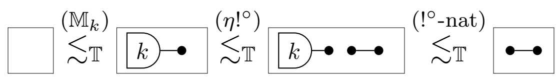
We write $\cong_{\mathbb{T}}$ for $\lesssim_{\mathbb{T}} \cap \gtrsim_{\mathbb{T}}$. The following result proves that the deduction relation is sound, i.e. it preserves all models.

Proposition 7.5. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $c, d \colon n \to m$ in $\mathbf{FOB}_{\mathbb{T}}$. If $c \lesssim_{\mathbb{T}} d$, then $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$ for all models $\mathcal{I}$ of $\mathbb{T}$.*

Proof. By induction on (3.6). For the rule (id), we have two cases: either $(c, d) \in \lesssim$ or $(c, d) \in \mathbb{I}$. For $\lesssim$, we conclude immediately by Proposition 6.9. For $(c, d) \in \mathbb{I}$, the inclusion $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$ holds by definition of model. The proofs for the other rules are trivial. □

Example 7.6 (Theory with constants)**.** Consider the theory $\mathbb{T}$ with $\Sigma = \{k \colon 0 \to 1\}$ and axioms $\mathbb{M}_k$. An interpretation $\mathcal{I}$ of Σ consists of a set X and a relation $k \subseteq \mathbb{1} \times X$. An interpretation is a model if and only if k is a function of type $\mathbb{1} \to X$.

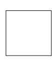
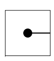
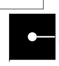
Intuitively, all models of such theories must have non-empty domains, witnessed by the presence of the constant k. Formally, non-emptiness can be proved by the following derivation:

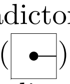
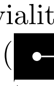
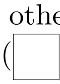


(7.1)

7.1. **Trivial vs Contradictory theories.** The following classes of theories are important for the subtleties of completeness. It is also a remarkable example of where the syntax of NPR_Σ can be more informative than traditional FOL notation.

Definition 7.7. Let $\mathbb{T}$ be a first-order theory.

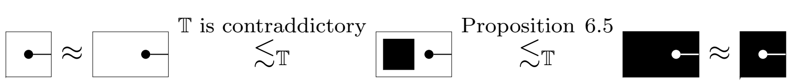
- $\mathbb{T}$ is *contradictory* if  $\lesssim_{\mathbb{T}}$ ;
- $\mathbb{T}$ is *trivial* if  $\lesssim_{\mathbb{T}}$ .

The distinction between contradictory and trivial theories is subtle. Triviality implies all models have empty domain: $\mathcal{I}^\sharp($$) = \{(\star, x) \mid x \in X\}$ is included in $\mathcal{I}^\sharp($$) = \varnothing$ iff $X = \varnothing$. On the other hand, contradictory theories cannot have a model, not even when $X = \varnothing$: since $\mathcal{I}^\sharp($$) = \{(\star, \star)\}$ and $\mathcal{I}^\sharp($$) = \varnothing$, independently of X.

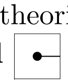
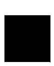
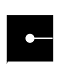
As expected, every contradictory theory is also trivial.

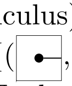
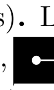
Lemma 7.8. *Let $\mathbb{T}$ be a theory. If $\mathbb{T}$ is contraddictory then it is trivial.*

Proof. Assume $\mathbb{T}$ to be contraddictory and consider the following derivation.



□

Remark 7.9. The difference between contradictory and trivial theories is not usually *seen* in FOL. Indeed, as we will see later in Remark 9.1, both  and  translate to $\top$ (truth) in FOL syntax, while  and  translate to $\bot$ (falsity).

Example 7.10 (The Trivial Theory of Propositional Calculus)**.** Let $\mathbb{P} = (\Sigma, \mathbb{I})$, where Σ contains only symbols $P, Q, R \ldots$ of type $0 \to 0$ and $\mathbb{I} = \{($, $)\}$. In any model of $\mathbb{P}$, the domain X must be $\varnothing$, because of the only axiom in $\mathbb{I}$, that makes $\mathbb{P}$ a trivial theory. A model is a mapping of each of the symbols in Σ to either $\{(\star, \star)\}$ or $\varnothing$. In other words,

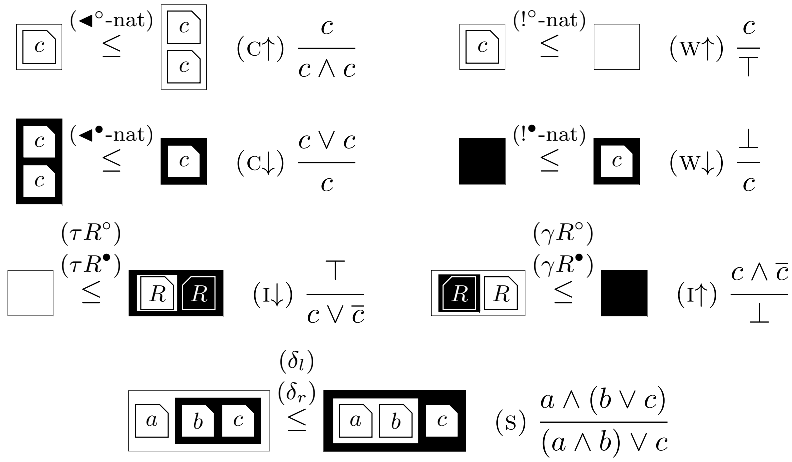
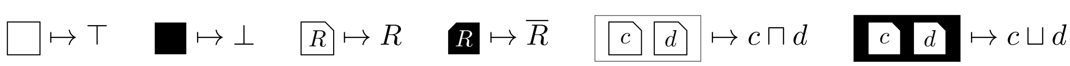


FIGURE 7. The axioms of fo-bicategories reduce to those above for diagrams of type $0 \to 0$.

$P, Q, R, \dots$ act as propositional variables and any model is just an assignment of boolean values.

In $\mathbb{P}$, like in any trivial theory, all diagrams are equal with the exception of those of type $0 \to 0$ (see Lemma E.1 in Appendix E). Diagrams of type $0 \to 0$ are exactly propositional formulas, as illustrated below (see Proposition E.2 in Appendix E for a formal statement).



Note that, by the axioms of symmetric monoidal categories, $;$ and $\otimes$ coincide on diagrams $0 \to 0$ and are associative, commutative and with unit id_0°.

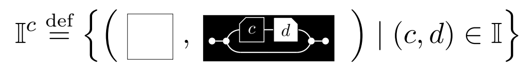
For arrows of type $0 \to 0$ our axiomatisation reduces to the one in Figure 7. Consider for instance (◀°-nat): by definition of $◀_0^\circ$ in Table 1, the two diagrams of (◀°-nat) in Figure 3 reduce to those in Figure 7 above. The rules (ν_l°), (ν_r°), $(\nu_l^\bullet)$ and $(\nu_r^\bullet)$ become redundant. Interestingly, the collapsed axiomatisation corresponds to the rules of the *deep inference calculus of structures* SKSg presented in [Brü03]. The correspondence is illustrated in Figure 7, where each axiom reports its associated SKSg rule on the right.

7.2. **Closed Theories and the Deduction Theorem.** Even though we did not establish yet a formal correspondence between formulas of FOL and diagrams of $\mathbf{FOB}_\Sigma$, one can already guess that dangling wires, either on the left or on the right of a diagram, play the role of free variables. Thus, diagrams $d\colon 0 \to 0$ can be thought of as *closed* formulas of FOL, which also play an important role in our proof of completeness.

Recall that a theory in FOL is usually defined as a set $\mathcal{T}$ of closed formulas that must hold in all models. With a slight abuse of notation, one can think of constructing a corresponding theory in $\mathbf{FOB}_\Sigma$, whose set of axioms is $\{(id_0^\circ, d) \mid d \in \mathcal{T}\}$. Since the semantics $\mathcal{I}^\sharp$ assigns to every diagram $d\colon 0 \to 0$ a relation $R \subseteq \mathbb{1} \times \mathbb{1}$, either $\{(\star, \star)\}$ (i.e., $id_{\mathbb{1}}^\circ$) representing true or $\varnothing$ (i.e., $id_{\mathbb{1}}^\bullet$) representing false, the fact that d must hold in any model is indeed forced by requiring the axiom (id_0°, d). This leads us to the definition of another relevant class of diagrammatic first-order theories, that we call *closed theories*.

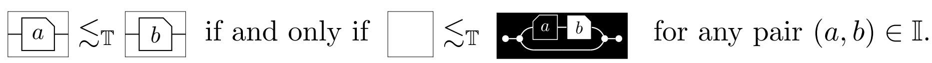
Definition 7.11. A theory $\mathbb{T} = (\Sigma, \mathbb{I})$ is said to be *closed* if all the pairs $(c, d) \in \mathbb{I}$ are of the form (id_0°, d).

For instance, the theory of sets and the theory of non-empty sets in Examples 7.2 and 7.3 are closed, while the other theories encountered so far are not. However, by means of Lemma 6.8, one can always translate an arbitrary theory $\mathbb{T} = (\Sigma, \mathbb{I})$ into a closed theory $\mathbb{T}^c = (\Sigma, \mathbb{I}^c)$ where



Proposition 7.12. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $a, b \colon n \to m$ in* $\mathbf{FOB}_\Sigma$*. Then $a \lesssim_{\mathbb{T}} b$ iff $a \lesssim_{\mathbb{T}^c} b$.*

Proof. By induction on the rules in (3.6). The base case (id) is given by means of Lemma 6.8 and in particular from the fact that:



The base case (r) and the inductive cases are trivial. □

This result allows us to safely restrict our attention to closed theories. However, note that this assumption is not actually needed for the proof of completeness. More interestingly, it tells us that while diagrammatic first-order theories, in general, appear to be rather different from the usual FOL theories, they can always be translated into closed theories which are essentially those of FOL.

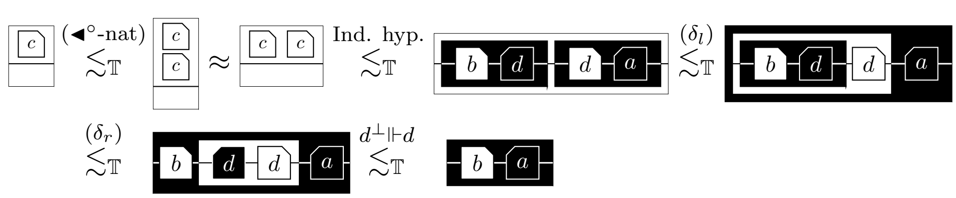
The fact that a closed formula d is derivable in $\mathcal{T}$, usually written as $\mathcal{T} \vdash d$, translates in $\mathbf{FOB}_\Sigma$ to $id_0^\circ \lesssim_{\mathbb{T}} d$. In particular, when d is an implication $c \Rightarrow b$, we have $id_0^\circ \lesssim_{\mathbb{T}} b \,;\, c^\perp$ that, by Lemma 5.7, is equivalent to $c \lesssim_{\mathbb{T}} b$.

In FOL it is trivial – by modus ponens – that if $\mathcal{T} \vdash c \Rightarrow b$ then $\mathcal{T} \cup \{c\} \vdash b$. In $\mathbf{FOB}_\Sigma$, this fact follows by transitivity of $\lesssim_{\mathbb{T}}$: fix $\mathbb{T}' = (\Sigma, \mathbb{I} \cup \{(id_0^\circ, c)\})$ and observe that $id_0^\circ \lesssim_{\mathbb{T}'} c \lesssim_{\mathbb{T}'} b$. The converse implication, namely if $\mathcal{T} \cup \{c\} \vdash b$ then $\mathcal{T} \vdash c \Rightarrow b$, is known in FOL as the *deduction theorem*. It can be generalised in NPR_Σ as follows.

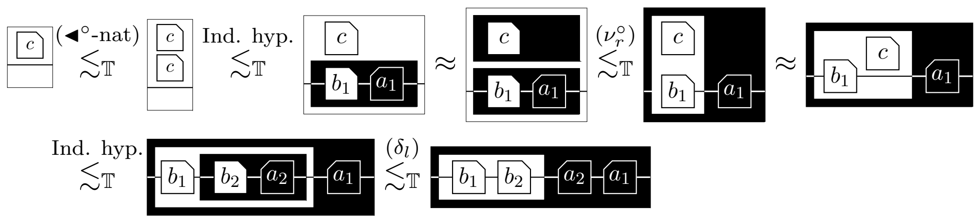
Theorem 7.13 (Deduction theorem)**.** *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $c \colon 0 \to 0$ in* $\mathbf{FOB}_\Sigma$*. Let $\mathbb{I}' = \mathbb{I} \cup \{(id_0^\circ, c)\}$ and let $\mathbb{T}'$ denote the theory $(\Sigma, \mathbb{I}')$. Then, for every $a, b \colon n \to m$ arrows of* $\mathbf{FOB}_\Sigma$*,*

Proof. By induction on the rules of (3.6). The base cases (id) and (r) are trivial. The inductive cases are listed below.

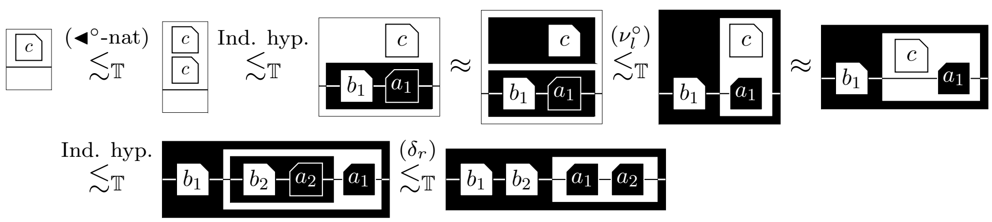
(t) Assume $a \lesssim_{\mathbb{T}'} d$ and $d \lesssim_{\mathbb{T}'} b$ for some $d \colon n \to m$. Observe that $a \lesssim_{\mathbb{T}'} b$ by (t) and $c \otimes id_n^\circ \lesssim_{\mathbb{T}} d \,;\, a^\perp$ and $c \otimes id_n^\circ \lesssim_{\mathbb{T}} b \,;\, d^\perp$ by inductive hypothesis. To conclude we need to show:



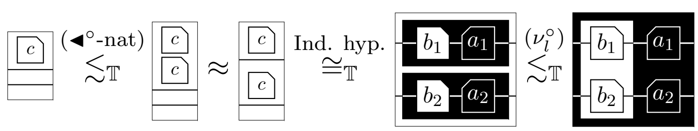
($\mathring{;}$) Assume $a = a_1 \mathring{;} a_2$ and $b = b_1 \mathring{;} b_2$ for some $a_1, b_1 \colon n \to l, a_2, b_2 \colon l \to m$ such that $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$. By induction hypothesis $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_1 \mathbin{\dot{;}} a_1^\perp$ and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_2 \mathbin{\dot{;}} a_2^\perp$. Note that:



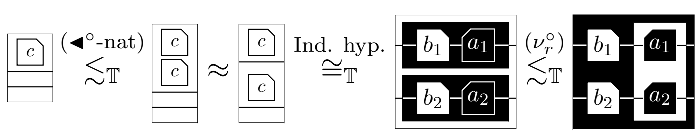
($\dot{;}$) Assume $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$ such that $a = a_1 \mathbin{\dot{;}} a_2$ and $b = b_1 \mathbin{\dot{;}} b_2$ for some $a_1, b_1 \colon n \to l, a_2, b_2 \colon l \to m$. Observe that $a_1 \mathbin{\dot{;}} a_2 \lesssim_{\mathbb{T}'} b_1 \mathbin{\dot{;}} b_2$ by ($\dot{;}$) and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_1 \mathbin{\dot{;}} a_1^\perp$ and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_2 \mathbin{\dot{;}} a_2^\perp$ by inductive hypothesis. To conclude we need to show:



($\otimes$) Assume $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$ such that $a = a_1 \otimes a_2$ and $b = b_1 \otimes b_2$ for some $a_1, b_1 \colon n' \to m', a_2, b_2 \colon n'' \to m''$. Observe that $a_1 \otimes a_2 \lesssim_{\mathbb{T}'} b_1 \otimes b_2$ by ($\otimes$) and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_1 \mathbin{\dot{;}} a_1^\perp$ and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_2 \mathbin{\dot{;}} a_2^\perp$ by inductive hypothesis. To conclude we need to show:



($\boldsymbol{\otimes}$) Assume $a_1 \lesssim_{\mathbb{T}'} b_1$ and $a_2 \lesssim_{\mathbb{T}'} b_2$ such that $a = a_1 \boldsymbol{\otimes} a_2$ and $b = b_1 \boldsymbol{\otimes} b_2$ for some $a_1, b_1 \colon n' \to m', a_2, b_2 \colon n'' \to m''$. Observe that $a_1 \boldsymbol{\otimes} a_2 \lesssim_{\mathbb{T}'} b_1 \boldsymbol{\otimes} b_2$ by ($\boldsymbol{\otimes}$) and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_1 \mathbin{\dot{;}} a_1^\perp$ and $c \otimes id^\circ_n \lesssim_{\mathbb{T}} b_2 \mathbin{\dot{;}} a_2^\perp$ by inductive hypothesis. To conclude we need to show:



□

Corollary 7.14. *Let* $\mathbb{T} = (\Sigma, \mathbb{I})$ *be a theory,* $c \colon 0 \to 0$ *in* $\mathbf{FOB}_\Sigma$ *and* $\mathbb{T}' = (\Sigma, \mathbb{I} \cup \{(id^\circ_0, \bar{c})\})$. *Then* $id^\circ_0 \lesssim_{\mathbb{T}} c$ *iff* $\mathbb{T}'$ *is contradictory.*

Proof. Suppose that $\mathbb{T}'$ is contradictory, namely $id^\circ_0 \lesssim_{\mathbb{T}'} id^\bullet_0$. By the deduction theorem (Theorem 7.13), $\bar{c} \lesssim_{\mathbb{T}} id^\bullet_0$ and thus $\overline{id^\bullet_0} \lesssim_{\mathbb{T}} \bar{\bar{c}}$, that is $id^\circ_0 \lesssim_{\mathbb{T}} c$. The the other direction is trivial: since $id^\circ_0 \lesssim_{\mathbb{T}'} c$ and $id^\circ_0 \lesssim_{\mathbb{T}'} \bar{c}$, then $id^\circ_0 \lesssim_{\mathbb{T}'} c \sqcap \bar{c} \lesssim_{\mathbb{T}'} \bot = id^\bullet_0$. □

7.3. **Functorial Semantics for First-Order Theories.** Recall that the notion of interpretation of a signature Σ in **Rel** has been generalised in Definition 6.11 to an arbitrary fo-bicategory. As expected, the same is possible also with the notion of model.

Definition 7.15. Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $\mathbf{C}$ a first-order bicategory. An interpretation $\mathcal{I}$ of Σ in $\mathbf{C}$ is a model iff, for all $(c, d) \in \mathbb{I}$, $\mathcal{I}^\sharp(c) \leq \mathcal{I}^\sharp(d)$.

For any theory $\mathbb{T} = (\Sigma, \mathbb{I})$, one can construct a fo-bicategory $\mathbf{FOB}_{\mathbb{T}}$ similarly to the construction of $\mathbf{FOB}_\Sigma$ illustrated in Section 6.1: we fix every homset $\mathbf{FOB}_{\mathbb{T}}[n, m] \stackrel{\text{def}}{=} \{[d]_{\cong_{\mathbb{T}}} \mid d \in \mathbf{FOB}_\Sigma[n, m]\}$ that is ordered by $\lesssim_{\mathbb{T}}$. Since, by definition, $\lesssim \subseteq \lesssim_{\mathbb{T}}$, $\mathbf{FOB}_{\mathbb{T}}$ is a fo-bicategory. Thus, one can consider an interpretation $\mathcal{Q}_{\mathbb{T}}$ of Σ in $\mathbf{FOB}_{\mathbb{T}}$: the domain X is 1 and $\rho(R) = [R^\circ]_{\cong_{\mathbb{T}}}$ for all $R \in \Sigma$. By Proposition 6.12, $\mathcal{Q}_{\mathbb{T}}$ induces a morphism of fo-bicategories $\mathcal{Q}^\sharp_{\mathbb{T}} \colon \mathbf{FOB}_\Sigma \to \mathbf{FOB}_{\mathbb{T}}$.

Proposition 7.16. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory, $\mathbf{C}$ a fo-bicategory and $\mathcal{I}$ an interpretation of Σ in $\mathbf{C}$. Then $\mathcal{I}$ is a model of $\mathbb{T}$ in $\mathbf{C}$ iff $\mathcal{I}^\sharp \colon \mathbf{FOB}_\Sigma \to \mathbf{C}$ factors through $\mathcal{Q}^\sharp_{\mathbb{T}} \colon \mathbf{FOB}_\Sigma \to \mathbf{FOB}_{\mathbb{T}}$, namely there exists a morphism of fo-bicategories $\mathcal{I}^\sharp_{\mathbb{T}} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$ such that the diagram below commutes. Moreover, $\mathcal{I}^\sharp_{\mathbb{T}}$ is the unique such morphism.*

$$\begin{array}{ccc} \mathbf{FOB}_\Sigma & \xrightarrow{\mathcal{Q}^\sharp_{\mathbb{T}}} & \mathbf{FOB}_{\mathbb{T}} \\ & \searrow^{\mathcal{I}^\sharp} & \downarrow{\scriptstyle \mathcal{I}^\sharp_{\mathbb{T}}} \\ & & \mathbf{C} \end{array}$$

Proof. First, observe that a simple inductive argument allows to prove that, for all diagrams c in $\mathbf{FOB}_\Sigma$,

$$\mathcal{Q}^\sharp_{\mathbb{T}}(c) = [c]_{\cong_{\mathbb{T}}}. \tag{7.2}$$

Now, suppose that there exists $\mathcal{I}^\sharp_{\mathbb{T}} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$ making the above diagram commute and consider $(c, d) \in \mathbb{I}$. By definition, $c \lesssim_{\mathbb{T}} d$ and, by (7.2),

$$\mathcal{Q}^\sharp_{\mathbb{T}}(c) \lesssim_{\mathbb{T}} \mathcal{Q}^\sharp_{\mathbb{T}}(d). \tag{7.3}$$

Then, the following derivation confirms that $\mathcal{I}$ is a model of $\mathbb{T}$ in $\mathbf{C}$.

$$\begin{aligned} \mathcal{I}^\sharp(c) &= \mathcal{I}^\sharp_{\mathbb{T}}(\mathcal{Q}^\sharp_{\mathbb{T}}(c)) && (\mathcal{I}^\sharp = \mathcal{Q}^\sharp_{\mathbb{T}}; \mathcal{I}^\sharp_{\mathbb{T}}) \\ &\leq \mathcal{I}^\sharp_{\mathbb{T}}(\mathcal{Q}^\sharp_{\mathbb{T}}(d)) && ((7.3) \text{ and } \mathcal{I}^\sharp_{\mathbb{T}} \text{ is a morphism}) \\ &= \mathcal{I}^\sharp(d) && (\mathcal{I}^\sharp = \mathcal{Q}^\sharp_{\mathbb{T}}; \mathcal{I}^\sharp_{\mathbb{T}}) \end{aligned}$$

Viceversa, suppose that $\mathcal{I}$ is a model of $\mathbb{T}$ in $\mathbf{C}$. Then by definition of model, for all $(c, d) \in \mathbb{I}$, $\mathcal{I}^\sharp(c) \leq \mathcal{I}^\sharp(d)$. A simple inductive argument on the rules in (3.6) confirms that, for all diagrams c, d in $\mathbf{FOB}_\Sigma$,

$$\text{if } c \lesssim_{\mathbb{T}} d \text{ then } \mathcal{I}^\sharp(c) \leq \mathcal{I}^\sharp(d).$$

In particular, if $c \cong_{\mathbb{T}} d$ then $\mathcal{I}^\sharp(c) = \mathcal{I}^\sharp(d)$. Therefore, we are allowed to define $\mathcal{I}^\sharp_{\mathbb{T}}([c]_{\cong_{\mathbb{T}}}) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c)$ for all arrows $[c]_{\cong_{\mathbb{T}}}$ of $\mathbf{FOB}_{\mathbb{T}}$ and $\mathcal{I}^\sharp_{\mathbb{T}}(n) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(n)$ for all objects n of $\mathbf{FOB}_{\mathbb{T}}$. The fact that $\mathcal{I}^\sharp_{\mathbb{T}}$ preserves the ordering follows immediately from the above implication. The fact that $\mathcal{I}^\sharp_{\mathbb{T}}$ preserves the structure of fo-bicategories follows easily from the fact that $\mathcal{I}^\sharp$ is a morphism. Therefore $\mathcal{I}^\sharp_{\mathbb{T}}$ is a morphism of fo-bicategories. The fact that the above diagram commutes is obvious by definition of $\mathcal{I}^\sharp_{\mathbb{T}}$ and (7.2).

