Computing Nearby Non-Trivial Smith Forms

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The Smith Normal Form

Smith Normal Form (SNF)

Any $\mathcal{A} \in \mathbb{R}[t]^{n\times n}$ is unimodularly equivalent to

$$\mathcal{S} = \text{diag}(s_1, s_2, \ldots, s_n) \quad \text{where} \quad s_j | s_{j+1} \quad \text{and} \quad s_j \in \mathbb{R}[t].$$

That is, there exists $\mathcal{U}, \mathcal{V} \in \mathbb{R}[t]^{n\times n}$ such that

$$\mathcal{U}\mathcal{A}\mathcal{V} = \mathcal{S} \quad \text{and} \quad \det(\mathcal{U}), \det(\mathcal{V}) \in \mathbb{R}\backslash\{0\}.$$

- The $\{s_j\}_{j=1}^{n}$ are the *invariant factors*
- Computing $\mathcal{S}$ is well understood in exact-arithmetic
- Analyze the SNF as a symbolic-numeric optimization problem

Smith Normal Forms

Example (Boring SNF over $\mathbb{R}[t]^{3\times3}$)

$$\mathcal{A} = \begin{pmatrix} t^3+3t+1 & 1 & t+1 \\ 0 & t^2+2t+2 & 0 \\ t+1 & t+1 & t^3+5t+1 \end{pmatrix} \text{ and } \mathrm{SNF}(\mathcal{A}) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \det(\mathcal{A}) \end{pmatrix}$$

$$\det(\mathcal{A}) = t^8 + 2t^7 + 10t^6 + 18t^5 + 34t^4 + 38t^3 + 40t^2 + 12t.$$

Example (Interesting SNF over $\mathbb{R}[t]^{3\times3}$)

$$\mathcal{B} = \begin{pmatrix} t+1 & t+1 & t-1 \\ 0 & t+1 & t^3 \\ 0 & 0 & t^2-1 \end{pmatrix} \text{ and } \mathrm{SNF}(\mathcal{B}) = \begin{pmatrix} 1 & & \\ & t+1 & \\ & & (t+1)(t^2-1) \end{pmatrix}$$

SNF Computation in a Floating Point Environment

When does $\mathcal{A}$ have a non-trivial Smith Normal Form?

- Small perturbations to $\mathcal{A}$ generically produce a trivial SNF
- How far is $\mathcal{A}$ from a matrix polynomial $\widehat{\mathcal{A}}$ with non-trivial SNF?
- Is there a radius of triviality?
  - I.e., if $\mathcal{A}$ is perturbed by a small amount is the SNF still trivial?

When is Computing the SNF Well-Posed?

Is there a nearest matrix polynomial $\widehat{\mathcal{A}}$ with an interesting SNF?

- Is $\widehat{\mathcal{A}}$ locally unique?
- How do we compute $\widehat{\mathcal{A}}$?
- How do perturbations to $\mathcal{A}$ affect $\widehat{\mathcal{A}}$?

Nearby SNF via Optimization

The McCoy Rank - Number of 1's in the SNF

Formally: McCoy rank of $\mathcal{A} \in \mathbb{R}[t]^{n\times n}$ is $\min_{\omega\in\mathbb{C}} \operatorname{rank}(\mathcal{A}(\omega))$.

Approximations Require a Norm

$\|\mathcal{A}_{ij}\|_2 = \sqrt{\sum_{0\le k\le \deg \mathcal{A}_{ij}} \mathcal{A}_{ijk}^2}$ and $\|\mathcal{A}\| = \|\mathcal{A}\|_F = \sqrt{\sum_{1\le i,j\le n} \|\mathcal{A}_{ij}\|_2^2}$.

Main Problem: Nearby Interesting SNF

Given $\mathcal{A} \in \mathbb{R}[t]^{n\times n}$ of McCoy rank at most $n-1$, find $\widehat{\mathcal{A}} \in \mathbb{R}[t]^{n\times n}$ that (locally) solves the optimization problem

$$\min \|\mathcal{A} - \widehat{\mathcal{A}}\| \text{ such that } \begin{cases} \operatorname{SNF}(\widehat{\mathcal{A}}) = \operatorname{diag}(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{n-1}, \hat{s}_n), \\ \deg(s_n) \ge \deg(\hat{s}_{n-1}) \ge 1. \end{cases}$$

Our Contributions

1. Tight lower bounds on the radius of triviality
2. Polynomial-time decision procedure for ill-posedness
3. Stability analysis on SNF via Optimization
4. Iterative algorithms with local quadratic convergence
    - Nearest matrix with reduced McCoy rank
    - Nearest matrix with McCoy rank at most $n - r$
    - Reasonable initial guess heuristics for both algorithms
    - Polynomial per-iteration cost
5. Implementation in `Maple`

Previous Work on Floating Point SNF Computations

Reduction to Degree One

Every matrix polynomial $\mathcal{A} \in \mathbb{R}[t]^{n\times n}$ can be *linearized* to

$$\mathcal{P} = \mathcal{P}_0 + t\mathcal{P}_1 \quad \text{for some} \quad \mathcal{P}_0, \mathcal{P}_1 \in \mathbb{R}^{nd\times nd}.$$

- Extract the SNF from Kronecker's Canonical Form
- $\mathrm{SNF}(\mathcal{P}) = \mathrm{diag}(1, 1, \ldots, 1, \mathrm{SNF}(\mathcal{A}))$

Backward Stable: Finds the SNF of a nearby matrix.