Uniqueness follows immediately from the fact that $\mathcal{Q}^\sharp_{\mathbb{T}} \colon \mathbf{FOB}_\Sigma \to \mathbf{FOB}_{\mathbb{T}}$ is an epi, namely all objects and arrows of $\mathbf{FOB}_{\mathbb{T}}$ are in the image of $\mathcal{Q}^\sharp_{\mathbb{T}}$. □

The assignment $\mathcal{I} \mapsto \mathcal{I}^\sharp_{\mathbb{T}}$ provides a bijective correspondence between models and morphisms.

Corollary 7.17. *To give a model of* $\mathbb{T}$ *in* $\mathbf{C}$ *is to give a fo-bicategory morphism* $\mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$.

Proof. To go from models to morphisms we use the assignment $\mathcal{I} \mapsto \mathcal{I}^\sharp_{\mathbb{T}}$ provided by Proposition 7.16. To transform morphisms into models, we need a slightly less straightforward assignment. Take a morphism of fo-bicategories $\mathcal{F} \colon \mathbf{FOB}_{\mathbb{T}} \to \mathbf{C}$ and consider $\mathcal{Q}^\sharp_{\mathbb{T}} ; \mathcal{F} \colon \mathbf{FOB}_\Sigma \to \mathbf{C}$. This gives rise to the interpretation $\mathcal{I}_{\mathcal{F}} = (X, \rho)$ defined as

$$X \stackrel{\text{def}}{=} \mathcal{Q}^\sharp_{\mathbb{T}} ; \mathcal{F}(1) \qquad \text{and} \qquad \rho(R) \stackrel{\text{def}}{=} \mathcal{Q}^\sharp_{\mathbb{T}} ; \mathcal{F}(R^\circ) \quad \text{for all } R \in \Sigma$$

Then, by Proposition 6.12, $\mathcal{I}^\sharp_{\mathcal{F}} = \mathcal{Q}^\sharp_{\mathbb{T}} ; \mathcal{F}$ and thus, by Proposition 7.16, $\mathcal{I}_{\mathcal{F}}$ is a model. Since $\mathcal{I}^\sharp_{\mathcal{F}} = \mathcal{Q}^\sharp_{\mathbb{T}} ; \mathcal{F}$, by the uniqueness provided by Proposition 7.16, $(\mathcal{I}_{\mathcal{F}})^\sharp_{\mathbb{T}} = \mathcal{F}$.

To conclude, we only need to prove that $\mathcal{I}_{(\mathcal{I}^\sharp_{\mathbb{T}})} = \mathcal{I}$. Since $\mathcal{Q}^\sharp_{\mathbb{T}} ; \mathcal{I}^\sharp_{\mathbb{T}} = \mathcal{I}^\sharp$, then $\mathcal{I}_{(\mathcal{I}^\sharp_{\mathbb{T}})}(R^\circ) = \mathcal{Q}^\sharp_{\mathbb{T}} ; \mathcal{I}^\sharp_{\mathbb{T}}(R^\circ) = \mathcal{I}^\sharp(R^\circ) = \rho(R)$ for all $R \in \Sigma$. Similarly for the domain X. □

8. Completeness

In this section we illustrate a proof of Theorem 3.4, asserting completeness of the axioms of first-order bicategories. Our proof is divided into two main statements

$$\text{if } \mathbb{T} \text{ is a non-trivial theory, then } \mathbb{T} \text{ has a model} \tag{Gödel}$$

and

$$\text{if } \mathbb{T} \text{ is a trivial and non-contradictory theory, then } \mathbb{T} \text{ has a model} \tag{Prop}$$

that immediately entail

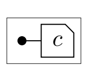
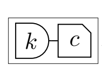
$$\text{if } \mathbb{T} \text{ is non-contradictory theory, then } \mathbb{T} \text{ has a model.} \tag{General}$$

Then, Theorem 3.4 easily follows by means of standard first-order logic arguments relying on the deduction theorem (Theorem 7.13). In Section 8.3 we illustrate a proof for (Prop) and in Section 8.4, one for Theorem 3.4. As expected, the proof for (Gödel) is more laborious and it is divided in two parts, illustrated in Sections 8.1 and 8.2.

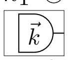
Before delving in the proof of (Gödel), it is convenient to have an overview. First, we must say that our proof is a faithful adaptation of the proof of Henkin's [Hen49] to NPR_Σ. Henkin's proof starts with the following two notions.

Definition 8.1. Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory.

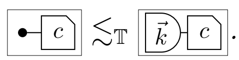
- $\mathbb{T}$ is *syntactically complete* if for all $c \colon 0 \to 0$ either $id_0^\circ \lesssim_{\mathbb{T}} c$ or $id_0^\circ \lesssim_{\mathbb{T}} \bar{c}$.
- $\mathbb{T}$ has *Henkin witnesses* if for all $c \colon 1 \to 0$ there is a map $k \colon 0 \to 1$ such that

 $\lesssim_{\mathbb{T}}$ . (H-wit)

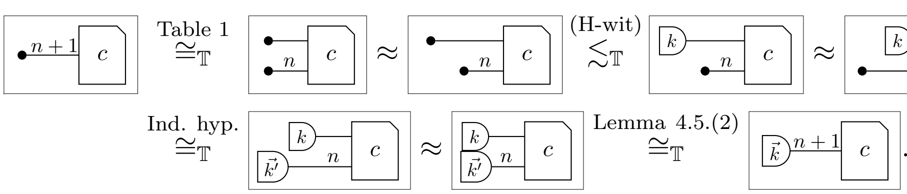
These properties do not hold for the theories we have considered so far. In terms of FOL, syntactic completeness means that closed formulas either hold in all models of the theory or in none. A Henkin witness is a term k such that $c(k)$ holds: a theory has Henkin witnesses if for every true formula $\exists x.c(x)$, there exists such a k. We shall see in Theorem 8.13 that non-trivial theories can be expanded to have Henkin witnesses, be non-contradictory and syntactically complete. The key idea of Henkin's proof, Theorem 8.5, is that these three properties yield a model, known as Henkin's model.

8.1. **Gödel completeness: Part I.** Before introducing Henkin's interpretation, it is convenient to recall that, by Lemma 4.5, in $\mathbf{FOB}_\Sigma$, any map $k\colon 0 \to n$ can be decomposed as $k_1 \otimes \ldots \otimes k_n$ where each $k_i\colon 0 \to 1$ is a map. We thus write such k as $\vec{k}$, depicted as , to make explicit the fact that it is a vector of n constants. This allows for a generalisation to n-ary maps of the Henkin witness property.

Lemma 8.2. *Let $\mathbb{T}$ be a theory with Henkin witnesses. For all $c\colon n \to 0$, there is a map $\vec{k}\colon 0 \to n$ such that*



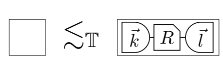
Proof. The proof goes by induction on n. For $n = 0$, take $\vec{k} = id_0^\circ$. For $n + 1$, we have that:



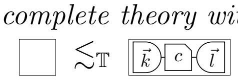
□

Now we have all the necessary equipment for being able to define a peculiar interpretation of monoidal signatures in **Rel**.

Definition 8.3. Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory. The *Henkin interpretation* $\mathcal{H}$ of Σ, consists of a set $X \overset{\text{def}}{=} \mathtt{Map}(\mathbf{FOB}_\mathbb{T})[0, 1]$ and a function ρ, defined for all $R\colon n \to m \in \Sigma$ as:

$$\rho(R) \overset{\text{def}}{=} \{(\vec{k}, \vec{l}) \in X^n \times X^m \mid$$  $$\}.$$

The domain is the set of constants of the theory. Indeed $\mathtt{Map}(\mathbf{FOB}_\mathbb{T})[0, 1]$ is the set of all maps $k\colon 0 \to 1$ in $\mathbf{FOB}_\mathbb{T}$. Then $R\colon n \to m$ is mapped to all pairs $(\vec{k}, \vec{l})$ of vectors that make R true in $\mathbb{T}$. The following characterisation of $\mathcal{H}^\sharp\colon \mathbf{FOB}_\Sigma \to \mathbf{Rel}$ is crucial.

Proposition 8.4. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-contradictory, syntactically complete theory with Henkin witnesses. Then, for any $c\colon n \to m$, $\mathcal{H}^\sharp(c) = \{(\vec{k}, \vec{l}) \in X^n \times X^m \mid$*  $\}$.

Proof. The proof goes by induction on c.

Consider the case $c = id_1^\circ$. Observe that, for all $k, l \in X$ (i.e., $\mathtt{Map}(\mathbf{FOB}_\mathbb{T})[0, 1]$),

$$id_0^\circ \lesssim_\mathbb{T} k \,;\, l^\dagger \text{ iff } k = l \tag{8.1}$$

Indeed, if $k = l$, then $id_0^\circ \lesssim_{\mathbb{T}} k \mathbin{;} l^\dagger$ by Proposition 4.6. Viceversa, if $id_0^\circ \lesssim_{\mathbb{T}} k \mathbin{;} l^\dagger$, then

$$
\begin{aligned}
k &\approx id_0^\circ \mathbin{;} k && \text{(SMC)}\\
&\lesssim_{\mathbb{T}} k \mathbin{;} l^\dagger \mathbin{;} k && (id_0^\circ \lesssim_{\mathbb{T}} k \mathbin{;} l^\dagger)\\
&= (k \mathbin{;} l^\dagger)^\dagger \mathbin{;} k && \text{(Table 1)}\\
&= l \mathbin{;} k^\dagger \mathbin{;} k && \text{(Table 2)}\\
&\lesssim_{\mathbb{T}} l && \text{(Proposition 4.6)}
\end{aligned}
$$

We thus have that

$$
\begin{aligned}
\mathcal{H}^\sharp(id_1^\circ) &= id_X^\circ && (3.4)\\
&= \{(k,l) \mid k = l\} && (2.2)\\
&= \{(k,l) \mid id_0^\circ \lesssim_{\mathbb{T}} k \mathbin{;} l^\dagger\} && (8.1)\\
&\approx \{(k,l) \mid id_0^\circ \lesssim_{\mathbb{T}} k \mathbin{;} id_1^\circ \mathbin{;} l^\dagger\} && \text{(SMC)}
\end{aligned}
$$

The proofs for the other constants of the white fragment follow analogous arguments. Note that none of the hypothesis about the theory is used here.

Instead, for the case $c = id_0^\bullet$ we use the hypothesis that $\mathbb{T}$ is not contradictory. Suppose that there exist map $k, l \colon 0 \to 0$ such that $id_0^\circ \lesssim_{\mathbb{T}} k \mathbin{;} id_0^\bullet \mathbin{;} l^\dagger$. By Lemma 4.5, id_0° is the only map of type $0 \to 0$ and thus, it should be the case that $id_0^\circ \lesssim_{\mathbb{T}} id_0^\circ \mathbin{;} id_0^\bullet \mathbin{;} id_0^\circ$. Since $id_0^\circ \mathbin{;} id_0^\bullet \mathbin{;} id_0^\circ \approx id_0^\bullet$, we have that $id_0^\circ \lesssim_{\mathbb{T}} id_0^\bullet$, against the hypothesis that $\mathbb{T}$ is non-contraddictory. Thus $\{(k,l) \mid id_0^\circ \lesssim_{\mathbb{T}} k \mathbin{;} id_0^\bullet \mathbin{;} l^\dagger\} = \varnothing$ which, by (3.4), is $\mathcal{H}^\sharp(id_0^\bullet)$.

The remaining constants of the black fragment follow a recurring pattern, using the hypothesis that $\mathbb{T}$ is syntactically complete. We show only the case $c = {}_n\boxed{R}^{\bullet}_m$.

$$
\begin{aligned}
\mathcal{H}^\sharp(\boxed{R}^{\bullet}) &= \{(\vec{l},\vec{k}) \in X^n \times X^m \mid (\vec{k},\vec{l}) \notin \mathcal{H}^\sharp(\boxed{R})\} && (3.4)\\
&= \{(\vec{l},\vec{k}) \in X^n \times X^m \mid \square \not\lesssim_{\mathbb{T}} \boxed{\vec{k}\; R\; \vec{l}}\} && \text{(Definition 8.3)}\\
&= \{(\vec{l},\vec{k}) \in X^n \times X^m \mid \square \lesssim_{\mathbb{T}} \boxed{\vec{k}\; R\; \vec{l}}^{\bullet}\} && (\mathbb{T}\text{ is syntactically complete})\\
&= \{(\vec{l},\vec{k}) \in X^n \times X^m \mid (\square)^\dagger \lesssim_{\mathbb{T}} (\boxed{\vec{k}\; R\; \vec{l}}^{\bullet})^\dagger\} && \text{(Table 2)}\\
&= \{(\vec{l},\vec{k}) \in X^n \times X^m \mid \square \lesssim_{\mathbb{T}} \boxed{\vec{l}\; R\; \vec{k}}^{\bullet}\} && \text{(Table 2)}\\
&= \{(\vec{l},\vec{k}) \in X^n \times X^m \mid \square \lesssim_{\mathbb{T}} \boxed{\vec{l}\; \boxed{R}^{\bullet}\; \vec{k}}\} && \text{(Proposition 6.5)}
\end{aligned}
$$

The mosts interesting part is the inductive case $c \mathbin{;} d$, where one exploits the hypothesis that $\mathbb{T}$ has Henkin witnesses. Suppose $c \colon n \to o$ and $d \colon o \to m$, then observe that the following holds:

$$
\begin{aligned}
&\{(\vec{k},\vec{l}) \in X^n \times X^m \mid \square \lesssim_{\mathbb{T}} \boxed{\vec{k}\; c\; d\; \vec{l}}\}\\
&= \{(\vec{k},\vec{l}) \in X^n \times X^m \mid \square \lesssim_{\mathbb{T}} \boxed{\bullet\!\!-\!\!\bullet \begin{smallmatrix} c\;\vec{k} \\ d\;\vec{l} \end{smallmatrix}}\} && (4.1)\\
&\subseteq \{(\vec{k},\vec{l}) \in X^n \times X^m \mid \exists \vec{t}\; \square \lesssim_{\mathbb{T}} \boxed{\vec{t}\!-\!\!\bullet \begin{smallmatrix} c\;\vec{k} \\ d\;\vec{l} \end{smallmatrix}}\} && \text{(Lemma 8.2)}
\end{aligned}
$$

$$= \{(\vec{k},\vec{l}) \in X^n \times X^m \mid \exists \vec{t}\ \boxed{} \lesssim_{\mathbb{T}} \boxed{\begin{matrix} \vec{t}\,\boxed{c}\,\vec{k} \\ \vec{t}\,\boxed{d}\,\vec{l} \end{matrix}} \} \qquad (\mathbb{M}_k)$$

$$= \{(\vec{k},\vec{l}) \in X^n \times X^m \mid \exists \vec{t}\ \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{t}\,\boxed{c}\,\vec{k}} \wedge \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{t}\,\boxed{d}\,\vec{l}} \} \qquad ((\blacktriangleleft^{\circ}\text{-nat}),(!^{\circ}\text{-nat}))$$

$$= \{(\vec{k},\vec{l}) \in X^n \times X^m \mid \exists \vec{t}\ \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{k}\,\boxed{c}\,\vec{t}} \wedge \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{t}\,\boxed{d}\,\vec{l}} \} \qquad (\text{Table 2.(a)})$$

$$= \{(\vec{k},\vec{t}) \in X^n \times X^o \mid \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{k}\,\boxed{c}\,\vec{t}} \} \fatsemi \{(\vec{t},\vec{l}) \in X^o \times X^m \mid \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{t}\,\boxed{d}\,\vec{l}} \} \qquad (2.1)$$

$$= \mathcal{H}^\sharp(\boxed{c}) \fatsemi \mathcal{H}^\sharp(\boxed{d}) \qquad (\text{Ind. hyp.})$$

$$= \mathcal{H}^\sharp(\boxed{c\,d}) \qquad (3.4)$$

The other inclusion is less interesting: the reader can use (3.4), the induction hypothesis, (2.1), ($\blacktriangleleft^{\circ}$-nat), ($!^{\circ}$-nat) and Proposition 4.8 to check it.

Similarly for the case $c \otimes d$: the reader can check it using (3.4), the induction hypothesis, (3.3), ($\blacktriangleleft^{\circ}$-nat),($!^{\circ}$-nat) and Lemma 4.5.

For the inductive case $c \mathbin{\boldsymbol{;}} d$, assume $c\colon n \to o$ and $d\colon o \to m$, then observe that:

$$\mathcal{H}^\sharp(\blacksquare\boxed{c\,d}) = \overline{\mathcal{H}^\sharp(\boxed{\blacksquare c\,\blacksquare d})} \qquad (\text{Lemma 6.13})$$

$$= \overline{\{(\vec{k},\vec{l}) \in X^n \times X^m \mid \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{k}\,\blacksquare c\,\blacksquare d\,\vec{l}}\}} \qquad (\text{Case } c \fatsemi d)$$

$$= \{(\vec{k},\vec{l}) \in X^n \times X^m \mid \boxed{} \not\lesssim_{\mathbb{T}} \boxed{\vec{k}\,\blacksquare c\,\blacksquare d\,\vec{l}}\}$$

$$= \{(\vec{k},\vec{l}) \in X^n \times X^m \mid \boxed{} \lesssim_{\mathbb{T}} \blacksquare\boxed{\vec{k}\,c\,d\,\vec{l}}\} \qquad (\mathbb{T} \text{ is syntactically complete})$$

$$= \{(\vec{k},\vec{l}) \in X^n \times X^m \mid \boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{k}\,\blacksquare\boxed{c\,d}\,\vec{l}} \} \qquad (\text{Proposition 6.5})$$

The proof above relies on Lemma 6.13 and the previous inductive case of $c \fatsemi d$. The case of $c \mathbin{\boldsymbol{\otimes}} d$ follows the exact same reasoning but, as expected, this time one has to exploit the proof of $c \otimes d$. □

Theorem 8.5. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-contradictory, syntactically complete theory with Henkin witnesses. Then $\mathcal{H}$ is a model.*

Proof. We show that $c \lesssim_{\mathbb{T}} d$ gives $\mathcal{H}^\sharp(c) \subseteq \mathcal{H}^\sharp(d)$. If $(\vec{k},\vec{l}) \in \mathcal{H}^\sharp(c)$ then $\boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{k}\,\boxed{c}\,\vec{l}}$ by Prop. 8.4. Since $c \lesssim_{\mathbb{T}} d$, $\boxed{} \lesssim_{\mathbb{T}} \boxed{\vec{k}\,\boxed{c}\,\vec{l}} \lesssim_{\mathbb{T}} \boxed{\vec{k}\,\boxed{d}\,\vec{l}}$ and by Prop. 8.4, $(\vec{k},\vec{l}) \in \mathcal{H}^\sharp(d)$. □

8.2. **Gödel completeness: Part II.** Theorem 8.5 states that any theory with Henkin witness that is syntactically complete and non-contradictory has a model. To prove (Gödel), we now need show that from a non-trivial theory $\mathbb{T} = (\Sigma, \mathbb{I})$ one can always generate a theory $\mathbb{T}' = (\Sigma', \mathbb{I}')$ which enjoy these three properties and such that $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq I'$ (formally stated in Theorem 8.13).

We begin by illustrating a procedure that allows to add Henkin witnesses. To add a witness for $c\colon 1 \to 0$, one adds a constant $k\colon 0 \to 1$ and the axiom $\mathbb{W}_k^c$ below, asserting that k is a witness.

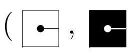
$$\mathbb{W}_k^c \stackrel{\text{def}}{=} \{(\boxed{}, \blacksquare\boxed{\begin{matrix} \boxed{k}\,\boxed{c} \\ \bullet\!-\!\boxed{c} \end{matrix}})\}$$

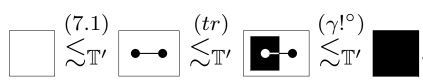
Now, we focus on proving the following key result.

Lemma 8.6 (Witness Addition)**.** *Let* $\mathbb{T} = (\Sigma, \mathbb{I})$ *be a theory and consider an arbitrary* $c\colon 1 \to 0$. *Let* $\mathbb{T}' = (\Sigma \cup \{k\colon 0 \to 1\}, \mathbb{I} \cup \mathbb{M}_k \cup \mathbb{W}^c_k)$. *If* $\mathbb{T}$ *is non-trivial then* $\mathbb{T}'$ *is non-trivial.*

Remark 8.7. Before proving Lemma 8.6, it is worth to observe that the distinction between trivial and contradictory theories is essential for the lemma. Indeed, under the conditions of Lemma 8.6, it does *not* hold that

if $\mathbb{T}$ is non-contradictory, then $\mathbb{T}'$ is non-contradictory.

As counter-example, take as $\mathbb{T}$ the theory consisting only of the trivialising axiom $(tr) \stackrel{\text{def}}{=}$ . By definition $\mathbb{T}$ is trivial but non-contradictory. Instead $\mathbb{T}'$ is contradictory:

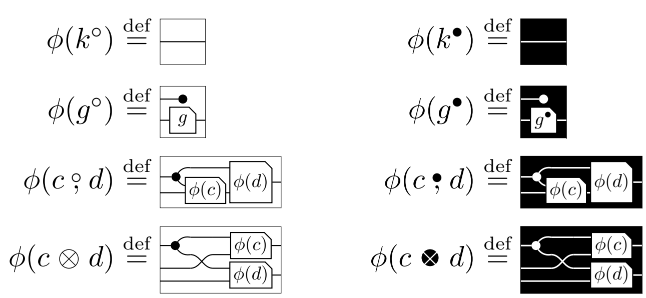


(8.2)

This shows that adding Henkin witnesses to a non-contradictory theory may end up in a contradictory theory. Therefore, the usual Henkin proof for FOL works just for our *non-trivial* theories.

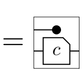
In order to prove Lemma 8.6 and then Theorem 8.13, we need to show that *adding* constants to a non-trivial theory results in a non-trivial theory. To do this, it is useful to have a procedure for *erasing* constants. This is defined as follows.

Definition 8.8. Let Σ be a signature and $\Sigma' = \Sigma \cup \{k\colon 0 \to 1\}$. The function $\phi\colon \mathbf{FOB}_{\Sigma'}[n, m] \to \mathbf{FOB}_{\Sigma}[1 + n, m]$ is inductively defined as follows:

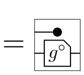
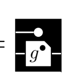
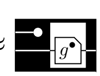
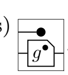


(def-ϕ)

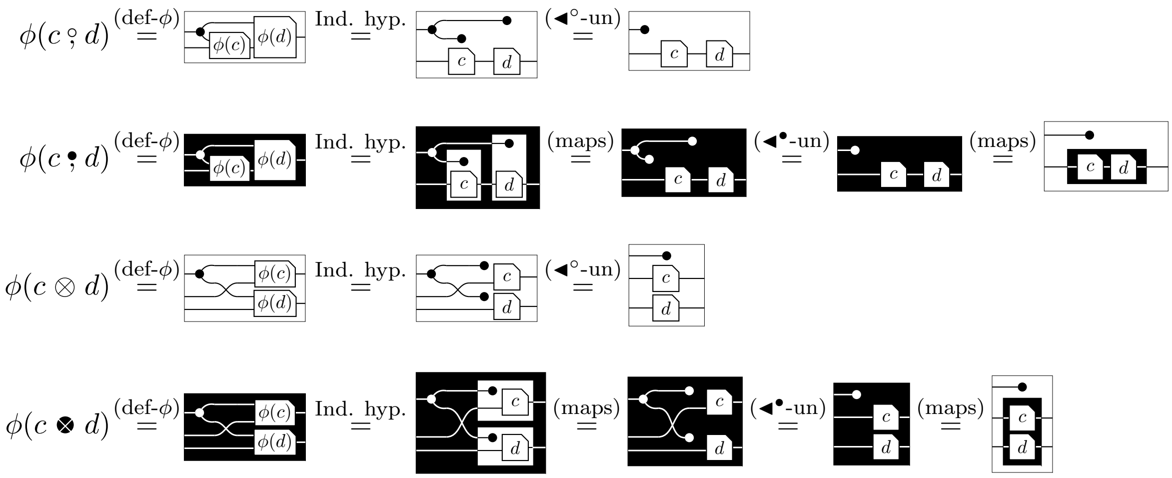
where $g^\circ \in \{◀^\circ_1, !^\circ_1, R^\circ, ¡^\circ_1, ▶^\circ_1, id^\circ_0, id^\circ_1, \sigma^\circ_{1,1}\}$ and $g^\bullet \in \{◀^\bullet_1, !^\bullet_1, R^\bullet, ¡^\bullet_1, ▶^\bullet_1, id^\bullet_0, id^\bullet_1, \sigma^\bullet_{1,1}\}$.

Lemma 8.9. *Let* $c\colon n \to m$ *be a diagram of* $\mathbf{FOB}_\Sigma$, *then* $\phi(c) =$ .

Proof. The proof goes by induction on the syntax.

The base cases are split in two groups. For all generators g° in $\mathsf{NPR}^\circ_\Sigma$, $\phi(g^\circ) =$  by definition, while for those $g^\bullet$ in $\mathsf{NPR}^\bullet_\Sigma$, $\phi(g^\bullet) =$  $\approx$  $\stackrel{\text{(maps)}}{=}$ .

The four inductive cases are shown below:

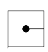
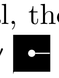


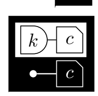
□

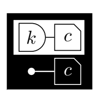
The proof of the following result goes by induction on (3.6) and relies on Lemma 8.9. The interested reader can find its proof in Appendix G.

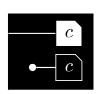
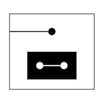
Lemma 8.10 (Constant Erasion)**.** *Let* $\mathbb{T} = (\Sigma, \mathbb{I})$ *be a theory and* $\mathbb{T}' = (\Sigma', \mathbb{I}')$ *be the theory where* $\Sigma' = \Sigma \cup \{k\colon 0 \to 1\}$ *and* $\mathbb{I}' = \mathbb{I} \cup \mathbb{M}_k$*. Then, for any* $c, d\colon n \to m$ *in* $\mathbf{FOB}_{\Sigma'}$ *if* $c \lesssim_{\mathbb{T}'} d$ *then* $\phi(c) \lesssim_{\mathbb{T}} \phi(d)$*.*

Now we are ready to prove Lemma 8.6, namely that witness addition preserves non-triviality.

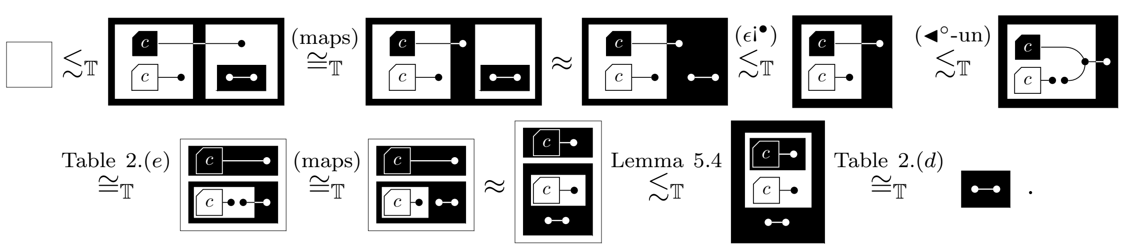
Proof of Lemma 8.6. We prove that if $\mathbb{T}'$ is trivial, then also $\mathbb{T}$ is trivial. Let $\mathbb{T}'' = \{\Sigma \cup k, \mathbb{I} \cup \mathbb{M}_k\}$ and assume $\mathbb{T}'$ to be trivial, i.e.  $\lesssim_{\mathbb{T}'}$ , then:

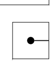
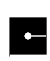
(1) by the Deduction Theorem (7.13) we have  $\lesssim_{\mathbb{T}''}$  ;

(2) thus, by Lemma 8.10, $\phi($  $) \lesssim_{\mathbb{T}} \phi($  $)$;

(3) and, by Definition 8.8 and Lemma 8.9,  $\lesssim_{\mathbb{T}}$  .

To conclude, apply Lemma 6.8 and observe that



which, by Lemma 6.8 again, is exactly that  $\lesssim_{\mathbb{T}}$ . Namely $\mathbb{T}$ is trivial. □

By iteratively using Lemma 8.6, one can transform a non-trivial theory into a non-trivial theory with Henkin witnesses. This was our main technical effort in this part of the proof. Now, the procedure to obtain a syntactically complete theory closely follows the standard well-known arguments (reported e.g. in [LP01]) and, for this reason, we defer the remaining proofs to Appendix G.

Proposition 8.11. *Let I be a linearly ordered set and for all $i \in I$ let $\mathbb{T}_i = (\Sigma_i, \mathbb{I}_i)$ be first order theories such that if $i \leq j$, then $\Sigma_i \subseteq \Sigma_j$ and $\mathbb{I}_i \subseteq \mathbb{I}_j$. Let $\mathbb{T}$ be the theory*

$$(\bigcup_{i\in I} \Sigma_i, \bigcup_{i\in I} \mathbb{I}_i).$$

.

(1) *If all $\mathbb{T}_i$ are non-contradictory, then $\mathbb{T}$ is non-contradictory.*
(2) *If all $\mathbb{T}_i$ are non-trivial, then $\mathbb{T}$ is non-trivial.*

By means of the above result and Zorn Lemma [Zor35], one can obtain the desired syntactically complete theory.

Proposition 8.12. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-contradictory theory. There exists a theory $\mathbb{T}' = (\Sigma, \mathbb{I}')$ that is syntactically complete, non-contradictory and $\mathbb{I} \subseteq \mathbb{I}'$.*

Now, we obtain the desired result by following the standard Henkin argument which iteratively apply Lemma 8.6, wisely combined with Propositions 8.11 and 8.12.

Theorem 8.13. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a non-trivial theory. There exists a theory $\mathbb{T}' = (\Sigma', \mathbb{I}')$ such that $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq \mathbb{I}'$; $\mathbb{T}'$ has Henkin witnesses; $\mathbb{T}'$ is syntactically complete; $\mathbb{T}'$ is non-contradictory.*

Theorems 8.13 and 8.5 give us a proof for (Gödel).

Proof of (Gödel). Let $\mathbb{T}' = (\Sigma', \mathbb{I}')$ be obtained via Theorem 8.13. Since $\mathbb{T}'$ has Henkin witnesses, is syntactically complete and non-contradictory, Theorem 8.5 ensures that $\mathcal{H}$ is a model for $\mathbb{T}'$. Since $\Sigma \subseteq \Sigma'$ and $\mathbb{I} \subseteq \mathbb{I}'$, then $\mathcal{H}$ is also a model for $\mathbb{T}$. □

8.3. **Propositional completeness.** Now, we would like to conclude Theorem 3.4 by means of (Gödel), but this is not possible since, for the former one needs a model for all non-contradictory theories, while (Gödel) provides it only for non-trivial ones. Thankfully, the Henkin interpretation $\mathcal{H}$ (Definition 8.3) gives us, once more, a model (Proposition 8.18) that allows us to prove

$$\text{if } \mathbb{T} \text{ is a trivial and non-contradictory theory, then } \mathbb{T} \text{ has a model.} \tag{Prop}$$

We commence by illustrating $\mathcal{H}$ for a trivial and non-contradictory theory.

Lemma 8.14. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory that is trivial and non-contradictory and let $\mathcal{H}$ be the Henkin interpretation of Σ. Then, the domain X of $\mathcal{H}$ is $\varnothing$ and*

$$\rho(R) = \begin{cases} \{(\star, \star)\} & \text{if } id_0^\circ \lesssim_{\mathbb{T}} R^\circ \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. Recall by Definition 8.3, that the domain X of $\mathcal{H}$ is defined as the set $\mathtt{Map}(\mathbf{FOB}_{\mathbb{T}})[0,1]$. This set should be necessarily empty since, if there exists some map $k\colon 0 \to 1$, then by (8.2) $\mathbb{T}$ would be contradictory, against the hypothesis. Thus $\mathtt{Map}(\mathbf{FOB}_{\mathbb{T}})[0,1] = \varnothing$.

By Lemma4.5.(2), one has also that $\texttt{Map}(\mathbf{FOB}_{\mathbb{T}})[0, n+1] = \varnothing$. We thus may only have maps in $\texttt{Map}(\mathbf{FOB}_{\mathbb{T}})[0, 0]$. By Lemma 4.5, there is only one map in $\texttt{Map}(\mathbf{FOB}_{\mathbb{T}})[0, 0]$, which is exactly $id_0^\circ \colon 0 \to 0$. Recall that by Definition 8.3,

$$\rho(R) = \{(\vec{k}, \vec{l}) \in X^n \times X^m \mid id_0^\circ \lesssim_{\mathbb{T}} \vec{k} \,;\, R^\circ \,;\, (\vec{k})^\dagger\}$$

for all $R \in \Sigma$. Since our only map is $id_0^\circ \colon 0 \to 0$, we have that

$$\rho(R) = \{(\star, \star) \in \mathbb{1} \times \mathbb{1} \mid id_0^\circ \lesssim_{\mathbb{T}} R^\circ\}.$$

□

Lemma 8.15. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory that is trivial and non-contradictory and let $c \colon n \to m+1$ and $d \colon n+1 \to m$ be arrows of $\mathbf{FOB}_{\mathbb{T}}$. Thus $\mathcal{H}^\sharp(c) = \varnothing$ and $\mathcal{H}^\sharp(d) = \varnothing$.*

Proof. Recall that for any interpretation $\mathcal{I}$ with domain X, $\mathcal{I}^\sharp(c) \subseteq X^n \times X^{m+1} = X^n \times X^m \times X$. By Lemma 8.14, the domain of $\mathcal{H}$ is $\varnothing$ and thus $\mathcal{H}^\sharp(c) \subseteq \varnothing^n \times \varnothing^m \times \varnothing$, i.e., $\mathcal{H}^\sharp(c) = \varnothing$. The proof for $\mathcal{H}^\sharp(d)$ is identical. □

Lemma 8.16. *Let $\mathbb{T}$ be a trivial, syntactically complete and non-contradictory theory. Let $c \colon 0 \to 0$ be an arrow of $\mathbf{FOB}_{\mathbb{T}}$. If $\mathcal{H}^\sharp(c) = \{(\star, \star)\}$ then $c =_{\mathbb{T}} id_0^\circ$.*

Proof. By induction on c. For the base cases, there are only four constants $c \colon 0 \to 0$.

- $c = id_0^\circ$. Then, trivially, $c =_{\mathbb{T}} id_0^\circ$.
- $c = id_0^\bullet$. Then, by (3.4), $\mathcal{H}^\sharp(c) = \varnothing$ against the hypothesis.
- $c = R^\circ$. Then, by (3.4), $\rho(R) = \mathcal{H}^\sharp(R^\circ) = \{(\star, \star)\}$. By Lemma 8.14, $id_0^\circ =_{\mathbb{T}} R^\circ$.
- $c = R^\bullet$. Then, by (3.4), $\overline{\rho(R)} = \mathcal{H}^\sharp(R^\bullet) = \{(\star, \star)\}$. Thus, by Lemma 8.14, $id_0^\circ \not\lesssim_{\mathbb{T}} R^\circ$. Since $\mathbb{T}$ is syntactically complete $id_0^\circ \lesssim_{\mathbb{T}} R^\bullet$.

We now consider the usual four inductive cases.

- $c = c_1 \otimes c_2$. Since $c \colon 0 \to 0$, then also c_1 and c_2 have type $0 \to 0$. By definition, $\mathcal{H}^\sharp(c) = \mathcal{H}^\sharp(c_1) \otimes \mathcal{H}^\sharp(c_2)$. By definition of $\otimes$ in **Rel** both $\mathcal{H}^\sharp(c_1)$ and $\mathcal{H}^\sharp(c_2)$ must be $\{(\star, \star)\}$. We can thus apply the inductive hypothesis to deduce that $c_1 =_{\mathbb{T}} id_0^\circ$ and $c_2 =_{\mathbb{T}} id_0^\circ$. Therefore $c = c_1 \otimes c_2 =_{\mathbb{T}} id_0^\circ \otimes id_0^\circ =_{\mathbb{T}} id_0^\circ$.
- $c = c_1 \,;\, c_2$. There are two possible cases: either $c_1 \colon 0 \to n+1$ and $c_2 \colon n+1 \to 0$, or $c_1 \colon 0 \to 0$ and $c_2 \colon 0 \to 0$. In the former case, we have by Lemma 8.15, that $\mathcal{H}^\sharp(c) = \mathcal{H}^\sharp(c_1) \,;\, \mathcal{H}^\sharp(c_2) = \varnothing \,;\, \varnothing = \varnothing$. Against the hypothesis. Thus the second case should hold: $c_1 \colon 0 \to 0$ and $c_2 \colon 0 \to 0$. In this case we just observe that $c_1 \,;\, c_2$ is, by the laws of symmetric monoidal categories, equal to $c_1 \otimes c_2$. We can thus reuse the point above.
- $c = c_1 \,\bullet\!\!\!\otimes\, c_2$. Since $c \colon 0 \to 0$, then also c_1 and c_2 have type $0 \to 0$. Consider the case where $\mathcal{H}^\sharp(c_1) = \varnothing = \mathcal{H}^\sharp(c_2)$. Thus $\mathcal{H}^\sharp(c) = \varnothing$, against the hypothesis. Therefore either $\mathcal{H}^\sharp(c_1) = \{(\star, \star)\}$ or $\mathcal{H}^\sharp(c_2) = \{(\star, \star)\}$. If $\mathcal{H}^\sharp(c_1) = \{(\star, \star)\}$, then by induction hypothesis $c_1 =_{\mathbb{T}} id_0^\circ$. Therefore $c = c_1 \,\bullet\!\!\!\otimes\, c_2 = c_1 \sqcup c_2 =_{\mathbb{T}} id_0^\circ \sqcup c_2 =_{\mathbb{T}} \top \sqcup c_2 =_{\mathbb{T}} \top =_{\mathbb{T}} id_0^\circ$. The case for $\mathcal{H}^\sharp(c_2) = \{(\star, \star)\}$ is symmetric.
- $c = c_1 \,;^\bullet\, c_2$. The proof is analogous to the case $c = c_1 \,;\, c_2$.

□

From the above result, one easily obtains its dual.

Lemma 8.17. *Let $\mathbb{T}$ be a trivial, syntactically complete and non-contradictory theory. Let $c \colon 0 \to 0$ be an arrow of $\mathbf{FOB}_{\mathbb{T}}$. If $\mathcal{H}^\sharp(c) = \varnothing$ then $c =_{\mathbb{T}} id_0^\bullet$.*

Proof. If $\mathcal{H}^\sharp(c) = \varnothing$, then by Lemma 6.13, $\mathcal{H}^\sharp(\overline{c}) = \overline{\varnothing} = \{(\star,\star)\}$. Thus by Lemma 8.16, $\overline{c} =_{\mathbb{T}} id_0^\circ$ and thus $c =_{\mathbb{T}} id_0^\bullet$. □

The following result is the key to prove (Prop).

Proposition 8.18. *Let $\mathbb{T}$ be a trivial, syntactically complete and non-contradictory theory. Then $\mathcal{H}$ is a model of $\mathbb{T}$.*

Proof. Recall that Proposition 7.5, $\mathcal{H}$ is a model iff for all $c, d\colon n \to m$ in $\mathbf{FOB}_\Sigma$, if $c \lesssim_{\mathbb{T}} d$, then $\mathcal{H}^\sharp(c) \subseteq \mathcal{H}^\sharp(d)$. We prove that if $\mathcal{H}^\sharp(c) \not\subseteq \mathcal{H}^\sharp(d)$, then $c \not\lesssim_{\mathbb{T}} d$.

If $c\colon n \to m+1$ or $c\colon n+1 \to m$, then by Lemma 8.15, $\mathcal{H}^\sharp(c) = \varnothing$ and thus it is not the case that $\mathcal{H}^\sharp(c) \not\subseteq \mathcal{H}^\sharp(d)$. Thus we need to consider only the case where $c, d\colon 0 \to 0$.

For $c, d\colon 0 \to 0$ if $\mathcal{H}^\sharp(c) \not\subseteq \mathcal{H}^\sharp(d)$, then $\mathcal{H}^\sharp(c) = \{(\star,\star)\}$ and $\mathcal{H}^\sharp(d) = \varnothing$. By Lemmas 8.16 and 8.17, we thus have $c =_{\mathbb{T}} id_0^\circ$ and $d =_{\mathbb{T}} id_0^\bullet$. Since $\mathbb{T}$ is non-contradictory, then $c \not\lesssim_{\mathbb{T}} d$. □

Analogously to the proof (Gödel), we can exploit Proposition 8.12, but now combined with the proposition above to prove (Prop).

Proof of (Prop). Since $\mathbb{T} = (\Sigma, \mathbb{I})$ is non-contradictory, by Proposition 8.12 there exists a syntactically complete non-contradictory theory $\mathbb{T}' = (\Sigma, \mathbb{I}')$ such that $\mathbb{I} \subseteq \mathbb{I}'$. Since $¡_1^\circ \lesssim_{\mathbb{T}} ¡_1^\bullet$, then $¡_1^\circ \lesssim_{\mathbb{T}'} ¡_1^\bullet$, namely $\mathbb{T}'$ is also trivial. We can thus use Proposition 8.18, to deduce that $\mathcal{H}$ is a model for $\mathbb{T}'$. Since $\mathbb{I} \subseteq \mathbb{I}'$, then $\mathcal{H}$ is also a model of $\mathbb{T}$. □

8.4. **General Completeness.** From (Prop) and (Gödel) we can prove general completeness

$$\text{if } \mathbb{T} \text{ is non-contradictory theory, then } \mathbb{T} \text{ has a model.} \tag{General}$$

Proof of (General). To prove (General) take $\mathbb{T}$ to be a non-contradictory theory. If $\mathbb{T}$ is trivial, then it has a model by (Prop). Otherwise, it has a model by (Gödel). □

Before proving Theorem 3.4, we illustrate the following result which simply rephrases standard arguments of completeness for first order logic.

Lemma 8.19. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $c\colon 0 \to 0$ be a diagram in $\mathbf{FOB}_{\mathbb{T}}$. If, for all models $\mathcal{I}$ of $\mathbb{T}$, $\{(\star,\star)\} \subseteq \mathcal{I}^\sharp(c)$, then $id_0^\circ \lesssim_{\mathbb{T}} c$.*

Proof. Suppose that $id_0^\circ \not\lesssim_{\mathbb{T}} c$. Then, by Corollary 7.14, $\mathbb{T}' = (\Sigma, \mathbb{I} \cup \{(id_0^\circ, \overline{c})\})$ is non-contradictory. Thus, by (General), $\mathbb{T}'$ has a model $\mathcal{I}$. Since $\mathbb{I} \subseteq \mathbb{I} \cup \{(id_0^\circ, \overline{c})\}$, $\mathcal{I}$ is also a model of $\mathbb{T}$. Now note that

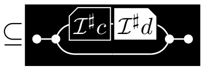
$$\begin{aligned} \{(\star,\star)\} &= \mathcal{I}^\sharp(id_0^\circ) && (3.4)\\ &\subseteq \mathcal{I}^\sharp(\overline{c}) && (\mathcal{I} \text{ is a model of } \mathbb{T}')\\ &= \overline{\mathcal{I}^\sharp(c)}. && (\text{Lemma 6.13}) \end{aligned}$$

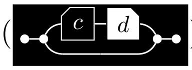
Thus $\mathcal{I}^\sharp(c) \subseteq \overline{\{(\star,\star)\}} = \varnothing$, against the hypothesis. □

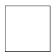
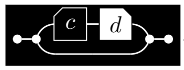
Using Lemma 6.8, we can extends the above result to arbitrary morphisms $c, d\colon n \to m$.

Proposition 8.20. *Let $\mathbb{T} = (\Sigma, \mathbb{I})$ be a theory and $c, d\colon n \to m$ be diagrams in $\mathbf{FOB}_{\mathbb{T}}$. If, for all models $\mathcal{I}$ of $\mathbb{T}$, $\mathcal{I}^\sharp(c) \subseteq \mathcal{I}^\sharp(d)$, then $c \lesssim_{\mathbb{T}} d$.*

Proof. Since **Rel** is a fo-bicategory, one can safely exploit Lemma 6.8 to show that

$\{(\star,\star)\} \subseteq$  (Lemma 6.8)

$= \mathcal{I}^\sharp($  $).$ (Lemma 6.13)

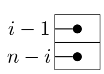
Thus by Lemma 8.19,  $\lesssim_{\mathbb{T}}$ . Again, by Lemma 6.8, $c \lesssim_{\mathbb{T}} d$. □

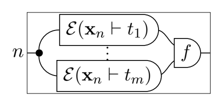
Proof of Theorem 3.4. From Proposition 8.20 when taking $\mathbb{I} = \varnothing$. □

9. First Order Logic with Equality

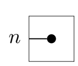
As we already mentioned in the introduction the white fragment of NPR_Σ is as expressive as the existential-conjunctive fragment of first order logic with equality (FOL). The semantic preserving encodings between the two fragments are illustrated in [BSS18]. From the fact that the full NPR_Σ can express negation, we get immediately semantic preserving encodings between NPR_Σ and the full FOL. In this section we illustrate anyway a translation $\mathcal{E}(\cdot)\colon \mathsf{FOL} \to \mathsf{NPR}_\Sigma$ to emphasise the subtle differences between the two.

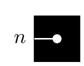
To ease the presentation, we consider FOL formulas φ to be typed in the context of a list of variables that are allowed (but not required) to appear in φ. Fixing $\mathbf{x}_n \stackrel{\text{def}}{=} \{x_1, \dots, x_n\}$ we write $\mathbf{x}_n \vdash \varphi$ if all free variables of φ are contained in $\mathbf{x}_n$. It is standard to present the syntax of FOL in two steps: first terms and then formulas. For every function symbol f of arity m in FOL, we have a symbol $f\colon m \to 1$ in the signature Σ together with the equations $\mathbb{M}_f$ forcing f to be interpreted as a function. The translation of $\mathbf{x}_n \vdash t$ to a $\mathbf{FOB}_\Sigma$ diagram $n \to 1$ is inductively given as follows.

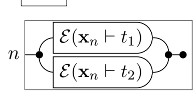
$\mathcal{E}(\mathbf{x}_n \vdash x_i) \stackrel{\text{def}}{=}$ 

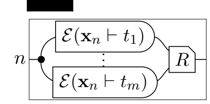
$\mathcal{E}(\mathbf{x}_n \vdash f(t_1, \dots, t_m)) \stackrel{\text{def}}{=}$ 

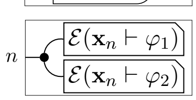
Formulas $\mathbf{x}_n \vdash \varphi$ translate to $\mathbf{FOB}_\Sigma$ diagrams $n \to 0$. For every n-ary predicate symbol R in FOL there is a symbol $R\colon n \to 0 \in \Sigma$. In order not to over-complicate the presentation with bureaucratic details, we assume that it is always the last variable that is quantified over. Additional variable manipulations can be introduced easily (see, for example, [BSS18]).

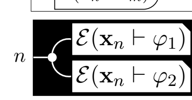
$\mathcal{E}(\mathbf{x}_n \vdash \top) \stackrel{\text{def}}{=}$ 

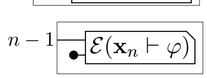
$\mathcal{E}((\mathbf{x}_n \vdash \bot)) \stackrel{\text{def}}{=}$ 

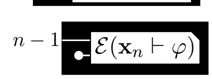
$\mathcal{E}(\mathbf{x}_n \vdash t_1 = t_2) \stackrel{\text{def}}{=}$ 

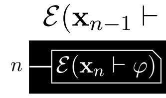
$\mathcal{E}(\mathbf{x}_n \vdash R(t_1, \dots, t_m)) \stackrel{\text{def}}{=}$ 

$\mathcal{E}(\mathbf{x}_n \vdash \varphi_1 \wedge \varphi_2) \stackrel{\text{def}}{=}$ 

$\mathcal{E}(\mathbf{x}_n \vdash \varphi_1 \vee \varphi_2) \stackrel{\text{def}}{=}$ 

$\mathcal{E}(\mathbf{x}_{n-1} \vdash \exists x_n.\, \varphi) \stackrel{\text{def}}{=}$ 

$\mathcal{E}(\mathbf{x}_{n-1} \vdash \forall x_n.\, \varphi) \stackrel{\text{def}}{=}$ 

$\mathcal{E}(\mathbf{x}_n \vdash \neg\varphi) \stackrel{\text{def}}{=}$ 

The above encoding should give the reader the spirit of the correspondence between NPR_Σ and traditional syntax. There is one aspect of the above translation that merits additional attention.

Remark 9.1. By the definition of $!^\circ_n$ in Table 1, we have that:

$$\mathcal{E}(\mathbf{x}_0 \vdash \top) \stackrel{\text{def}}{=} \square \qquad\qquad \mathcal{E}(\mathbf{x}_0 \vdash \bot) \stackrel{\text{def}}{=} \blacksquare$$

Thus $\top$ and $\bot$ translate to, respectively id°_0, $id^\bullet_0$ in the absence of free variables or to $!^\circ_n$, $!^\bullet_n$, respectively, when $n > 0$. This can be seen as an ambiguity in the traditional FOL syntax, which obscures the distinction between inconsistent and trivial theories in traditional accounts, and as a side effect requires the assumption on non-empty models in formal statements of Gödel completeness. Instead, the syntax of NPR_Σ ensures that this pitfall is side-stepped.

To conclude our analysis of the relationship between NPR_Σ and FOL we show how to translate from diagrams of NPR_Σ to formulas of FOL.

Note that in general terms of NPR_Σ feature "dangling" wires both on the left and on the right of a term. While this is inconsequential from the point of view of expressivity, since terms can always be "rewired" using the compact closed structure of cartesian bicategories, this separation is convenient for composing terms in a flexible manner. Therefore, in the translation in Figure 8, we keep two separate lists of free variables in the context, denoted as $n; m$, where n and m are the lenghts of the two lists.

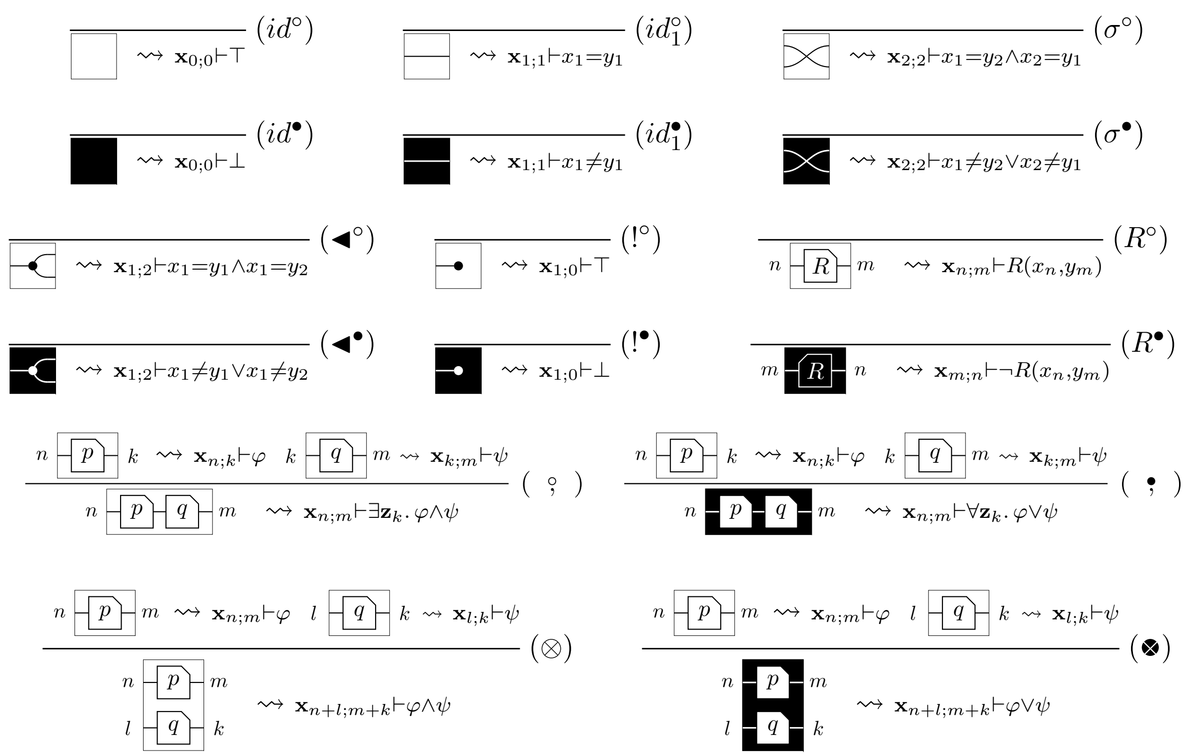


FIGURE 8. Encoding of NPR_Σ diagrams as FOL formulas.

10. Back to Peirce, Tarski and Quine

In this section we illustrate how to encode the calculus of binary relations (Section 10.1), Quine's predicate functor logic (Section 10.2) and Peirce's existential graphs (Section 10.3) within the calculus of neo-Peircean relations.

10.1. **A positive answer to Tarski's question.** In Section 2, we have recalled Tarski's question about axiomatizability of $\leq_{\mathsf{CR}}$. Our unconventional answer is the following:

By leaving the traditional (cartesian) syntax of the calculus of binary relations, for the diagrammatic (monoidal) syntax of the calculus of neo-Peircean relations, one has a complete axiomatisation.

To make this formal, in Table 3 we inductively define an encoding $\mathcal{E}(\cdot)\colon \mathsf{CR}_\Sigma \to \mathsf{NPR}_\Sigma$ assigning to each expressions E of CR_Σ a term $\mathcal{E}(E)\colon 1 \to 1$ of NPR_Σ.

$$\mathcal{E}(id^\circ) \stackrel{\text{def}}{=} id_1^\circ \quad \mathcal{E}(E_1 \fatsemi E_2) \stackrel{\text{def}}{=} \mathcal{E}(E_1) \fatsemi \mathcal{E}(E_2) \quad \mathcal{E}(\top) \stackrel{\text{def}}{=} !_1^\circ \fatsemi ¡_1^\circ \quad \mathcal{E}(E_1 \cap E_2) \stackrel{\text{def}}{=} \blacktriangleleft_1^\circ \fatsemi (\mathcal{E}(E_1) \otimes \mathcal{E}(E_2)) \fatsemi \blacktriangleright_1^\circ$$

$$\mathcal{E}(id^\bullet) \stackrel{\text{def}}{=} id_1^\bullet \quad \mathcal{E}(E_1 \pmb{;} E_2) \stackrel{\text{def}}{=} \mathcal{E}(E_1) \pmb{;} \mathcal{E}(E_2) \quad \mathcal{E}(\bot) \stackrel{\text{def}}{=} !_1^\bullet \pmb{;} ¡_1^\bullet \quad \mathcal{E}(E_1 \cup E_2) \stackrel{\text{def}}{=} \blacktriangleleft_1^\bullet \pmb{;} (\mathcal{E}(E_1) \circledast \mathcal{E}(E_2)) \pmb{;} \blacktriangleright_1^\bullet$$

$$\mathcal{E}(R) \stackrel{\text{def}}{=} R^\circ \qquad \mathcal{E}(\overline{E}) \stackrel{\text{def}}{=} \overline{\mathcal{E}(E)} \qquad \mathcal{E}(E^\dagger) \stackrel{\text{def}}{=} \mathcal{E}(E)^\dagger$$

Table 3. The encoding $\mathcal{E}(\cdot)\colon \mathsf{CR}_\Sigma \to \mathsf{NPR}_\Sigma$

Note that the occurrences of $(\cdot)^\dagger$ and $\overline{(\cdot)}$ on the right-hand side of the equations are those defined in (4.1) and (6.1). As expected, $\mathcal{E}(\cdot)$ preserves the semantics.

Proposition 10.1. *For all expressions E of CR_Σ and interpretations $\mathcal{I}$, $\langle E \rangle_{\mathcal{I}} = \mathcal{I}^\sharp(\mathcal{E}(E))$.*

We report the straightforward inductive proof of the above result in Appendix H.

Corollary 10.2. *For all E_1, E_2, $E_1 \leq_{\mathsf{CR}} E_2$ iff $\mathcal{E}(E_1) \lesssim \mathcal{E}(E_2)$.*

Proof.

$$\begin{aligned}
E_1 \leq_{\mathsf{CR}} E_2 &\iff \forall \mathcal{I}.\ \langle E_1 \rangle_{\mathcal{I}} \subseteq \langle E_2 \rangle_{\mathcal{I}} && \text{(Definition of } \leq_{\mathsf{CR}}) \\
&\iff \forall \mathcal{I}.\ \mathcal{I}^\sharp(\mathcal{E}(E_1)) \subseteq \mathcal{I}^\sharp(\mathcal{E}(E_2)) && \text{(Proposition 10.1)} \\
&\iff \mathcal{E}(E_1) \leqq \mathcal{E}(E_2) && \text{(Definition of } \leqq) \\
&\iff \mathcal{E}(E_1) \lesssim \mathcal{E}(E_2) && \text{(Theorem 3.4)}
\end{aligned}$$

□

In other words, one can check inclusions of expressions of CR_Σ by translating them into NPR_Σ via $\mathcal{E}(\cdot)$ and then using the axioms in Figures 3, 4, 5 and 6.

10.2. **Quine's Predicate Functor Logic.** Inspired by combinatory logic, Quine [Qui71] introduced *predicate functor logic*, PFL_Σ for short, as a quantifier-free treatment of first order logic with equality. Several flavours of the logic have been proposed by Quine and others, here we focus on the treatment by Kuhn [Kuh83]. Using the terminology of that thread of research, for each $n \geq 0$ there is a collection of atomic n-ary predicates, corresponding to traditional FOL predicate symbols together with an additional binary predicate I (identity). The term (predicate) constructors are called *functors* – here the terminology is unrelated to the notion of functor in category theory. These are divided into unary operations

$\mathbf{p}, \mathbf{P}, [,]$ called *combinatory functors* that, in the absence of explicit variables, capture the combinatorial aspects of handling variable lists as well as (existential) quantification. To get full expressivity of FOL, there are two additional *alethic functors*: a binary conjunction and unary negation.

$$P \quad ::= \quad R \mid I \mid \mathbf{p}P \mid \mathbf{P}P \mid [P \mid]P \mid P \cap P \mid \neg P, \quad \text{where } R \in \Sigma$$

$$\frac{-}{I\colon 2} \qquad \frac{ar(R) = n}{R\colon n} \qquad \frac{P\colon n \; n \geq 2}{\mathbf{p}P\colon n} \qquad \frac{P\colon 1}{\mathbf{p}P\colon 2} \qquad \frac{P\colon 0}{\mathbf{p}P\colon 2} \qquad \frac{P\colon n}{\mathbf{P}P\colon n}$$

$$\frac{P_1\colon n \;\; P_2\colon m \;\; n \geq m}{P_1 \cap P_2\colon n} \qquad \frac{P_1\colon n \;\; P_2\colon m \;\; n < m}{P_1 \cap P_2\colon m} \qquad \frac{P\colon n}{\neg P\colon n} \qquad \frac{P\colon n}{[P\colon n+1} \qquad \frac{P\colon n+1}{]P\colon n} \qquad \frac{P\colon 0}{]P\colon 0}$$

$$\langle R \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid (\tau_1, \dots, \tau_n) \in \rho(R)\} \qquad \langle I \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau_1 = \tau_2\} \qquad \langle]P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau_2 \cdot \tau_3 \cdots \in \langle P \rangle_{\mathcal{I}})\}$$

$$\langle P_1 \cap P_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle P_1 \rangle_{\mathcal{I}} \cap \langle P_2 \rangle_{\mathcal{I}} \qquad \langle \neg P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau \notin \langle P \rangle_{\mathcal{I}}\} \qquad \langle [P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{x_0 \cdot \tau_1 \cdot \tau_2 \cdots \mid x_0 \in X, \tau_1 \cdot \tau_2 \cdots \in \langle P \rangle_{\mathcal{I}}\}$$

$$\langle \mathbf{P}P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau_n \cdot \tau_2 \cdots \tau_{n-1} \cdot \tau_1 \cdot \tau_{n+1} \cdots \in \langle P \rangle_{\mathcal{I}}\} \qquad \langle \mathbf{p}P \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \{\tau \mid \tau_2 \cdot \tau_1 \cdots \in \langle P \rangle_{\mathcal{I}}\}$$

TABLE 4. PFL_Σ: (top) syntax; (mid) typing rules; (bottom) semantics w.r.t. an interpretation $\mathcal{I} = (X, \rho)$.

The syntax is reported on the top of Table 4 where R belong to Σ, a set of symbols with an associated arity. Similarly to NPR_Σ, only the predicates that can be typed according to the rules in Table 4 are considered. The semantics, on the bottom, is defined w.r.t. an interpretation $\mathcal{I}$ consisting of a *non-empty* set X and a set $\rho(R) \subseteq X^n$ for all $R \in \Sigma$ of arity n. For all predicates P, $\langle P \rangle_{\mathcal{I}}$ is a subset of $X^\omega \stackrel{\text{def}}{=} \{\tau_1 \cdot \tau_2 \cdots \mid \tau_i \in X \text{ for all } i \in \mathbb{N}^+\}$.

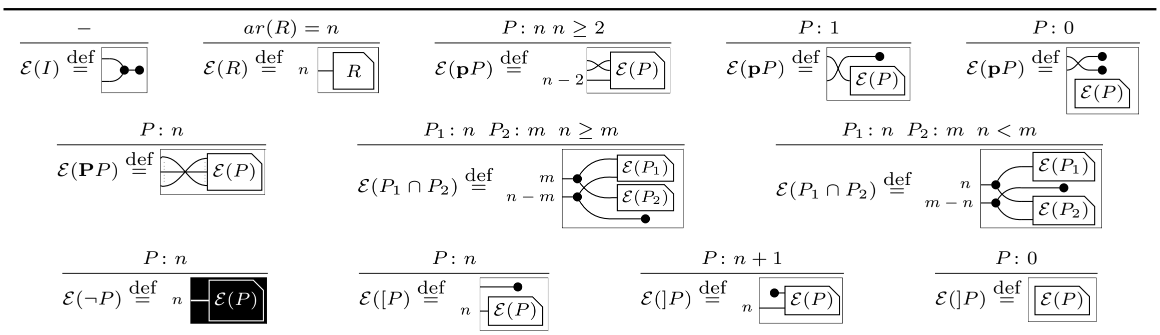
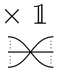


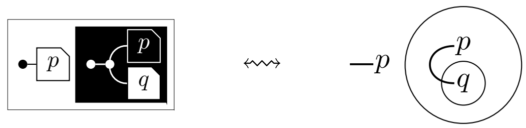
TABLE 5. The encoding $\mathcal{E}(\cdot)\colon \mathsf{PFL}_\Sigma \to \mathsf{NPR}_\Sigma$.

From $\mathcal{I} = (X, \rho)$, one can define an interpretation of NPR_Σ $\mathcal{I}_p \stackrel{\text{def}}{=} (X, \rho_p)$ where $\rho_p(R) \stackrel{\text{def}}{=} \{(x, \star) \mid x \in \rho(R)\} \subseteq X^n \times \mathbb{1}$ for all $R \in \Sigma$ of arity n. The encoding of PFL_Σ into NPR_Σ is given in Table 5 where  is a suggestive representation for the permutation formally defined as $\sigma^\circ_{1,n-1} ; (\sigma^\circ_{n-2,1} \otimes id^\circ_1)$ for $n \geq 2$, id°_n for $n < 2$. The following result (proved in Appendix H) ensures that the encoding preserve the semantics.

Proposition 10.3. *Let $P\colon n$ be a predicate of PFL_Σ. Then*

$$\langle P \rangle_{\mathcal{I}} = \{\tau \mid ((\tau_1, \dots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\}.$$

10.3. **Peirce's Existential Graphs.** The diagrammatic notation of $\mathbf{FOB}_\Sigma$ is closely related to system β of Peirce's EG [Pei20, Rob73]. Consider the two diagrams below corresponding to the closed FOL formula $\exists x.\, p(x) \wedge \forall y.\, p(y) \to q(y)$.



In existential graph notation the circle enclosure (dubbed 'cut' by Peirce) signifies negation. To move from EG to diagrams of $\mathbf{FOB}_\Sigma$ it suffices to treat lines and predicate symbols in the obvious way and each cut as a color switch.

A string diagrammatic approach to existential graphs appeared in [HS20]. This calculus exploits the white fragment of $\mathbf{FOB}_\Sigma$ with a primitive negation operator rendered as Peirce's cut, namely a circle around diagrams. However, this inhibits a fully compositional treatment since, for instance, negation is not functorial. As an example consider Peirce's (de)iteration rule in Figure 9: in $\mathbf{FOB}_\Sigma$ on the left, and in [HS20] on the right.

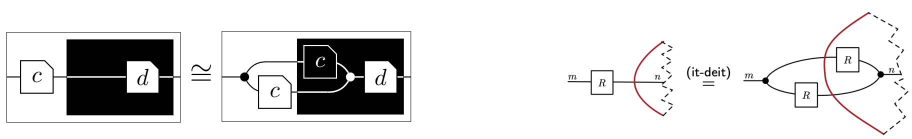
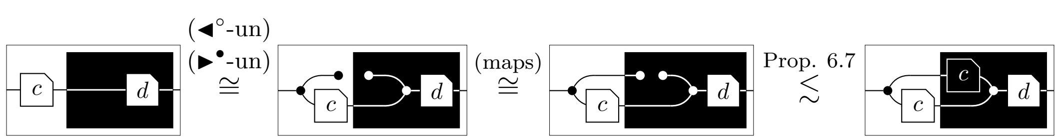


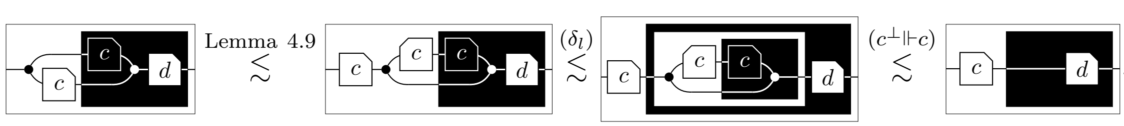
FIGURE 9. Peirce's (de)iteration rule in $\mathbf{FOB}_\Sigma$ (left) and in [HS20] (right).

Note that the diagrams on the right require open cuts, a notational trick, allowing to express the rule for arbitrary contexts, i.e. any diagram eventually appearing inside the cut. In $\mathbf{FOB}_\Sigma$ this ad-hoc treatment of contexts is not needed as negation is not a primitive operation, but a derived one. Moreover, observe that in both Peirce's EG and the calculus in [HS20], the (de)iteration rule is taken as an axiom, while in $\mathbf{FOB}_\Sigma$ the rule is derivable, as shown below.

Proof of Peirce's (de)iteration rule in $\mathbf{FOB}_\Sigma$. We prove the two inclusions separately:



and



□

11. A tribute to Charles S. Peirce

We have chosen the name "Neo-Peircean Relations" to emphasize several connections with the work of Charles S. Peirce.

First of all, NPR_{Σ} and the calculus of relations in 'Note B' [Pei83] share the same underlying philosophy: they both propose a relational analogue to Boole's algebra of classes.

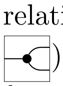
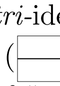
Second, Peirce's presentation in 'Note B' contains already several key ingredients of NPR_{Σ}. As we have stressed, it singles out the two forms of composition ($\mathbin{\mathring{;}}$ and $\mathbin{\bullet\!;}$), presents linear distributivity ((δ_l) and (δ_r)) and linear adjunctions ((τR°), ($\tau R^{\bullet}$), (γR°), and ($\gamma R^{\bullet}$)), and even the cyclic conditions of Lemma 6.8.(2)-(3). With respect to the rules for linear distributivity and linear adjunction, Peirce states that the latter are "highly important" and that the former are "so constantly used that hardly anything can be done without them" (p. 192 & 190).

At around the same time as 'Note B' Pierce gave a systematic study of residuation [Pei89, see "On the Logic of Relatives"] and listed a set of equivalent expressions that includes Lemma 5.7:

$$c \mathbin{\mathring{;}} a \leq b \text{ iff } c \leq b \mathbin{\bullet\!;} a^{\perp}.$$

In Peirce's words:

> "Hence the rule is that having a formula of the form [$c \mathbin{\mathring{;}} a \leq b$], the three letters may be cyclically advanced one place in the order of writing, those which are carried from one side of the copula to the other being both negatived and converted." [Pei89, p. 341]

Peirce took the principal defect of the presentation in 'Note B' to be its focus on binary relations [Pei66, 8:831]. He went on to emphasize the *teri-* or *tri-*identity relation (in NPR_{Σ}, ), arising from adding a 'branch' to the identity relation (), as the key to moving from binary to arbitrary relations. Having the advantage now of "treating triadic and higher relations as easily as dyadic relations... it's superiority to the human mind as an instrument of logic", he writes, "is overwhelming" [Pei22, p. 173].

By moving from binary to arbitrary relations, Peirce felt the importance of a graphical syntax and developed the existential graphs.

> "One of my earliest works was an enlargement of Boole's idea so as to take into account ideas of relation, — or at least of all ideas of existential relation. . . *I was finally led to prefer what I call a diagrammatic syntax.* It is a way of setting down on paper any assertion, however intricate" [MS515, emphasis in original, 1911].

We refer the reader to [HS20] for a detailed explanation of Peirce's topological intuitions behind the Frobenius equations and the correspondence of some inference rules for EGs with those of (co)cartesian bicategories. Moreover, we now know that Peirce continued to study and draw graphs of residuation [HP21] and — as affirmed in Fig. 7 — we know the rules for propositional EGs comprise a deep inference system [MP17].

In short, Peirce's development of EGs shares many of the features that NPR_{Σ} has over other approaches, such as Tarski's presentation of relation algebra. We like to think that if Peirce had known category theory then he would have presented NPR_{Σ}.

12. Concluding Remarks

We introduced NPR_Σ, a calculus of relations with the expressivity of first order logic. We showed that it enjoys a sound and complete axiomatisation that arises through the interaction of two well-known categorical structures: cartesian bicategories and linear bicategories. We characterised these and dubbed the result fo-bicategories. Below we list some further related and future work.

In § 10 we already compared NPR_Σ to [HS20]. Other diagrammatic calculi, reminiscent of Peirce's EGs, appear in [MZ16] and [BT98]. The categorical treatment goes, respectively, through the notions of chiralities and doctrines. The formers consider a pair of categories $(\mathbf{Rel}_\bullet, \mathbf{Rel}_\circ)$ that are significantly different from our $\mathbf{Rel}^\circ$ and $\mathbf{Rel}^\bullet$. Instead fo-bicategies are equivalent to boolean hyperdoctrines, as recently illustrated in [BGT24]. The connection with allegories [FS90] is also worth to be explored: since cartesian bicategories are equivalent to unitary pretabular allegories, Proposition 6.7 suggests that fo-bicategories are closely related to Peirce allegories [OS97].

It is worth remarking that NPR_Σ only deals with *classical* FOL, as hinted by the fact that the homsets of a fo-bicategory are Boolean algebras (Proposition 6.7). Hopefully, the intuitionistic case might be handled by relaxing some of the conditions of Definition 6.1.

To conclude it is worth mentioning a further research direction. We plan to extend to FOL, the combinatorial characterisation of its regular fragment in terms of hypergraphs [CM77] and the associated rewriting approach [BGK+22]. Some relevant insights may come from [RC24]. Moreover, we foresee the possibility of defining a deep inference system having as rules the inequalities of our axiomatisation and compare its proof theory with [Brü06, Ral19, HSW21].

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Appendix A. Complete axiomatisation of NPR_Σ

$$
\begin{array}{rcl@{\qquad}rcl}
\blacktriangleleft^{\circ}_1 \fatsemi (id^{\circ}_1 \otimes \blacktriangleleft^{\circ}_1) & \overset{(\blacktriangleleft^{\circ}\text{-as})}{=} & \blacktriangleleft^{\circ}_1 \fatsemi (\blacktriangleleft^{\circ}_1 \otimes id^{\circ}_1) & (id^{\circ}_1 \otimes \blacktriangleright^{\circ}_1) \fatsemi \blacktriangleright^{\circ}_1 & \overset{(\blacktriangleright^{\circ}\text{-as})}{=} & (\blacktriangleright^{\circ}_1 \otimes id^{\circ}_1) \fatsemi \blacktriangleright^{\circ}_1 \\
\blacktriangleleft^{\circ}_1 \fatsemi (id^{\circ}_1 \otimes !^{\circ}_1) & \overset{(\blacktriangleleft^{\circ}\text{-un})}{=} & id^{\circ}_1 & (id^{\circ}_1 \otimes \text{¡}^{\circ}_1) \fatsemi \blacktriangleright^{\circ}_1 & \overset{(\blacktriangleright^{\circ}\text{-un})}{=} & id^{\circ}_1 \\
\blacktriangleleft^{\circ}_1 \fatsemi \sigma^{\circ}_{1,1} & \overset{(\blacktriangleleft^{\circ}\text{-co})}{=} & \blacktriangleleft^{\circ}_1 & \sigma^{\circ}_{1,1} \fatsemi \blacktriangleright^{\circ}_1 & \overset{(\blacktriangleright^{\circ}\text{-co})}{=} & \blacktriangleright^{\circ}_1
\end{array}
$$

$$
(\blacktriangleleft^{\circ}_1 \otimes id^{\circ}_1) \fatsemi (id^{\circ}_1 \otimes \blacktriangleright^{\circ}_1) \overset{(\mathrm{F}^{\circ})}{=} (id^{\circ}_1 \otimes \blacktriangleleft^{\circ}_1) \fatsemi (\blacktriangleright^{\circ}_1 \otimes id^{\circ}_1) \qquad \blacktriangleleft^{\circ}_1 \fatsemi \blacktriangleright^{\circ}_1 \overset{(\mathrm{S}^{\circ})}{=} id^{\circ}_1
$$

$$
\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\text{¡}^{\circ}_1 \fatsemi !^{\circ}_1 & \overset{(\epsilon !^{\circ})}{\leq} & id^{\circ}_0 & \blacktriangleright^{\circ}_1 \fatsemi \blacktriangleleft^{\circ}_1 & \overset{(\epsilon \blacktriangleleft^{\circ})}{\leq} & (id^{\circ}_1 \otimes id^{\circ}_1) & c \fatsemi \blacktriangleleft^{\circ}_m & \overset{(\blacktriangleleft^{\circ}\text{-nat})}{\leq} & \blacktriangleleft^{\circ}_n \fatsemi (c \otimes c) \\
id^{\circ}_1 & \overset{(\eta !^{\circ})}{\leq} & !^{\circ}_1 \fatsemi \text{¡}^{\circ}_1 & id^{\circ}_1 & \overset{(\eta \blacktriangleleft^{\circ})}{\leq} & \blacktriangleleft^{\circ}_1 \fatsemi \blacktriangleright^{\circ}_1 & c \fatsemi !^{\circ}_m & \overset{(!^{\circ}\text{-nat})}{\leq} & !^{\circ}_n
\end{array}
$$

$$
\begin{array}{rcl@{\qquad}rcl}
\blacktriangleleft^{\bullet}_1 \mathbin{\boldsymbol{;}} (id^{\bullet}_1 \boldsymbol{\otimes} \blacktriangleleft^{\bullet}_1) & \overset{(\blacktriangleleft^{\bullet}\text{-as})}{=} & \blacktriangleleft^{\bullet}_1 \mathbin{\boldsymbol{;}} (\blacktriangleleft^{\bullet}_1 \boldsymbol{\otimes} id^{\bullet}_1) & (id^{\bullet}_1 \boldsymbol{\otimes} \blacktriangleright^{\bullet}_1) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_1 & \overset{(\blacktriangleright^{\bullet}\text{-as})}{=} & (\blacktriangleright^{\bullet}_1 \boldsymbol{\otimes} id^{\bullet}_1) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_1 \\
\blacktriangleleft^{\bullet}_1 \mathbin{\boldsymbol{;}} (id^{\bullet}_1 \boldsymbol{\otimes} !^{\bullet}_1) & \overset{(\blacktriangleleft^{\bullet}\text{-un})}{=} & id^{\bullet}_1 & (id^{\bullet}_1 \boldsymbol{\otimes} \text{¡}^{\bullet}_1) \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_1 & \overset{(\blacktriangleright^{\bullet}\text{-un})}{=} & id^{\bullet}_1 \\
\blacktriangleleft^{\bullet}_1 \mathbin{\boldsymbol{;}} \sigma^{\bullet}_{1,1} & \overset{(\blacktriangleleft^{\bullet}\text{-co})}{=} & \blacktriangleleft^{\bullet}_1 & \sigma^{\bullet}_{1,1} \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_1 & \overset{(\blacktriangleright^{\bullet}\text{-co})}{=} & \blacktriangleright^{\bullet}_1
\end{array}
$$

$$
(\blacktriangleleft^{\bullet}_1 \boldsymbol{\otimes} id^{\bullet}_1) \mathbin{\boldsymbol{;}} (id^{\bullet}_1 \boldsymbol{\otimes} \blacktriangleright^{\bullet}_1) \overset{(\mathrm{F}^{\bullet})}{=} (id^{\bullet}_1 \boldsymbol{\otimes} \blacktriangleleft^{\bullet}_1) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\bullet}_1 \boldsymbol{\otimes} id^{\bullet}_1) \qquad \blacktriangleleft^{\bullet}_1 \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_1 \overset{(\mathrm{S}^{\bullet})}{=} id^{\bullet}_1
$$

$$
\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
!^{\bullet}_1 \mathbin{\boldsymbol{;}} \text{¡}^{\bullet}_1 & \overset{(\epsilon <^{\bullet})}{\leq} & id^{\bullet}_1 & \blacktriangleleft^{\bullet}_1 \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_1 & \overset{(\epsilon \blacktriangleright^{\bullet})}{\leq} & id^{\bullet}_1 & \blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} (c \boldsymbol{\otimes} c) & \overset{(\blacktriangleleft^{\bullet}\text{-nat})}{\leq} & c \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_m \\
id^{\bullet}_0 & \overset{(\eta <^{\bullet})}{\leq} & \text{¡}^{\bullet}_1 \mathbin{\boldsymbol{;}} !^{\bullet}_1 & \blacktriangleright^{\bullet}_1 \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_1 & \overset{(\eta \blacktriangleright^{\bullet})}{\leq} & (id^{\bullet}_1 \boldsymbol{\otimes} id^{\bullet}_1) & !^{\bullet}_n & \overset{(!^{\bullet}\text{-nat})}{\leq} & c \mathbin{\boldsymbol{;}} !^{\bullet}_m
\end{array}
$$

$$
a \fatsemi (b \mathbin{\boldsymbol{;}} c) \overset{(\delta_l)}{\leq} (a \fatsemi b) \mathbin{\boldsymbol{;}} c \qquad (a \mathbin{\boldsymbol{;}} b) \fatsemi c \overset{(\delta_r)}{\leq} a \mathbin{\boldsymbol{;}} (b \fatsemi c)
$$

$$
\begin{array}{rcl@{\quad}rcl@{\qquad}rcl@{\quad}rcl}
id^{\circ}_{n+m} & \overset{(\tau\sigma^{\circ})}{\leq} & \sigma^{\circ}_{n,m} \mathbin{\boldsymbol{;}} \sigma^{\bullet}_{m,n} & \sigma^{\bullet}_{n,m} \fatsemi \sigma^{\circ}_{m,n} & \overset{(\gamma\sigma^{\circ})}{\leq} & id^{\bullet}_{n+m} & id^{\circ}_n & \overset{(\tau R^{\circ})}{\leq} & R^{\circ} \mathbin{\boldsymbol{;}} R^{\bullet} & R^{\bullet} \fatsemi R^{\circ} & \overset{(\gamma R^{\circ})}{\leq} & id^{\bullet}_m \\
id^{\circ}_{n+m} & \overset{(\tau\sigma^{\bullet})}{\leq} & \sigma^{\bullet}_{n,m} \mathbin{\boldsymbol{;}} \sigma^{\circ}_{m,n} & \sigma^{\circ}_{n,m} \fatsemi \sigma^{\bullet}_{m,n} & \overset{(\gamma\sigma^{\bullet})}{\leq} & id^{\bullet}_{n+m} & id^{\circ}_m & \overset{(\tau R^{\bullet})}{\leq} & R^{\bullet} \mathbin{\boldsymbol{;}} R^{\circ} & R^{\circ} \fatsemi R^{\bullet} & \overset{(\gamma R^{\bullet})}{\leq} & id^{\bullet}_n
\end{array}
$$

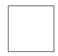
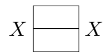
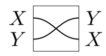
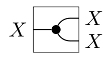
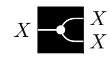
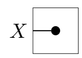
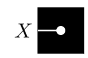
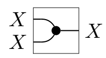
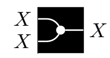
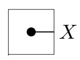
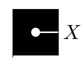
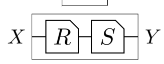
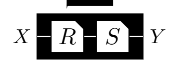
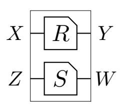
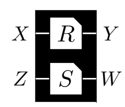
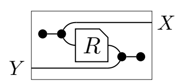
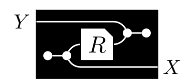
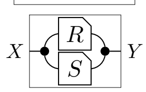
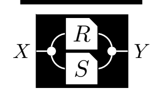
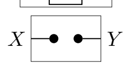
$$
\begin{array}{rcl@{\qquad}rcl}
id^{\circ}_{n+m} & \overset{(\boldsymbol{\otimes}^{\circ})}{\leq} & id^{\circ}_n \boldsymbol{\otimes} id^{\circ}_m & id^{\bullet}_n \otimes id^{\bullet}_m & \overset{(\otimes^{\bullet})}{\leq} & id^{\bullet}_{n+m} \\
(a \mathbin{\boldsymbol{;}} b) \otimes (c \mathbin{\boldsymbol{;}} d) & \overset{(\nu^{\circ}_l)}{\leq} & (a \otimes c) \mathbin{\boldsymbol{;}} (b \boldsymbol{\otimes} d) & (a \boldsymbol{\otimes} c) \fatsemi (b \otimes d) & \overset{(\nu^{\bullet}_l)}{\leq} & (a \fatsemi b) \boldsymbol{\otimes} (c \fatsemi d) \\
(a \mathbin{\boldsymbol{;}} b) \otimes (c \mathbin{\boldsymbol{;}} d) & \overset{(\nu^{\circ}_r)}{\leq} & (a \boldsymbol{\otimes} c) \mathbin{\boldsymbol{;}} (b \otimes d) & (a \otimes c) \fatsemi (b \boldsymbol{\otimes} d) & \overset{(\nu^{\bullet}_r)}{\leq} & (a \fatsemi b) \boldsymbol{\otimes} (c \fatsemi d)
\end{array}
$$

$$
\begin{array}{rcl@{\quad}rcl@{\qquad}rcl@{\quad}rcl}
id^{\circ}_n & \overset{(\tau \blacktriangleleft^{\circ})}{\leq} & \blacktriangleleft^{\circ}_n \mathbin{\boldsymbol{;}} \blacktriangleright^{\bullet}_n & \blacktriangleright^{\bullet}_n \fatsemi \blacktriangleleft^{\circ}_n & \overset{(\gamma \blacktriangleleft^{\circ})}{\leq} & id^{\bullet}_{n+n} & id^{\circ}_{n+n} & \overset{(\tau \blacktriangleright^{\circ})}{\leq} & \blacktriangleright^{\circ}_n \mathbin{\boldsymbol{;}} \blacktriangleleft^{\bullet}_n & \blacktriangleleft^{\bullet}_n \fatsemi \blacktriangleright^{\circ}_n & \overset{(\gamma \blacktriangleright^{\circ})}{\leq} & id^{\bullet}_n \\
id^{\circ}_n & \overset{(\tau !^{\circ})}{\leq} & !^{\circ}_n \mathbin{\boldsymbol{;}} \text{¡}^{\bullet}_n & \text{¡}^{\bullet}_n \fatsemi !^{\circ}_n & \overset{(\gamma !^{\circ})}{\leq} & id^{\bullet}_0 & id^{\circ}_0 & \overset{(\tau <^{\circ})}{\leq} & \text{¡}^{\circ}_n \mathbin{\boldsymbol{;}} !^{\bullet}_n & !^{\bullet}_n \fatsemi \text{¡}^{\circ}_n & \overset{(\gamma <^{\circ})}{\leq} & id^{\bullet}_n \\
id^{\circ}_n & \overset{(\tau \blacktriangleleft^{\bullet})}{\leq} & \blacktriangleleft^{\bullet}_n \mathbin{\boldsymbol{;}} \blacktriangleright^{\circ}_n & \blacktriangleright^{\circ}_n \fatsemi \blacktriangleleft^{\bullet}_n & \overset{(\gamma \blacktriangleleft^{\bullet})}{\leq} & id^{\bullet}_{n+n} & id^{\circ}_{n+n} & \overset{(\tau \blacktriangleright^{\bullet})}{\leq} & \blacktriangleright^{\bullet}_n \mathbin{\boldsymbol{;}} \blacktriangleleft^{\circ}_n & \blacktriangleleft^{\circ}_n \fatsemi \blacktriangleright^{\bullet}_n & \overset{(\gamma \blacktriangleright^{\bullet})}{\leq} & id^{\bullet}_n \\
id^{\circ}_n & \overset{(\tau !^{\bullet})}{\leq} & !^{\bullet}_n \mathbin{\boldsymbol{;}} \text{¡}^{\circ}_n & \text{¡}^{\circ}_n \fatsemi !^{\bullet}_n & \overset{(\gamma !^{\bullet})}{\leq} & id^{\bullet}_0 & id^{\circ}_0 & \overset{(\tau <^{\bullet})}{\leq} & \text{¡}^{\bullet}_n \mathbin{\boldsymbol{;}} !^{\circ}_n & !^{\circ}_n \fatsemi \text{¡}^{\bullet}_n & \overset{(\gamma <^{\bullet})}{\leq} & id^{\bullet}_0
\end{array}
$$

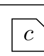
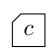
$$
\begin{array}{rcl@{\quad}rcl}
(\blacktriangleleft^{\bullet}_n \otimes id^{\circ}_n) \fatsemi (id^{\circ}_n \otimes \blacktriangleright^{\circ}_n) & \overset{(\mathrm{F}^{\bullet}{}_{\circ})}{=} & (id^{\circ}_n \otimes \blacktriangleleft^{\circ}_n) \fatsemi (\blacktriangleright^{\bullet}_n \otimes id^{\circ}_n) & (\blacktriangleleft^{\circ}_n \otimes id^{\circ}_n) \fatsemi (id^{\circ}_n \otimes \blacktriangleright^{\bullet}_n) & \overset{(\mathrm{F}^{\circ}{}_{\bullet})}{=} & (id^{\circ}_n \otimes \blacktriangleleft^{\bullet}_n) \fatsemi (\blacktriangleright^{\circ}_n \otimes id^{\circ}_n) \\
(\blacktriangleleft^{\circ}_n \boldsymbol{\otimes} id^{\bullet}_n) \mathbin{\boldsymbol{;}} (id^{\bullet}_n \boldsymbol{\otimes} \blacktriangleright^{\bullet}_n) & \overset{(\mathrm{F}_{\bullet}{}^{\circ})}{=} & (id^{\bullet}_n \boldsymbol{\otimes} \blacktriangleleft^{\bullet}_n) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\circ}_n \boldsymbol{\otimes} id^{\bullet}_n) & (\blacktriangleleft^{\bullet}_n \boldsymbol{\otimes} id^{\bullet}_n) \mathbin{\boldsymbol{;}} (id^{\bullet}_n \boldsymbol{\otimes} \blacktriangleright^{\circ}_n) & \overset{(\mathrm{F}_{\circ}{}^{\bullet})}{=} & (id^{\bullet}_n \boldsymbol{\otimes} \blacktriangleleft^{\circ}_n) \mathbin{\boldsymbol{;}} (\blacktriangleright^{\bullet}_n \boldsymbol{\otimes} id^{\bullet}_n)
\end{array}
$$

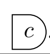
Figure 10. Axioms for NPR_Σ. Here a, b, c, d are properly typed terms.

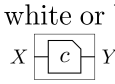
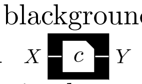
Appendix B. Dictionary

White fragment			Black fragment		
Empty	$id^{\circ}_{\mathbb{1}}$		Empty	$id^{\bullet}_{\mathbb{1}}$	
Identity	id°_{X}		Identity	$id^{\bullet}_{X}$	
Symmetry	$\sigma^{\circ}_{X,Y}$		Symmetry	$\sigma^{\bullet}_{X,Y}$	
Copier	$◀^{\circ}_{X}$		Copier	$◀^{\bullet}_{X}$	
Discharger	$!^{\circ}_{X}$		Discharger	$!^{\bullet}_{X}$	
Cocopier	$▶^{\circ}_{X}$		Cocopier	$▶^{\bullet}_{X}$	
Codischarger	$¡^{\circ}_{X}$		Codischarger	$¡^{\bullet}_{X}$	
Composition	$R \fatsemi S$		Composition	$R \,;\!\bullet\, S$	
Monoidal product	$R \otimes S$		Monoidal product	$R \circledast S$	
Dagger	$R^{\dagger}$		Dagger	$R^{\ddagger}$	
Meet	$R \sqcap S$		Join	$R \sqcup S$	
Top	$\top$		Bottom	$\bot$	

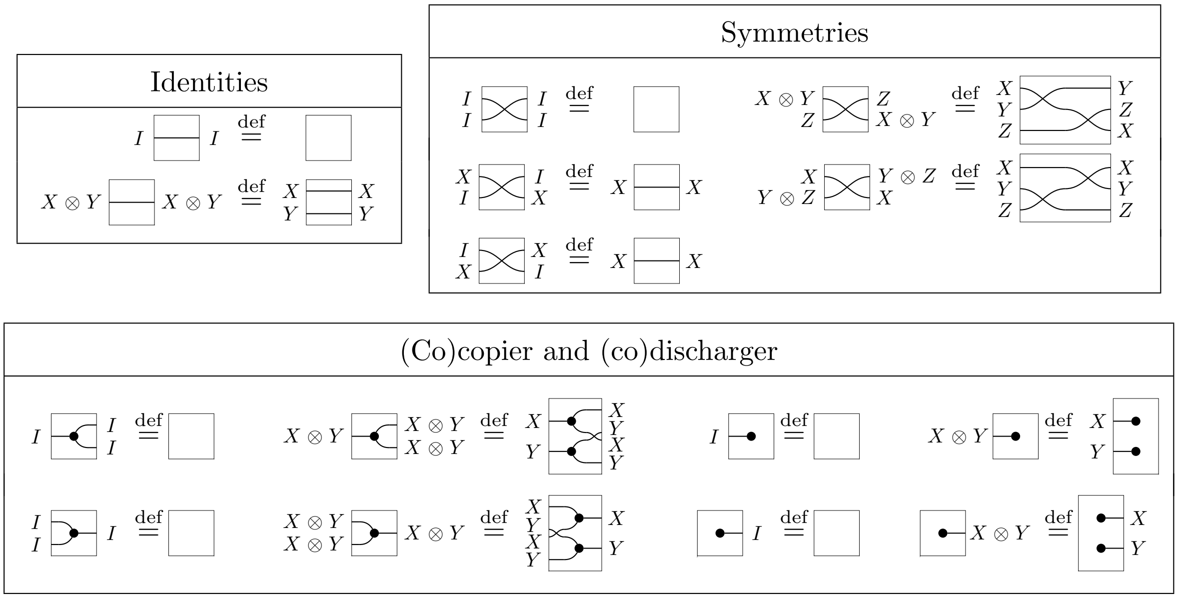
Diagrammatic conventions

For a generic arrow c , we draw its
- negation $\overline{c}$ as 
- opposite $c^{\dagger}$ as 
- linear adjoint $c^{\perp}$ as ,

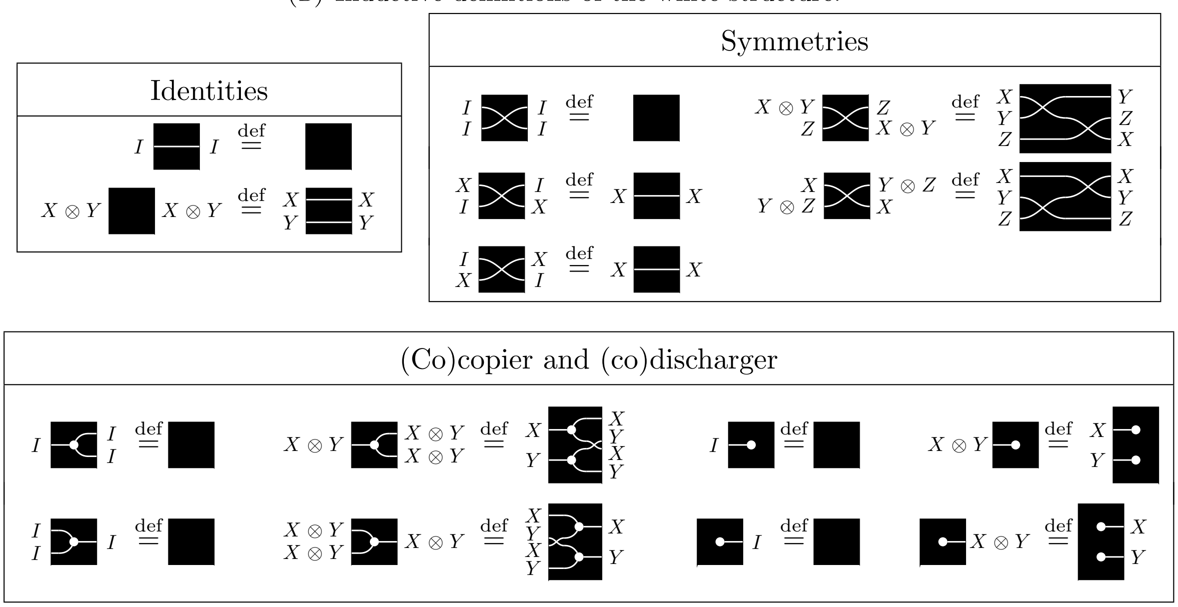
and if it is a map as .

Note: each of the above can be drawn on either a white or black blackground. For instance, a generic arrow $c\colon X \to Y$ can be drawn both as  and . The choice of background serves only to distinguish between the compositions in the two fragments.

(A) Correspondence between term and graphical notation; and diagrammatic conventions.



(B) Inductive definitions of the white structure.



(C) Inductive definitions of the black structure.

TABLE 6

Appendix C. Appendix to Section 5

Proof of Proposition 5.10. First, we prove that for all $a, b \colon X \to Y$ it holds that

(0) if $a \leq b$ then $a^\perp \geq b^\perp$

The proof is illustrated below.

(0)

$$\begin{aligned} b^\perp &= b^\perp \mathbin{\overset{\circ}{,}} id^\circ_Y \\ &\leq b^\perp \mathbin{\overset{\circ}{,}} (a \mathbin{\overset{\bullet}{,}} a^\perp) && (a^\perp \Vdash a) \\ &\leq (b^\perp \mathbin{\overset{\circ}{,}} a) \mathbin{\overset{\bullet}{,}} a^\perp && (\delta_l) \\ &\leq (b^\perp \mathbin{\overset{\circ}{,}} b) \mathbin{\overset{\bullet}{,}} a^\perp && (a \leq b) \\ &\leq id^\bullet_Y \mathbin{\overset{\bullet}{,}} a^\perp && (b^\perp \Vdash b) \\ &= a^\perp \end{aligned}$$

We next illustrate that for all $a \colon X \to Y$ and $b \colon Y \to Z$

(1) $(id^\circ_X)^\perp = id^\bullet_X$
(2) $(id^\bullet_X)^\perp = id^\circ_X$
(3) $(a \mathbin{\overset{\circ}{,}} b)^\perp = b^\perp \mathbin{\overset{\bullet}{,}} a^\perp$
(4) $(a \mathbin{\overset{\bullet}{,}} b)^\perp = b^\perp \mathbin{\overset{\circ}{,}} a^\perp$

The proofs are dispayed below.

(1) Observe that $id^\circ_X = id^\circ_X \mathbin{\overset{\bullet}{,}} id^\bullet_X$ and $id^\bullet_X \mathbin{\overset{\circ}{,}} id^\circ_X = id^\bullet_X$. Thus, by Lemma 5.6, $(id^\circ_X)^\perp = id^\bullet_X$.
(2) Similarly, $id^\circ_X = id^\bullet_X \mathbin{\overset{\bullet}{,}} id^\circ_X$ and $id^\circ_X \mathbin{\overset{\circ}{,}} id^\bullet_X = id^\bullet_X$. Again, by Lemma 5.6, $(id^\bullet_X)^\perp = id^\circ_X$.
(3) The following two derivations

$$\begin{aligned} id^\circ_X &\leq a \mathbin{\overset{\bullet}{,}} a^\perp && (a^\perp \Vdash a) \\ &= (a \mathbin{\overset{\circ}{,}} id^\circ_Y) \mathbin{\overset{\bullet}{,}} a^\perp \\ &\leq (a \mathbin{\overset{\circ}{,}} (b \mathbin{\overset{\bullet}{,}} b^\perp)) \mathbin{\overset{\bullet}{,}} a^\perp && (b^\perp \Vdash b) \\ &\leq ((a \mathbin{\overset{\circ}{,}} b) \mathbin{\overset{\bullet}{,}} b^\perp) \mathbin{\overset{\bullet}{,}} a^\perp && (\delta_l) \\ &= (a \mathbin{\overset{\circ}{,}} b) \mathbin{\overset{\bullet}{,}} (b^\perp \mathbin{\overset{\bullet}{,}} a^\perp) \end{aligned}$$

$$\begin{aligned} (b^\perp \mathbin{\overset{\bullet}{,}} a^\perp) \mathbin{\overset{\circ}{,}} (a \mathbin{\overset{\circ}{,}} b) &= ((b^\perp \mathbin{\overset{\bullet}{,}} a^\perp) \mathbin{\overset{\circ}{,}} a) \mathbin{\overset{\circ}{,}} b \\ &\leq (b^\perp \mathbin{\overset{\bullet}{,}} (a^\perp \mathbin{\overset{\circ}{,}} a)) \mathbin{\overset{\circ}{,}} b && (\delta_r) \\ &\leq (b^\perp \mathbin{\overset{\bullet}{,}} id^\bullet_Y) \mathbin{\overset{\circ}{,}} b && (a^\perp \Vdash a) \\ &= b^\perp \mathbin{\overset{\circ}{,}} b \\ &\leq id^\bullet_Z && (b^\perp \Vdash b) \end{aligned}$$

show that $(b^\perp \mathbin{\overset{\bullet}{,}} a^\perp) \Vdash (a \mathbin{\overset{\circ}{,}} b)$. Thus, by Lemma 5.6, $(a \mathbin{\overset{\circ}{,}} b)^\perp = b^\perp \mathbin{\overset{\bullet}{,}} a^\perp$.

(4) The following two derivations

$$\begin{aligned} id^\circ_X &\leq a \mathbin{\overset{\bullet}{,}} a^\perp && (a^\perp \Vdash a) \\ &= a \mathbin{\overset{\bullet}{,}} (id^\circ_Y \mathbin{\overset{\circ}{,}} a^\perp) \\ &\leq a \mathbin{\overset{\bullet}{,}} ((b \mathbin{\overset{\bullet}{,}} b^\perp) \mathbin{\overset{\circ}{,}} a^\perp) && (b^\perp \Vdash b) \\ &\leq a \mathbin{\overset{\bullet}{,}} (b \mathbin{\overset{\bullet}{,}} (b^\perp \mathbin{\overset{\circ}{,}} a^\perp)) && (\delta_r) \\ &= (a \mathbin{\overset{\bullet}{,}} b) \mathbin{\overset{\bullet}{,}} (b^\perp \mathbin{\overset{\circ}{,}} a^\perp) \end{aligned}$$

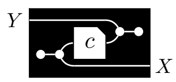
$$\begin{aligned} (b^\perp \mathbin{\overset{\circ}{,}} a^\perp) \mathbin{\overset{\circ}{,}} (a \mathbin{\overset{\bullet}{,}} b) &= b^\perp \mathbin{\overset{\circ}{,}} (a^\perp \mathbin{\overset{\circ}{,}} (a \mathbin{\overset{\bullet}{,}} b)) \\ &= b^\perp \mathbin{\overset{\circ}{,}} ((a^\perp \mathbin{\overset{\circ}{,}} a) \mathbin{\overset{\bullet}{,}} b) && (\delta_l) \\ &\leq b^\perp \mathbin{\overset{\circ}{,}} (id^\bullet_Y \mathbin{\overset{\bullet}{,}} b) && (a^\perp \Vdash a) \\ &= b^\perp \mathbin{\overset{\circ}{,}} b \\ &\leq id^\bullet_Z && (b^\perp \Vdash b) \end{aligned}$$

show that $(b^\perp \mathbin{\overset{\circ}{,}} a^\perp) \Vdash (a \mathbin{\overset{\bullet}{,}} b)$. Thus, by Lemma 5.6, $(a \mathbin{\overset{\bullet}{,}} b)^\perp = b^\perp \mathbin{\overset{\circ}{,}} a^\perp$.

The remaining cases are illustrated in the main text. □

Appendix D. Appendix to Section 6

D.1. **Proof of Proposition 6.3.** In this appendix, we illustrate a proof of Proposition 6.3. In order to prove that $(\cdot)^{\dagger}\colon \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ is a morphism of fo-bicategories, it is convenient to observe that in a cocartesian bicategory there is an analogue operation $(\cdot)^{\ddagger}\colon \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ that is an identity on object isomorphism, mapping every arrow $c\colon X \to Y$ to $c^{\ddagger}\colon Y \to X$ defined as below on the left.

$c^{\ddagger} \stackrel{\text{def}}{=}$ 

Properties of $(\cdot)^{\ddagger}\colon (\mathbf{C}, \blacktriangleleft^{\circ}, \blacktriangleright^{\circ}, \blacktriangleleft^{\bullet}, \blacktriangleright^{\bullet}) \to (\mathbf{C}^{\mathrm{op}}, \blacktriangleright^{\circ}, \blacktriangleleft^{\circ}, \blacktriangleright^{\bullet}, \blacktriangleleft^{\bullet})$			
if $c \leq d$ then $c^{\ddagger} \leq d^{\ddagger}$		$(c^{\ddagger})^{\ddagger} = c$	
$(c \mathbin{;} d)^{\ddagger} = d^{\ddagger} \mathbin{;} c^{\ddagger}$	$(id^{\circ}_X)^{\ddagger} = id^{\circ}_X$	$(\blacktriangleright^{\circ}_X)^{\ddagger} = \blacktriangleleft^{\circ}_X$	$(¡^{\circ}_X)^{\ddagger} = !^{\circ}_X$
$(c \otimes d)^{\ddagger} = c^{\ddagger} \otimes d^{\ddagger}$	$(\sigma^{\circ}_{X,Y})^{\ddagger} = \sigma^{\circ}_{Y,X}$	$(\blacktriangleleft^{\circ}_X)^{\ddagger} = \blacktriangleright^{\circ}_X$	$(!^{\circ}_X)^{\ddagger} = ¡^{\circ}_X$
$(c \mathbin{;}^{\bullet} d)^{\ddagger} = d^{\ddagger} \mathbin{;}^{\bullet} c^{\ddagger}$	$(id^{\bullet}_X)^{\ddagger} = id^{\bullet}_X$	$(\blacktriangleright^{\bullet}_X)^{\ddagger} = \blacktriangleleft^{\bullet}_X$	$(¡^{\bullet}_X)^{\ddagger} = !^{\bullet}_X$
$(c \otimes^{\bullet} d)^{\ddagger} = c^{\ddagger} \otimes^{\bullet} d^{\ddagger}$	$(\sigma^{\bullet}_{X,Y})^{\ddagger} = \sigma^{\bullet}_{Y,X}$	$(\blacktriangleleft^{\bullet}_X)^{\ddagger} = \blacktriangleright^{\bullet}_X$	$(!^{\bullet}_X)^{\ddagger} = ¡^{\bullet}_X$

Table 7

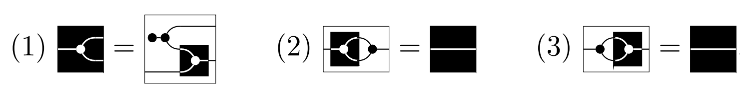
Proposition D.1. *$(\cdot)^{\ddagger}\colon \mathbf{C} \to \mathbf{C}^{op}$ is an isomorphism of cocartesian bicategories, that is the rules in the first three rows of Table 7 hold.*

Proof. See Theorem 2.4 in [CW87]. □

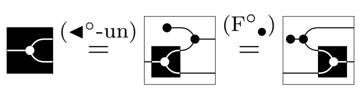
In a fo-bicategory, it turns out that $(\cdot)^{\dagger}$ and $(\cdot)^{\ddagger}$ coincide. For example, in **Rel**, both operations correspond to taking the converse of a relation. To establish this result, it is helpful to first examine their behavior on identities (Lemma D.2) and their interaction with $(\cdot)^{\perp}$ (Lemma D.3).

Lemma D.2. *For all X it holds that $id^{\bullet}_X = (id^{\bullet}_X)^{\dagger}$ and $id^{\circ}_X = (id^{\circ}_X)^{\ddagger}$.*

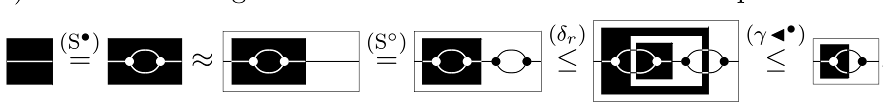
Proof. First we prove the following equalities:



The proofs are shown below.

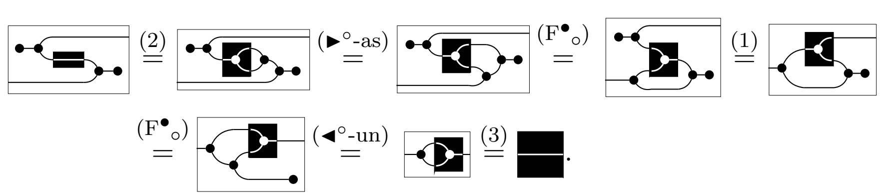
(1) is proved as follows: 

(2) is $(\gamma \blacktriangleright^{\circ})$ for the left-to-right inclusion and the other inclusion is proved below:



(3) is proved analogously, but exploiting (δ_l) and $(\gamma \blacktriangleleft^{\circ})$.

Now we are ready to show that $id^{\bullet}_X = (id^{\bullet}_X)^{\dagger}$.



The proof for $id^{\circ}_X = (id^{\circ}_X)^{\ddagger}$ is analogous. ☐

Lemma D.3. *For all* $a \colon X \to Y$ *it holds that* $(a^{\dagger})^{\perp} = (a^{\perp})^{\ddagger}$.

Proof. The proof follows from the fact that $(\blacktriangleleft^{\circ}, !^{\circ})$ is right linear adjoint to $(\blacktriangleright^{\bullet}, \text{¡}^{\bullet})$, Proposition 5.10 and the definition of $(\cdot)^{\dagger}$ and $(\cdot)^{\ddagger}$. ☐

Lemma D.4. *For all* $a \colon X \to Y$ *it holds that* $a^{\dagger} = a^{\ddagger}$.

Proof. We prove the inclusion $a^{\dagger} \leq a^{\ddagger}$ (left) by means of Lemma 5.7 and the other inclusion (right) directly:

$$\begin{aligned}
(a^{\ddagger} \mathbin{\overset{\bullet}{,}} (a^{\dagger})^{\perp}) &= a^{\ddagger} \mathbin{\overset{\bullet}{,}} (a^{\perp})^{\ddagger} && \text{(Lemma D.3)} \\
&= (a^{\perp} \mathbin{\overset{\bullet}{,}} a)^{\ddagger} && \text{(Table 7)} \\
&\geq (id^{\circ}_Y)^{\ddagger} && (a^{\perp} \Vdash a) \\
&= id^{\circ}_Y && \text{(Lemma D.2)}
\end{aligned}$$

$$\begin{aligned}
a^{\ddagger} &= ((a^{\dagger})^{\dagger})^{\ddagger} && ((\cdot)^{\dagger} \text{ is an iso}) \\
&\leq ((a^{\dagger})^{\ddagger})^{\ddagger} && (a^{\dagger} \leq a^{\ddagger}) \\
&= a^{\dagger} && ((\cdot)^{\ddagger} \text{ is an iso})
\end{aligned}$$

☐

Lemma D.5. $(\cdot)^{\dagger} \colon \mathbf{C} \to \mathbf{C}^{op}$ *is an isomorphisms of fo-bicategories, namely all the laws in Table 2.(a) hold.*

Proof. Follows from Lemma D.4 and the fact that $(\cdot)^{\dagger}$ preserves the positive structure (Proposition 4.6) and $(\cdot)^{\ddagger}$ preserve the negative structure (Proposition D.1). For instance, to prove that $(a \mathbin{\overset{\bullet}{,}} b)^{\dagger} = b^{\dagger} \mathbin{\overset{\bullet}{,}} a^{\dagger}$, it is enough to observe that $(a \mathbin{\overset{\bullet}{,}} b)^{\dagger} = (a \mathbin{\overset{\bullet}{,}} b)^{\ddagger}$ and that $(a \mathbin{\overset{\bullet}{,}} b)^{\ddagger} = b^{\ddagger} \mathbin{\overset{\bullet}{,}} a^{\ddagger}$. ☐

Corollary D.6. $(c^{\dagger})^{\perp} = (c^{\perp})^{\dagger}$.

Proof. By Lemma D.5 and Lemma 5.11. ☐

Lemma D.7. *For all* $a \colon X \to Y$ *it holds that* $(a^{\perp})^{\perp} = a$.

Proof. The following two derivations

$$\begin{aligned}
id^{\circ}_Y &= (id^{\circ}_Y)^{\dagger} && \text{(Proposition 4.6)} \\
&\leq (a^{\dagger} \mathbin{\overset{\bullet}{,}} (a^{\dagger})^{\perp})^{\dagger} && ((a^{\dagger})^{\perp} \Vdash a^{\dagger}) \\
&= (a^{\dagger} \mathbin{\overset{\bullet}{,}} (a^{\perp})^{\dagger})^{\dagger} && \text{(Corollary D.6)} \\
&= ((a^{\perp} \mathbin{\overset{\bullet}{,}} a)^{\dagger})^{\dagger} && \text{(Lemma D.5)} \\
&= a^{\perp} \mathbin{\overset{\bullet}{,}} a && \text{(Proposition 4.6)}
\end{aligned}$$

$$\begin{aligned}
id^{\bullet}_X &= (id^{\bullet}_X)^{\dagger} && \text{(Lemma D.5)} \\
&\geq ((a^{\dagger})^{\perp} \mathbin{\overset{\circ}{,}} a^{\dagger})^{\dagger} && ((a^{\dagger})^{\perp} \Vdash a^{\dagger}) \\
&= ((a^{\perp})^{\dagger} \mathbin{\overset{\circ}{,}} a^{\dagger})^{\dagger} && \text{(Corollary D.6)} \\
&= ((a \mathbin{\overset{\circ}{,}} a^{\perp})^{\dagger})^{\dagger} && \text{(Proposition 4.6)} \\
&= a \mathbin{\overset{\circ}{,}} a^{\perp} && \text{(Proposition 4.6)}
\end{aligned}$$

prove that the right linear adjoint of $a^{\perp}$ is a. Thus, by Lemma 5.6, $(a^{\perp})^{\perp} = a$. ☐

Lemma D.8. $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{co})^{op}$ *is an isomorphisms of fo-bicategories, namely all the laws in Table 2.(b) hold.*

Proof. By Proposition 5.10, $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ is a morphism of linear bicategories. Observe that $(\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ carries the structure of a cartesian bicategory where the positive comonoid is $(\blacktriangleright^{\bullet}, \text{¡}^{\bullet})$ and the positive monoid is $(\blacktriangleleft^{\bullet}, !^{\bullet})$. By Definition 5.1.4, one has that $(\blacktriangleleft^{\circ})^{\perp} = \blacktriangleright^{\bullet}$, $(!^{\circ})^{\perp} = \text{¡}^{\bullet}$ and $(\blacktriangleright^{\circ})^{\perp} = \blacktriangleleft^{\bullet}$, $(\text{¡}^{\circ})^{\perp} = !^{\bullet}$. Thus $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ is a morphism of cartesian bicategories.

By Lemma D.7, we also immediately know that $(\blacktriangleleft^{\bullet})^{\perp} = \blacktriangleright^{\circ}$, $(!^{\bullet})^{\perp} = ¡^{\circ}$ and $(\blacktriangleright^{\bullet})^{\perp} = \blacktriangleleft^{\circ}$, $(¡^{\bullet})^{\perp} = !^{\circ}$. Thus, $(\cdot)^{\perp} \colon \mathbf{C} \to (\mathbf{C}^{\mathsf{co}})^{\mathsf{op}}$ is a morphism of cocartesian bicategories. Thus, it is a morphism of fo-bicategories.

The fact that it is an isomorphism is immediate by Lemma D.7. □

Proof of Proposition 6.3. By Lemmas D.5 and D.8. □

D.2. **Appendix to Section 6.1.** In order to prove Proposition 6.10 is convenient to use the following function on diagrams and then prove that it maps every diagram in its right (Lemma D.10) and left (Lemma D.13) linear adjoint.

Definition D.9. The function $\alpha \colon \mathsf{NPR}_{\Sigma} \to \mathsf{NPR}_{\Sigma}$ is inductively defined as follows.

$$\begin{array}{ccccc}
\alpha(c \mathbin{\fatsemi} d) \stackrel{\text{def}}{=} \alpha(d) \mathbin{\fatsemi^{\bullet}} \alpha(c) & \alpha(id_1^{\circ}) \stackrel{\text{def}}{=} id_1^{\bullet} & \alpha(\blacktriangleright_1^{\circ}) \stackrel{\text{def}}{=} \blacktriangleleft_1^{\bullet} & \alpha(¡_1^{\circ}) \stackrel{\text{def}}{=} !_1^{\bullet} & \alpha(R^{\circ}) \stackrel{\text{def}}{=} R^{\bullet} \\
\alpha(c \otimes d) \stackrel{\text{def}}{=} \alpha(c) \mathbin{\otimes^{\bullet}} \alpha(d) & \alpha(\sigma_{1,1}^{\circ}) \stackrel{\text{def}}{=} \sigma_{1,1}^{\bullet} & \alpha(\blacktriangleleft_1^{\circ}) \stackrel{\text{def}}{=} \blacktriangleright_1^{\bullet} & \alpha(!_1^{\circ}) \stackrel{\text{def}}{=} ¡_1^{\bullet} &
\end{array}$$

$$\begin{array}{ccccc}
\alpha(c \mathbin{\fatsemi^{\bullet}} d) \stackrel{\text{def}}{=} \alpha(d) \mathbin{\fatsemi} \alpha(c) & \alpha(id_1^{\bullet}) \stackrel{\text{def}}{=} id_1^{\circ} & \alpha(\blacktriangleright_1^{\bullet}) \stackrel{\text{def}}{=} \blacktriangleleft_1^{\circ} & \alpha(¡_1^{\bullet}) \stackrel{\text{def}}{=} !_1^{\bullet} & \alpha(R^{\bullet}) \stackrel{\text{def}}{=} R^{\circ} \\
\alpha(c \mathbin{\otimes^{\bullet}} d) \stackrel{\text{def}}{=} \alpha(c) \otimes \alpha(d) & \alpha(\sigma_{1,1}^{\bullet}) \stackrel{\text{def}}{=} \sigma_{1,1}^{\circ} & \alpha(\blacktriangleleft_1^{\bullet}) \stackrel{\text{def}}{=} \blacktriangleright_1^{\circ} & \alpha(!_1^{\bullet}) \stackrel{\text{def}}{=} ¡_1^{\circ} &
\end{array}$$

Lemma D.10. *For all terms $c \colon n \to m$ in NPR_{Σ}, $id_n^{\circ} \lesssim c \mathbin{\fatsemi^{\bullet}} \alpha(c)$ and $\alpha(c) \mathbin{\fatsemi} c \lesssim id_m^{\bullet}$.*

Proof. The proof goes by induction on c. For the base cases of black and white (co)monoid, it is immediate by the axioms in the first block of Figure 6.For R°, $R^{\bullet}$, σ° and $\sigma^{\bullet}$, it is immediate by the axioms in the bottom Figure 5. For id° and $id^{\bullet}$ is trivial. For the inductive cases of $\fatsemi$, $\fatsemi^{\bullet}$, $\otimes$ and $\otimes^{\bullet}$ one can reuse exactly the proof of Proposition 5.10. □

Lemma D.11. *For all term $c \colon n \to m$ in NPR_{Σ}, $\alpha(\alpha(c)) = c$.*

Proof. The proof goes by induction on c. For the base cases, it is immediate by Definition D.9. For the inductive cases, one have just to use the definition and the inductive hypothesis. For instance $\alpha(\alpha(a \mathbin{\fatsemi} b))$ is, by Definition D.9, $\alpha(\alpha(a) \mathbin{\fatsemi^{\bullet}} \alpha(b))$ which, by Definition D.9, is $\alpha(\alpha(a)) \mathbin{\fatsemi} \alpha(\alpha(b))$ that, by induction hypothesis, is $a \mathbin{\fatsemi} b$. □

Lemma D.12. *For all terms $c, d \colon n \to m$ in NPR_{Σ}, if $c \lesssim d$, then $\alpha(d) \lesssim \alpha(c)$.*

Proof. Observe that the axioms in Figures 3, 4, 5 and 6 are closed under α, namely if $c \leq d$ is an axiom also $\alpha(d) \leq \alpha(c)$ is an axiom. □

Lemma D.13. *For all terms $c \colon n \to m$ in NPR_{Σ}, $id_m^{\circ} \lesssim \alpha(c) \mathbin{\fatsemi^{\bullet}} c$ and $c \mathbin{\fatsemi} \alpha(c) \lesssim id_n^{\bullet}$.*

Proof. By Lemma D.10, it holds that

$$id_n^{\circ} \lesssim c \mathbin{\fatsemi^{\bullet}} \alpha(c) \text{ and } \alpha(c) \mathbin{\fatsemi} c \lesssim id_m^{\bullet}.$$

By Lemma D.12, one can apply α to all the sides of the two inequalities to get

$$\alpha(c \mathbin{\fatsemi^{\bullet}} \alpha(c)) \lesssim \alpha(id_n^{\circ}) \text{ and } \alpha(id_m^{\bullet}) \lesssim \alpha(\alpha(c) \mathbin{\fatsemi} c).$$

That, by Definition D.9 gives exactly

$$\alpha(\alpha(c)) \mathbin{\fatsemi} \alpha(c) \lesssim id_n^{\bullet} \text{ and } id_m^{\circ} \lesssim \alpha(c) \mathbin{\fatsemi^{\bullet}} \alpha(\alpha(c)).$$

By Lemma D.11, one can conclude that

$$c \mathbin{\fatsemi} \alpha(c) \lesssim id_n^{\bullet} \text{ and } id_m^{\circ} \lesssim \alpha(c) \mathbin{\fatsemi^{\bullet}} c.$$

□

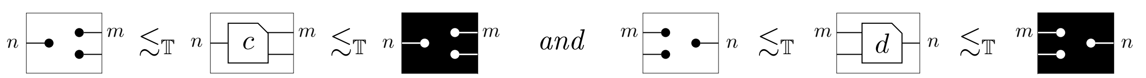
Proof of Proposition 6.10. By Lemmas D.10 and D.13, the diagram $\alpha(c)$ is both the right and the left linear adjoint of any diagram c. Thus $\mathbf{FOB}_\Sigma$ is a closed linear bicategory.

Next, we show that $(\mathbf{FOB}^\circ_\Sigma, \blacktriangleleft^\circ, \blacktriangleright^\circ)$ is a cartesian bicategory: for all objects $n \in \mathbb{N}$, $\blacktriangleleft^\circ_n$, $!^\circ_n$, $\blacktriangleright^\circ_n$ and $¡^\circ_n$ are inductively defined as in Table 1. Observe that such definitions guarantees that the coherence conditions in Definition 4.1.(5) are satisfied. The conditions in Definition 4.1.(1).(2).(3).(4) are the axioms in Figure 3 (and appear in the term version in Figure 10) that we have used to generate $\lesssim$.

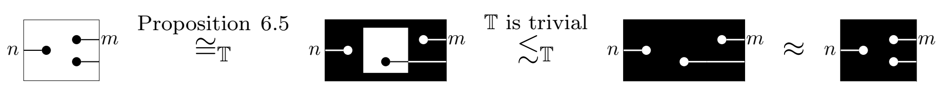
Similarly, $(\mathbf{FOB}^\bullet_\Sigma, \blacktriangleleft^\bullet, \blacktriangleright^\bullet)$ is a cocartesian bicategory: for all objects $n \in \mathbb{N}$, $\blacktriangleleft^\bullet_n$, $!^\bullet_n$, $\blacktriangleright^\bullet_n$ and $¡^\bullet_n$ are inductively defined as in Table 1. Again, the coherence conditions are satisfied by construction. The other conditions are the axioms in Figure 4 (and appear in the term version in Figure 10) that, by construction, are in $\lesssim$. To conclude that $\mathbf{FOB}_\Sigma$ is a first order bicategory we have to check that the conditions in Definition 6.1.(4),(5). But these are exactly the axioms in Figure 6 (and appear in the term version in Figure 10). □

Appendix E. Additional results on the trivial theories of propositional calculus

Lemma E.1. *Let $\mathbb{T}$ be a trivial theory and $c\colon n \to m+1, d\colon m+1 \to n$ be arrows of $\mathbf{FOB}_\mathbb{T}$. Then it holds that:*

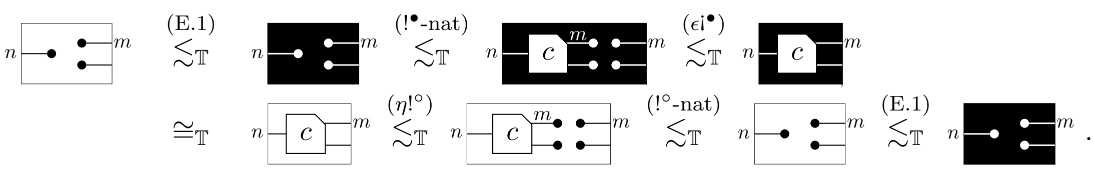


Proof. First observe that the following holds:



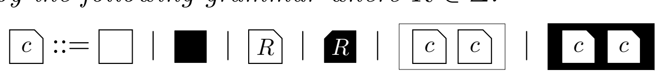
(E.1)

Then, a simple derivation proves the statement:



The proof for d follows a similar reasoning. □

Proposition E.2. *For every diagram $a\colon 0 \to 0$ in $\mathbf{FOB}_\mathbb{P}$ there exists a $\cong_\mathbb{P}$-equivalent diagram generated by the following grammar where $R \in \Sigma$.*



Proof. By induction on $a\colon 0 \to 0$. Observe that there are only four base cases: id°_0, $id^\bullet_0$, R° and $R^\bullet$. These already appear in the grammar above. We have the usual four inductive cases:

(1) $a = c \mathbin{;} d$. There are two sub-cases: either $c, d\colon 0 \to 0$ or $c\colon 0 \to n+1$ and $d\colon n+1 \to 0$. In the former we can use the inductive hypothesis to get c' and d' generated by the above grammar such that $c' \cong_\mathbb{P} c$ and $d' \cong_\mathbb{P} d$. Thus a is $\cong_\mathbb{P}$-equivalent to $c' \mathbin{;} d'$ that is generated by the above grammar.

Consider now the case where $c\colon 0 \to n+1$ and $d\colon n+1 \to 0$. By Lemma E.1, $c \cong_\mathbb{P} ¡^\circ_{n+1}$ and $d \cong_\mathbb{P} !^\bullet_{n+1}$. By axiom $(\gamma!^\bullet)$, $¡^\circ_{n+1} \mathbin{;} !^\bullet_{n+1} \cong_\mathbb{P} id^\bullet_0$. Thus $a \cong_\mathbb{P} id^\bullet_0$.

(2) $a = c \otimes d$. Note that, in this case both c and d must have type $0 \to 0$. Thus we can use the inductive hypothesis to get c' and d' generated by the above grammar such that $c' \cong_{\mathbb{P}} c$ and $d' \cong_{\mathbb{P}} d$. Thus $a \cong_{\mathbb{P}} c' \otimes d' \approx c' \mathbin{⨾} d'$. Note that $c' \mathbin{⨾} d'$ is generated by the above grammar.
(3) $a = c \mathbin{\bullet\!\!;} d$. The proof follows symmetrical arguments to the case $c \mathbin{⨾} d$.
(4) $a = c \mathbin{\otimes\!\!\!\!\bullet} d$. The proof follows symmetrical arguments to the case $c \otimes d$.

□

Appendix F. Proofs of Section 7

Lemma F.1. *Let $\mathbb{T}$ be a theory. If $\mathbb{T}$ is contradictory then it is trivial.*

Proof. Assume $\mathbb{T}$ to be contradictory and consider the following derivation.

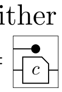
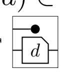
$$\begin{aligned} \text{¡}_1^{\circ} &= id_0^{\circ} \mathbin{⨾} \text{¡}_1^{\circ} \\ &\leq id_0^{\bullet} \mathbin{⨾} \text{¡}_1^{\circ} && (\mathbb{T}\text{ contradictory}) \\ &= id_0^{\bullet} \mathbin{\bullet\!\!;} \text{¡}_1^{\bullet} && (\text{Proposition 6.5}) \\ &= \text{¡}_1^{\bullet} \end{aligned}$$

□

Appendix G. Appendix to Section 8

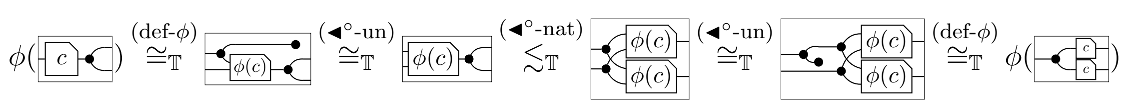
Proof of Lemma 8.10. The proof goes by induction on the rules in (3.6).

For the rule (id) we have three cases: either $(c, d) \in \mathbb{I}$ or $(c, d) \in \lesssim_{\Sigma'}$ or $(c, d) \in \mathbb{M}_k$.

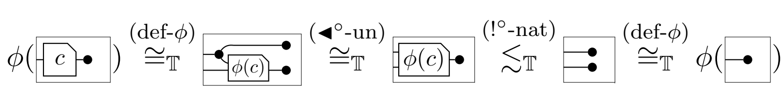
If $(c, d) \in \mathbb{I}$ then, by Lemma 8.9, $\phi(c) =$  $\lesssim_{\mathbb{T}}$  $= \phi(d)$.

If $(c, d) \in \lesssim_{\Sigma'}$ then (c, d) has been obtained by instantiating the axioms in Figures 3,4 and 5 with diagrams containing k. Therefore, we need to show that ϕ preserves these axioms. In the following we show only a few of them. The remaining ones follow similar reasonings.

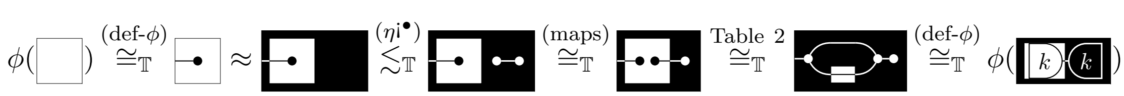
For ($\blacktriangleleft^{\circ}$-nat) the following holds:

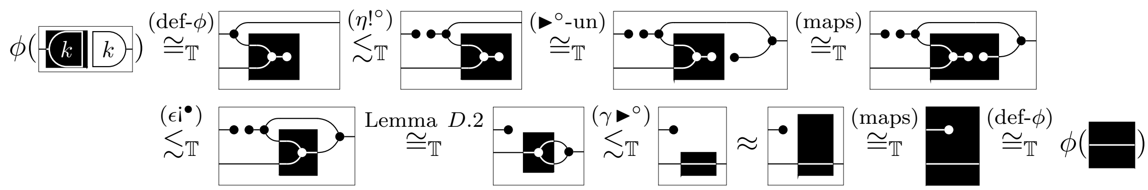


For (!$^{\circ}$-nat) the following holds:

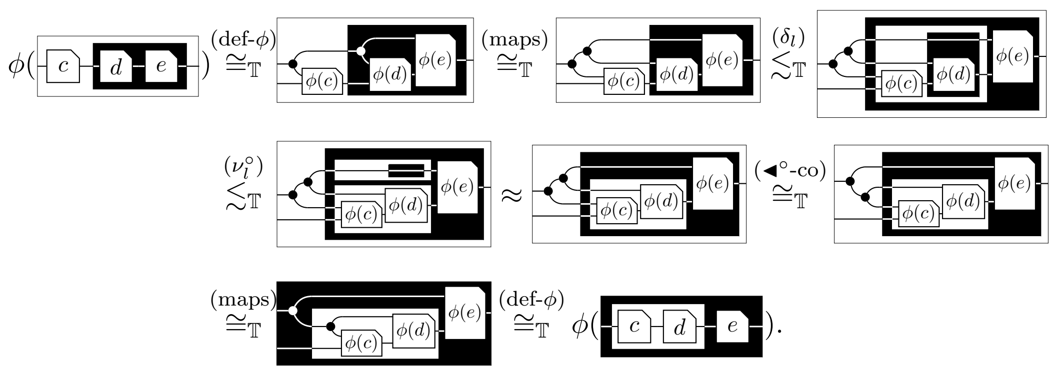


For (τR°) and (γR°) the following holds:

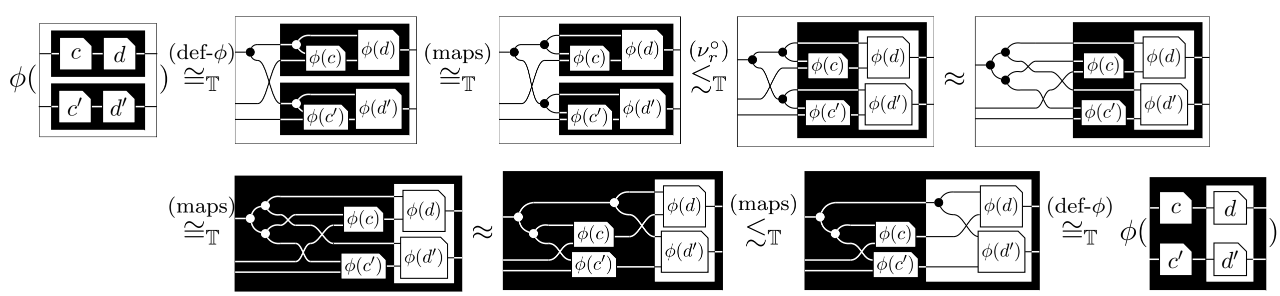




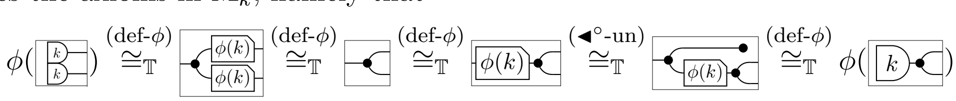
For (δ_l) the following holds:



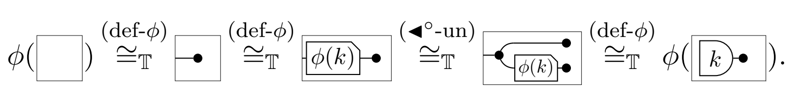
For (ν_r°) the following holds:



Similar to the previous argument, if $(c, d) \in \mathbb{M}_k$ then it is enough to show that ϕ preserves the axioms in $\mathbb{M}_k$, namely that



and



The base case (r) is trivial, while the proof for the remaining rules follows a straightforward inductive argument. □

Proof of Proposition 8.11. By using the well-known fact that $\mathtt{pc}(\cdot)$ preserves chains, one can easily see that

$$\lesssim_{\mathbb{T}} = \bigcup_{i \in I} \lesssim_{\mathbb{T}_i} \tag{G.1}$$

The interested reader can find all the details in Appendix I.1, Lemma I.12.

(1) Suppose that $\mathbb{T}$ is contradictory. By definition $id_0^\circ \lesssim_{\mathbb{T}} id_0^\bullet$ and then, by (G.1), $(id_0^\circ, id_0^\bullet) \in \bigcup_{i \in I} \lesssim_{\mathbb{T}_i}$. Thus there exists an $i \in I$ such that $id_0^\circ \lesssim_{\mathbb{T}_i} id_0^\bullet$. Against the hypothesis.

(2) Suppose that $\mathbb{T}$ is trivial. By definition $\mathsf{i}_1^\circ \lesssim_{\mathbb{T}} \mathsf{i}_1^\bullet$ and then, by (G.1), $(\mathsf{i}_1^\circ, \mathsf{i}_1^\bullet) \in \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}$. Thus there exists an $i \in I$ such that $\mathsf{i}_1^\circ \lesssim_{\mathbb{T}_i} \mathsf{i}_1^\bullet$. Against the hypothesis.

□

Proof of Proposition 8.12. The proof of this proposition relies on Zorn Lemma [Zor35] which states that if, in a non empty poset L every chain has a least upper bound, then L has at least one maximal element.

We consider the set Γ of all non-contradictory theories on Σ that include $\mathbb{I}$, namely

$$\Gamma \stackrel{\text{def}}{=} \{\mathbb{T} = (\Sigma, \mathbb{J}) \mid \mathbb{T} \text{ is non-contradictory and } \mathbb{I} \subseteq \mathbb{J}\}.$$

Observe that the set Γ is non empty since there is at least $\mathbb{T}$ which belongs to Γ.

Let $\Lambda \subseteq \Gamma$ be a chain, namely $\Lambda = \{\mathbb{T}_i = (\Sigma, \mathbb{J}_i) \in \Gamma \mid i \in I\}$ for some linearly ordered set I and if $i \leq j$, then $\mathbb{J}_i \subseteq \mathbb{J}_j$. By Proposition 8.11, the theory $(\Sigma, \bigcup_{i\in I} \mathbb{J}_i)$ is non-contradictory and thus it belongs to Γ.

We can thus use Zorn Lemma: the set Γ has a maximal element $\mathbb{T}' = (\Sigma, \mathbb{I}')$. By definition of Γ, $\mathbb{I} \subseteq \mathbb{I}'$ and, moreover, $\mathbb{T}'$ is non-contradictory.

We only need to prove that $\mathbb{T}'$ is syntactically complete, i.e., for all $c\colon 0 \to 0$, either $id_0^\circ \lesssim_{\mathbb{T}'} c$ or $id_0^\circ \lesssim_{\mathbb{T}'} \overline{c}$. Assume that $id_0^\circ \not\lesssim_{\mathbb{T}} c$. Thus $\mathbb{I}'$ is *strictly* included into $\mathbb{I}' \cup \{(id_0^\circ, c)\}$. By maximality of $\mathbb{T}'$ in Γ, we have that the theory $\mathbb{T}'' = (\Sigma, \mathbb{I}' \cup \{(id_0^\circ, c)\})$ is contradictory, i.e., $id_0^\circ \lesssim_{\mathbb{T}''} id_0^\bullet$. By the deduction theorem (Theorem 7.13), $c \lesssim_{\mathbb{T}'} id_0^\bullet$. Therefore $id_0^\circ \lesssim_{\mathbb{T}'} \overline{c}$. □

Proof of Theorem 8.13. This proof reuses the well-known arguments reported e.g. in [LP01].

We first illustrate a procedure to add Henkin witnesses without losing the property of being non-trivial.

Take an enumeration of diagrams in $\mathbf{FOB}_\Sigma[1, 0]$ and write c_i for the i-th diagram.

For all natural numbers $n \in \mathbb{N}$, we define

$$\Sigma^n \stackrel{\text{def}}{=} \Sigma \cup \{k_i\colon 0 \to 1 \mid i \leq n\} \qquad \mathbb{I}^n \stackrel{\text{def}}{=} \mathbb{I} \cup \mathbb{M}_{k_i} \cup \textstyle\bigcup_{i\leq n} \mathbb{W}_{k_i}^{c_i} \qquad \mathbb{T}^n \stackrel{\text{def}}{=} (\Sigma^n, \mathbb{I}^n).$$

By applying Lemma 8.6 n-times, one has that $\mathbb{T}^n$ is non-trivial. Define now

$$\Sigma_0 \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \Sigma^i \qquad \mathbb{I}_0 \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \mathbb{I}^j \qquad \mathbb{T}_0 \stackrel{\text{def}}{=} (\Sigma_0, \mathbb{I}_0).$$

Since $\mathbb{T}^0 \subseteq \mathbb{T}^1 \subseteq \cdots \subseteq \mathbb{T}^n \subseteq \dots$ are all non-trivial, then by Proposition 8.11.2, we have that $\mathbb{T}_0$ is non-trivial. One must not jump to the conclusion that $\mathbb{T}_0$ has Henkin witnesses: all the diagrams in $\mathbf{FOB}_\Sigma[1, 0]$ have Henkin witnesses, but in $\mathbb{T}_0$ we have more diagrams, since we have added the constants k_i to Σ_0.

We thus repeat the above construction, but now for diagrams in $\mathbf{FOB}_{\Sigma_0}[1, 0]$. We define

$$\Sigma_1 \stackrel{\text{def}}{=} \Sigma_0 \cup \{k_c \mid c \in \mathbf{FOB}_{\Sigma_0}[1, 0]\} \qquad \mathbb{I}_1 \stackrel{\text{def}}{=} \mathbb{I}_0 \cup \mathbb{M}_{k_c} \cup \mathbb{W}_{k_c}^c \qquad \mathbb{T}_1 \stackrel{\text{def}}{=} (\Sigma_1, \mathbb{I}_1).$$

The theory $\mathbb{T}_1$ is non-trivial but has Henkin witnesses only for the diagrams in $\mathbf{FOB}_{\Sigma_0}$.

Thus, for all natural numbers $n \in \mathbb{N}$, we define

$$\Sigma_{n+1} \stackrel{\text{def}}{=} \Sigma_n \cup \{k_c \mid c \in \mathbf{FOB}_{\Sigma_n}[1, 0]\} \qquad \mathbb{I}_{n+1} \stackrel{\text{def}}{=} \mathbb{I}_n \cup \mathbb{M}_{k_c} \cup \mathbb{W}_{k_c}^c \qquad \mathbb{T}_{n+1} \stackrel{\text{def}}{=} (\Sigma_{n+1}, \mathbb{I}_{n+1})$$

and

$$\Sigma' \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \Sigma_i \qquad \mathbb{I}' \stackrel{\text{def}}{=} \bigcup_{i\in\mathbb{N}} \mathbb{I}_i \qquad \mathbb{T}' \stackrel{\text{def}}{=} (\Sigma', \mathbb{I}').$$

Since $\mathbb{T}_0 \subseteq \mathbb{T}_1 \subseteq \cdots \subseteq \mathbb{T}_n \subseteq \dots$ are all non-trivial, then by Proposition 8.11.2, we have that $\mathbb{T}'$ is also non-trivial. Now $\mathbb{T}'$ has Henkin witnesses: if $c \in \mathbf{FOB}_{\Sigma'}[0, 1]$, then there exists

$n \in \mathbb{N}$ such that $c \in \mathbf{FOB}_{\Sigma_n}[0,1]$. By definition of $\mathbb{I}_n$, it holds that $\mathbb{W}^c_{k_c} \subseteq \mathbb{I}_{n+1}$ and thus $\mathbb{W}^c_{k_c} \subseteq \mathbb{I}'$.

Summarising, we manage to build a theory $\mathbb{T}' = (\Sigma', \mathbb{I}')$ that has Henkin witnesses and it is non-trivial. By Lemma F.1, $\mathbb{T}'$ is non-contradictory. We can thus use Proposition 8.12, to obtain a theory $\mathbb{T}'' = (\Sigma', \mathbb{I}'')$ that is syntactically complete and non-contradictory. Observe that $\mathbb{T}''$ has Henkin witnesses, since the signature Σ' is the same as in $\mathbb{T}'$ and $\mathbb{I}' \subseteq \mathbb{I}''$. $\square$

Appendix H. Appendix to Section 10

Proof of Proposition 10.1. The proof is by induction on E. The base cases are trivial. The inductive cases are shown below.

$$
\begin{aligned}
\mathcal{I}^\sharp(\mathcal{E}(E_1 \fatsemi E_2)) &= \mathcal{I}^\sharp(\mathcal{E}(E_1) \fatsemi \mathcal{E}(E_2)) && \text{(Table 3)}\\
&= \mathcal{I}^\sharp(\mathcal{E}(E_1)) \fatsemi \mathcal{I}^\sharp(\mathcal{E}(E_2)) && (3.4)\\
&= \langle E_1 \rangle_{\mathcal{I}} \fatsemi \langle E_2 \rangle_{\mathcal{I}} && \text{(Ind. hyp.)}\\
&= \langle E_1 \fatsemi E_2 \rangle_{\mathcal{I}} && (2.4)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^\sharp(\mathcal{E}(E_1 \bullet E_2)) &= \mathcal{I}^\sharp(\mathcal{E}(E_1) \bullet \mathcal{E}(E_2)) && \text{(Table 3)}\\
&= \mathcal{I}^\sharp(\mathcal{E}(E_1)) \bullet \mathcal{I}^\sharp(\mathcal{E}(E_2)) && (3.4)\\
&= \langle E_1 \rangle_{\mathcal{I}} \bullet \langle E_2 \rangle_{\mathcal{I}} && \text{(Ind. hyp.)}\\
&= \langle E_1 \bullet E_2 \rangle_{\mathcal{I}} && (2.4)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^\sharp(\mathcal{E}(E_1 \cap E_2)) &= \mathcal{I}^\sharp(\blacktriangleleft^{\circ}_{1} \fatsemi (\mathcal{E}(E_1) \otimes \mathcal{E}(E_2)) \fatsemi \blacktriangleright^{\circ}_{1}) && \text{(Table 3)}\\
&= \mathcal{I}^\sharp(\blacktriangleleft^{\circ}_{1}) \fatsemi (\mathcal{I}^\sharp(\mathcal{E}(E_1)) \otimes \mathcal{I}^\sharp(\mathcal{E}(E_2))) \fatsemi \mathcal{I}^\sharp(\blacktriangleright^{\circ}_{1}) && (3.4)\\
&= \blacktriangleleft^{\circ}_{X} \fatsemi (\mathcal{I}^\sharp(\mathcal{E}(E_1)) \otimes \mathcal{I}^\sharp(\mathcal{E}(E_2))) \fatsemi \blacktriangleright^{\circ}_{X} && (3.4)\\
&= \blacktriangleleft^{\circ}_{X} \fatsemi (\langle E_1 \rangle_{\mathcal{I}} \otimes \langle E_2 \rangle_{\mathcal{I}}) \fatsemi \blacktriangleright^{\circ}_{X} && \text{(Ind. hyp.)}\\
&= \langle E_1 \rangle_{\mathcal{I}} \cap \langle E_2 \rangle_{\mathcal{I}} && (4.2)\\
&= \langle E_1 \cap E_2 \rangle_{\mathcal{I}} && (2.4)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^\sharp(\mathcal{E}(E_1 \cup E_2)) &= \mathcal{I}^\sharp(\blacktriangleleft^{\bullet}_{1} \bullet (\mathcal{E}(E_1) \boldsymbol{\otimes} \mathcal{E}(E_2)) \bullet \blacktriangleright^{\bullet}_{1}) && \text{(Table 3)}\\
&= \mathcal{I}^\sharp(\blacktriangleleft^{\bullet}_{1}) \bullet (\mathcal{I}^\sharp(\mathcal{E}(E_1)) \boldsymbol{\otimes} \mathcal{I}^\sharp(\mathcal{E}(E_2))) \bullet \mathcal{I}^\sharp(\blacktriangleright^{\bullet}_{1}) && (3.4)\\
&= \blacktriangleleft^{\bullet}_{X} \bullet (\mathcal{I}^\sharp(\mathcal{E}(E_1)) \boldsymbol{\otimes} \mathcal{I}^\sharp(\mathcal{E}(E_2))) \bullet \blacktriangleright^{\bullet}_{X} && (3.4)\\
&= \blacktriangleleft^{\bullet}_{X} \bullet (\langle E_1 \rangle_{\mathcal{I}} \boldsymbol{\otimes} \langle E_2 \rangle_{\mathcal{I}}) \bullet \blacktriangleright^{\bullet}_{X} && \text{(Ind. hyp.)}\\
&= \langle E_1 \rangle_{\mathcal{I}} \cup \langle E_2 \rangle_{\mathcal{I}} && (4.3)\\
&= \langle E_1 \cup E_2 \rangle_{\mathcal{I}} && (2.4)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^\sharp(\mathcal{E}(E^\dagger)) &= \mathcal{I}^\sharp((\mathcal{E}(E))^\dagger) && \text{(Table 3)}\\
&= (\mathcal{I}^\sharp(\mathcal{E}(E)))^\dagger && \text{(Lemma 4.7)}\\
&= \langle E \rangle_{\mathcal{I}}^\dagger && \text{(Ind. hyp.)}\\
&= \langle E^\dagger \rangle_{\mathcal{I}} && (2.4)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^\sharp(\mathcal{E}(\overline{E})) &= \mathcal{I}^\sharp(\overline{(\mathcal{E}(E))}) && \text{(Table 3)}\\
&= \mathcal{I}^\sharp(((\mathcal{E}(E))^\perp)^\dagger) && \text{(Definition of } \overline{(\cdot)}\text{)}\\
&= (\mathcal{I}^\sharp(\mathcal{E}(E))^\perp)^\dagger && \text{(Lemmas 4.7, 5.11)}\\
&= ((\langle E\rangle_{\mathcal{I}})^\perp)^\dagger && \text{(Ind. hyp.)}\\
&= \overline{\langle E\rangle_{\mathcal{I}}} && \text{(Definition of } \overline{(\cdot)}\text{)}\\
&= \langle \overline{E}\rangle_{\mathcal{I}} && (2.4)
\end{aligned}
$$

□

Proof of Proposition 10.3. The proof goes by induction on the typing rules. For the base cases we have the following:

- $I\colon 2$. By definition $\langle I\rangle_{\mathcal{I}} = \{\tau \mid \tau_1 = \tau_2\}$ and $\mathcal{I}_p^\sharp(\mathcal{E}(I)) = \{((x_1, x_2), \star) \mid x_1 = x_2\}$. Thus $\langle I\rangle_{\mathcal{I}} = \{\tau \mid ((\tau_1, \tau_2), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(I))\}$.
- $R\colon n$. Assume $ar(R) = n$. By definition $\langle R\rangle_{\mathcal{I}} = \{\tau \mid (\tau_1, \ldots, \tau_n) \in \rho(R)\}$ and $\mathcal{I}_p^\sharp(\mathcal{E}(R)) = \{((x_1, \ldots, x_n), \star) \mid (x_1, \ldots x_n) \in \rho(R)\}$. Thus $\langle R\rangle_{\mathcal{I}} = \{\tau \mid ((\tau_1, \ldots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(R))\}$.

The inductive cases follow always the same argument. We report below only the most interesting ones.

- $P_1 \cap P_2$. Assume $P_1\colon n$, $P_2\colon m$ and $n \geq m$.

$$
\begin{aligned}
\langle P_1 \cap P_2\rangle_{\mathcal{I}} &= \langle P_1\rangle_{\mathcal{I}} \cap \langle P_2\rangle_{\mathcal{I}} && \text{(Definition of } \langle\cdot\rangle_{\mathcal{I}}\text{)}\\
&= \begin{array}{l}\{\tau \mid ((\tau_1, \ldots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_1))\}\\ \quad \cap \{\tau \mid ((\tau_1, \ldots, \tau_m), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_2))\}\end{array} && \text{(Ind. hyp.)}\\
&= \begin{array}{l}\{\tau \mid ((\tau_1, \ldots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_1))\\ \quad \wedge ((\tau_1, \ldots, \tau_m), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_2))\}\end{array}\\
&= \{\tau \mid ((\tau_1, \ldots, \tau_n), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P_1 \cap P_2)\} && \text{(Definition of } \mathcal{E}(\cdot) \text{ and } \mathcal{I}_p^\sharp(\cdot)\text{)}
\end{aligned}
$$

- $\mathbf{p}P\colon 2$. Assume $P\colon 1$.

$$
\begin{aligned}
\langle \mathbf{p}P\rangle_{\mathcal{I}} &= \{\tau \mid \tau_2, \tau_1, \tau_3, \tau_4 \cdots \in \langle P\rangle_{\mathcal{I}}\} && \text{(Definition of } \langle\cdot\rangle_{\mathcal{I}}\text{)}\\
&= \{\tau \mid \tau_2, \tau_1, \cdots \in \{\tau \mid (\tau_1, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\} \ \} && \text{(Ind. hyp.)}\\
&= \{\tau \mid (\tau_2, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\}\\
&= \{\tau \mid ((\tau_1, \tau_2), \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(\mathbf{p}P))\} && \text{(Definition of } \mathcal{E}(\cdot) \text{ and } \mathcal{I}_p^\sharp(\cdot)\text{)}
\end{aligned}
$$

- $]P\colon 0$. Assume $P\colon 0$.

$$
\begin{aligned}
\langle]P\rangle_{\mathcal{I}} &= \{\tau \mid \tau_2, \tau_3, \cdots \in \langle P\rangle_{\mathcal{I}}\} && \text{(Definition of } \langle\cdot\rangle_{\mathcal{I}}\text{)}\\
&= \{\tau \mid \tau_2, \tau_3, \cdots \in \{\tau \mid (\star, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\} \ \} && \text{(Ind. hyp.)}\\
&= \{\tau \mid (\star, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(P))\}\\
&= \{\tau \mid (\star, \star) \in \mathcal{I}_p^\sharp(\mathcal{E}(]P))\} && \text{(Definition of } \mathcal{E}(\cdot) \text{ and } \mathcal{I}_p^\sharp(\cdot)\text{)}
\end{aligned}
$$

□

Appendix I. Some well known facts about chains in a lattice

A *chain* on a complete lattice $(L, \sqsubseteq)$ is a family $\{x_i\}_{i\in I}$ of elements of L indexed by a linearly oredered set I such that $x_i \sqsubseteq x_j$ whenever $i \leq j$. A monotone map $f \colon L \to L$ is said to *preserve chains* if

$$f(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} f(x_i)$$

We write $id \colon L \to L$ for the identity function and $f \sqcup g \colon L \to L$ for the pointwise join of $f \colon L \to L$ and $g \colon L \to L$, namely $f \sqcup g(x) \stackrel{\text{def}}{=} f(x) \sqcup g(x)$ for all $x \in L$. For all natural numbers $n \in \mathbb{N}$, we define $f^n \colon L \to L$ inductively as $f^0 = id$ and $f^{n+1} = f^n ; f$. We fix $f^\omega \stackrel{\text{def}}{=} \bigsqcup_{n\in\mathbb{N}} f^n$.

Lemma I.1. *Let $f, g \colon L \to L$ be monotone maps preserving chains. Then*
(1) *$id \colon L \to L$ preserves chains;*
(2) *$f \sqcup g \colon L \to L$ preserves chains;*
(3) *$f^\omega \colon L \to L$ preserves chains.*

Proof. (1) Trivial.
(2) By hypothesis we have that $f(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} f(x_i)$ and $g(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} g(x_i)$. Thus

$$\begin{aligned} f \sqcup g(\bigsqcup_{i\in I} x_i) &= f(\bigsqcup_{i\in I} x_i) \sqcup g(\bigsqcup_{i\in I} x_i) \\ &= \bigsqcup_{i\in I} f(x_i) \sqcup \bigsqcup_{i\in I} g(x_i) \\ &= \bigsqcup_{i\in I} (f(x_i) \sqcup g(x_i)) \\ &= \bigsqcup_{i\in I} (f \sqcup g)(x_i) \end{aligned}$$

(3) We prove $f^n(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} f^n(x_i)$ for all $n \in \mathbb{N}$. We proceed by induction on n.
For $n = 0$, $f^0(\bigsqcup_{i\in I} x_i) = \bigsqcup_{i\in I} x_i = \bigsqcup_{i\in I} f^0(x_i)$.
For $n + 1$, we use the hypothesis that f preserves chain and thus

$$\begin{aligned} f^{n+1}((\bigsqcup_{i\in I} x_i) &= f(\, f^{n+1}((\bigsqcup_{i\in I} x_i)\,) \\ &= f(\bigsqcup_{i\in I} f^n(x_i)\,) && \text{(induction hypothesis)} \\ &= \bigsqcup_{i\in I} f(\, f^n(x_i)\,) \\ &= \bigsqcup_{i\in I} f^{n+1}(x_i) \end{aligned}$$

□

Lemma I.2. *Let $f, g \colon L \to L$ be monotone maps preserving chains such that $g \sqsubseteq f$. Then $f^\omega ; g \sqsubseteq f^\omega$*

Proof. For all $x \in L$, $f^\omega ; g(x) = g(\bigsqcup_{n\in\mathbb{N}} f^n(x)) = \bigsqcup_{n\in\mathbb{N}} g(f^n(x)) \sqsubseteq \bigsqcup_{n\in\mathbb{N}} f^{n+1}(x) \sqsubseteq \bigsqcup_{n\in\mathbb{N}} f^n(x) = f^\omega(x)$. □

Lemma I.3. *Let $f\colon L \to L$ be a monotone map preserving chains. Thus $f^\omega = f^\omega ; f^\omega$*

Proof. $f^\omega = f^\omega ; id \sqsubseteq f^\omega ; f^\omega$. For the other direction we prove that $f^\omega ; f^n \sqsubseteq f^\omega$ for all $n \in \mathbb{N}$. We proceed by induction on n. For $n = 0$ is trivial. For $n+1$, $f^\omega ; f^{n+1} = f^\omega ; f^n ; f \sqsubseteq f^\omega ; f \sqsubseteq f^\omega$. For the last inequality we use Lemma I.2. □

Lemma I.4. *Let $f, g\colon L \to L$ be monotone maps preserving chains. Then $(f \sqcup g)^\omega = (f^\omega \sqcup g)^\omega$*

Proof. Since $f = f^1 \sqsubseteq f^\omega$ and since $(\cdot)^\omega$ is monotone, it holds that $(f \sqcup g)^\omega \sqsubseteq (f^\omega \sqcup g)^\omega$.

For the other inclusion, we prove that $(f^\omega \sqcup g)^n \sqsubseteq (f \sqcup g)^\omega$ for all $n \in \mathbb{N}$. We proceed by induction on $n \in \mathbb{N}$. For $n = 0$, $(f^\omega \sqcup g)^0 = id \sqsubseteq (f \sqcup g)^\omega$.

For $n + 1$, observe that $f^\omega \sqsubseteq (f \sqcup g)^\omega$ and than $g \sqsubseteq (f \sqcup g)^\omega$. Thus

$$(f^\omega \sqcup g) \sqsubseteq (f \sqcup g)^\omega \tag{I.1}$$

We conclude with the following derivation.

$$\begin{aligned}
(f^\omega \sqcup g)^{n+1} &= (f^\omega \sqcup g)^n ; (f^\omega \sqcup g) \\
&\sqsubseteq (f \sqcup g)^\omega ; (f^\omega \sqcup g) && \text{(Induction Hypothesis)} \\
&\sqsubseteq (f \sqcup g)^\omega ; (f \sqcup g)^\omega && \text{((I.1))} \\
&= (f \sqcup g)^\omega && \text{(Lemma I.3)}
\end{aligned}$$

□

I.1. **Some well known facts about precongruence closure.** Let $X = \{X[n,m]\}_{n,m \in \mathbb{N}}$ be a family of sets indexes by pairs of natural numbers $(n,m) \in \mathbb{N} \times \mathbb{N}$. A well-typed relation $\mathbb{R}$ is a family of relation $\{R_{n,m}\}_{n,m \in \mathbb{N}}$ such that each $R_{n,m} \subseteq X[n,m] \times X[n,m]$. We consider the set WTRel_X of well typed relations over X. It is easy to see that WTRel_X forms a complete lattice with join given by union $\cup$. Hereafter we fix an arbitrary well-typed relation $\mathbb{I}$ and the well-typed identity relation Δ.

We define the following monotone maps for all $\mathbb{R} \in \mathsf{WTRel}_X$:

- $(id)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as the identity function;
- $(\mathbb{I})\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as the constant function $\mathbb{R} \mapsto \mathbb{I}$;
- $(r)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as the constant function $\mathbb{R} \mapsto \Delta$;
- $(t)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as $\mathbb{R} \mapsto \{(x,z) \mid \exists y.(x,y) \in \mathbb{R} \wedge (y,z) \in \mathbb{R}\}$;
- $(s)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as $\mathbb{R} \mapsto \{(x,y) \mid (y,x) \in \mathbb{R}\}$;
- $(⨾)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as $\mathbb{R} \mapsto \{(x_1 ⨾ y_1, x_2 ⨾ y_2) \mid (x_1,x_2) \in \mathbb{R} \wedge (y_1,y_2) \in \mathbb{R}\}$;
- $(\otimes)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ defined as $\mathbb{R} \mapsto \{(x_1 \otimes y_1, x_2 \otimes y_2) \mid (x_1,x_2) \in \mathbb{R} \wedge (y_1,y_2) \in \mathbb{R}\}$;

Observe that the function (id), (r), (t), $(⨾)$ and $(\otimes)$ are exactly the inference rules used in the definition of $\mathtt{pc}(\cdot)$ given in (3.6). Indeed the function $\mathtt{pc}(\cdot)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ can be decomposed as

$$\mathtt{pc}(\cdot) = (\,(id) \cup (r) \cup (t) \cup (⨾) \cup (\otimes)\,)^\omega$$

where f^ω stands the ω-iteration of a map f defined in the standard way (see Appendix I for a definition).

Similarly the congruence closure $\mathtt{c}(\cdot)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ can be decomposed as

$$\mathtt{c}(\cdot) = (\,(id) \cup (r) \cup (t) \cup (s) \cup (⨾) \cup (\otimes)\,)^\omega$$

These decompositions allow us to prove several facts in a modular way. For instance, to prove that $\mathtt{pc}(\cdot)$ preserves chains is enough to prove the following.

Lemma I.5. *The monotone maps* (id), $(\mathbb{I})$, (r), (s), (t), $(;)$ *and* $(\otimes)$ *defined above preserve chains.*

Proof. All the proofs are straightforward, we illustrate as an example the one for $(\otimes)$.

Let I be a linearly ordered set and $\{\mathbb{R}_i\}_{i\in I}$ be a family of well-typed relations such that if $i \leq j$, then $R_i \subseteq R_j$. We need to prove that $(\otimes)(\bigcup_{i\in I}\mathbb{R}_i) = \bigcup_{i\in I}(\otimes)(\mathbb{R}_i)$.

The inclusion $(\otimes)(\bigcup_{i\in I}\mathbb{R}_i) \supseteq \bigcup_{i\in I}(\otimes)(\mathbb{R}_i)$ trivially follows from monotonicity of $(\otimes)$ and the universal property of union. For the inclusion $(\otimes)(\bigcup_{i\in I}\mathbb{R}_i) \subseteq \bigcup_{i\in I}(\otimes)(\mathbb{R}_i)$, we take an arbitrary $(a,b) \in (\otimes)(\bigcup_{i\in I}\mathbb{R}_i)$. By definition of $(\otimes)$, there exist x_1, x_2, y_1, y_2 such that

$$a = x_1 \otimes y_1 \quad b = x_2 \otimes y_2 \quad (x_1, x_2) \in \bigcup_{i\in I}\mathbb{R}_i \quad (y_1, y_2) \in \bigcup_{i\in I}\mathbb{R}_i$$

By definition of union, there exist $i, j \in I$ such that $(x_1, y_1) \in R_i$ and $(x_2, y_2) \in R_j$. Since I is linearly ordered, there are two cases: either $i \leq j$ or $i \geq j$.

If $i \leq j$, then $R_i \subseteq R_j$ and thus $(x_1, y_1) \in R_j$. By definition of $(\otimes)$, we have $(x_1 \otimes x_2, y_1 \otimes y_2) \in R_j$ and thus $(a,b) \in R_j$. Since $R_j \subseteq \bigcup_{i\in I}\mathbb{R}_i$, then $(a,b) \in \bigcup_{i\in I}\mathbb{R}_i$. The case for $j \leq i$ is symmetric. □

Proposition I.6. *The monotone maps* $\mathtt{pc}(\cdot), \mathtt{c}(\cdot)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ *preserve chains.*

Proof. Follows immediately from Lemma I.5 and Lemma I.1 in Appendix I. □

Lemma I.7. *For all well-typed relations* $\mathbb{J}$, *the map* $\mathtt{pc}(\mathbb{J} \cup \cdot)\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ *preserves chains.*

Proof. Follows immediately from Lemma I.5 and Lemma I.1 in Appendix I. □

Lemma I.8. *For all well-typed relations* $\mathbb{I}$ *and* $\mathbb{J}$, $\mathtt{pc}(\mathbb{I} \cup \mathbb{J}) = \mathtt{pc}(\mathtt{pc}(\mathbb{I}) \cup \mathbb{J})$

Proof. Let $(\mathbb{J})\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ be the constant function to $\mathbb{J}$ and define $f, g\colon \mathsf{WTRel}_X \to \mathsf{WTRel}_X$ as

$$f \stackrel{\text{def}}{=} (id) \cup (r) \cup (t) \cup (;) \cup (\otimes) \qquad g \stackrel{\text{def}}{=} (\mathbb{J})$$

From Lemma I.5 and Lemma I.1, both f and g preserve chains. Observe that $f^{\omega}(\mathbb{I}) = \mathtt{pc}(\mathbb{I})$, that $(f \cup g)^{\omega} = \mathtt{pc}(\mathbb{I} \cup \mathbb{J})$ and that $(f^{\omega} \cup g)^{\omega}(\mathbb{I}) = \mathtt{pc}(\mathtt{pc}(\mathbb{I}) \cup \mathbb{J})$. Conclude with Lemma I.4 in Appendix I. □

Lemma I.9. *Let* $\mathbb{T} = (\Sigma, \mathbb{I})$ *be a first order theory. Then* $\lesssim_{\mathbb{T}} = \mathtt{pc}(\mathbb{FOB} \cup \mathbb{I})$

Proof. By definition $\lesssim_{\mathbb{T}} = \mathtt{pc}(\lesssim \cup \mathbb{I})$. Recall that $\lesssim = \mathtt{pc}(\mathbb{FOB})$. Thus $\lesssim_{\mathbb{T}} = \mathtt{pc}(\mathtt{pc}(\mathbb{FOB}) \cup \mathbb{I})$. By Lemma I.8, $\lesssim_{\mathbb{T}} = \mathtt{pc}(\mathbb{FOB} \cup \mathbb{I})$.

□

Lemma I.10. *Let* I *be a linearly ordered set and, for all* $i \in I$, *let* $\mathbb{T}_i = (\Sigma, \mathbb{I}_i)$ *be first order theories such that if* $i \leq j$, *then* $\mathbb{I}_i \subseteq \mathbb{I}_j$. *Let* $\mathbb{T}$ *be the theory* $(\Sigma, \bigcup_{i\in I}\mathbb{I}_i)$. *Then* $\lesssim_{\mathbb{T}} = \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}$.

Proof. By definition $\lesssim_{\mathbb{T}} = \mathtt{pc}(\lesssim \cup \bigcup_{i\in I}\mathbb{I}_i)$. Since $\mathbb{I}_i$ form a chain, by Lemma I.7, $\mathtt{pc}(\lesssim \cup \bigcup_{i\in I}\mathbb{I}_i) = \bigcup_{i\in I}\mathtt{pc}(\lesssim \cup \mathbb{I}_i)$. The latter is, by definition, $\bigcup_{i\in I} \lesssim_{\mathbb{I}_i}$. □

Lemma I.11. *Let* I *be a linearly ordered set and, for all* $i \in I$, *let* $\mathbb{T}_i = (\Sigma_i, \mathbb{I})$ *be first order theories such that if* $i \leq j$, *then* $\Sigma_i \subseteq \Sigma_j$. *Let* $\mathbb{T}$ *be the theory* $(\bigcup_{i\in I}\Sigma_i, \mathbb{I})$. *Then* $\lesssim_{\mathbb{T}} = \bigcup_{i\in I} \lesssim_{\mathbb{T}_i}$.

Proof. By Lemma I.5, the monotone map $\texttt{pcr}(\cdot) \stackrel{\text{def}}{=} (\,(id) \cup (\mathbb{I}) \cup (t) \cup (\text{⨟}) \cup (\otimes)\,)^{\omega}$ preserves chains. Let Δ_i be the well-typed identity relation on $\mathbf{FOB}_{\Sigma_i}$. Observe that $\lesssim_{\mathbb{T}_i} = \texttt{pcr}(\Delta_i)$ and that $\lesssim_{\mathbb{T}} = \texttt{pcr}(\bigcup_{i \in I} \Delta_i)$. To summarise:

$$\begin{aligned} \lesssim_{\mathbb{T}} &= \texttt{pcr}(\bigcup_{i \in I} \Delta_i) \\ &= \bigcup_{i \in I} \texttt{pcr}(\Delta_i) && \text{(preserve chains)} \\ &= \bigcup_{i \in I} \lesssim_{\mathbb{T}_i} \end{aligned}$$

□

Lemma I.12. *Let I be a linearly ordered set and, for all $i \in I$, let $\mathbb{T}_i = (\Sigma_i, \mathbb{I}_i)$ be first order theories such that if $i \leq j$, then $\Sigma_i \subseteq \Sigma_j$ and $\mathbb{I}_i \subseteq \mathbb{I}_j$. Let $\mathbb{T}$ be the theory $(\bigcup_{i \in I} \Sigma_i, \bigcup_{i \in I} \mathbb{I}_i)$. Then $\lesssim_{\mathbb{T}} = \bigcup_{i \in I} \lesssim_{\mathbb{T}_i}$.*

Proof. Immediate by Lemma I.11 and Lemma I.10. □