- Full Rank Case: QZ Algorithm
  - Wilkinson (1979)
- Singular Case: Fast Staircase/Deflation Algorithms
  - Beelen and Van Dooren (1984,1988)
- Current: GUPTRI
  - Demmel and Edelman (1995)

Applications of Approximate Smith Form

- Structured stability of polynomial eigenvalue problems
- Matrix polynomial eigenvalue least squares problems
  - Occurs frequently in control systems engineering
  - Decide if the SNF can be inferred numerically

Our goal is different: Find a nearby matrix with a non-trivial SNF.

- Structured backward stability analysis of SNF computations
- Detect irrecoverable failures of existing algorithms
  - SNF of a nearby matrix may be meaningless
  - Problem is **not always continuous**
  - We compute a nearby matrix with an interesting SNF

Reduction to Approximate GCD

Example (Find Nearest 2×2 matrix with a non-trivial SNF)

$C = \text{diag}\left(t^2 - 2t + 1, t^2 + 2t + 2\right)$ find a lower McCoy rank $\widetilde{C}$.

Approximate GCD of C_{11} and C_{22} (Karmarkar and Lakshman '96)

$$\inf \left\{ \|C_{11} - \widetilde{C}_{11}\|_2^2 + \|C_{22} - \widetilde{C}_{22}\|_2^2 \right\} \quad \text{s.t.} \quad \gcd(\widetilde{C}_{11}, \widetilde{C}_{22}) \neq 1.$$

Assume: $\widetilde{C}_{11} = (c_{11}t + c_{10})(h_1 t + 1)$ and $\widetilde{C}_{22} = (c_{21}t + c_{20})(h_1 t + 1)$.

The distance to a matrix with a non-trivial SNF is

$$\inf_{h_1 \in \mathbb{R}} \frac{5{h_1}^4 - 4{h_1}^3 + 14h_1 + 2}{{h_1}^4 + {h_1}^2 + 1} = 2 \text{ when } h_1 = 0.$$

Thus $\gcd(\widetilde{C}_{11}, \widetilde{C}_{22}) = 1$ at the infima.

Reducing Approximate SNF to Approximate GCD

- We can define the SNF in terms of the minors

$$s_j = \frac{\delta_j}{\delta_{j+1}} \text{ where } \delta_j = \text{GCD}\{\text{all } j \times j \text{ minors of } \mathcal{A}\}$$

- Requiring $\delta_j \neq 1 \implies \mathcal{A}$ has McCoy rank at most $n-j-1$
- Use Sylvester matrices and approximate GCD techniques
  - δ_j's are approximate GCD's of several polynomials
  - Coefficient structure is multi-linear in the entries of $\mathcal{A}$

Lemma

$\mathcal{A}$ has McCoy rank at most $n-2$ iff entries of the adjoint matrix have a non-trivial GCD.

We compute the adjoint matrix quickly and robustly!

Distance lower bounds via unstructured SVDs

- Embed matrix polynomials into scalar matrices over $\mathbb{R}$

Generalized Sylvester matrices

Let $a \in \mathbb{R}[t]$ with $\deg a \le d$.

$$\phi_r(a) = \begin{pmatrix} a_0 & \cdots & a_d & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_d \end{pmatrix} \in \mathbb{R}^{r\times(r+d)}.$$

- Let $\mathbf{f} = (f_1, \ldots, f_k) \in \mathbb{R}[t]^k$ be ordered by decreasing degree
- Take $\mathbf{d} = (\deg(f_1), \ldots, \deg(f_k))$, $r = \deg f_1$ and $d = \max\{\deg_{f_j}\}_{j=2}^k$

$$\underbrace{\text{Syl}(\mathbf{f}) = \text{Syl}_{\mathbf{d}}(\mathbf{f})}_{\text{Generalized Sylvester Matrix}} = \begin{pmatrix} \phi_r(f_1) \\ \phi_d(f_2) \\ \vdots \\ \phi_d(f_k) \end{pmatrix} \in \mathbb{R}^{(r+(k-1)d)\times(r+d)}.$$

Generalized Sylvester Matrices

Theorem

$\gcd(\mathbf{f}) = 1 \iff \mathrm{Syl}(\mathbf{f})$ *has full rank.*

Problem: What if the degrees of $\mathbf{f}$ can increase?

- Degrees of $\mathbf{f}$ can be at-most $\mathbf{d}' = \left(d'_1, \ldots, d'_k\right)$
- Spurious solutions can occur due to over-padding of zeros
- Define $\mathrm{rev}_{d'_j}(f_j) = t^{d_j} f(t^{-1})$
- Define $\mathrm{rev}_{\mathbf{d}'}(\mathbf{f})$ in the obvious way

Theorem

If $\mathrm{Syl}_{\mathbf{d}'}(\mathbf{f})$ *is rank deficient then* $\gcd(\mathbf{f}) = 1$ *iff* $\mathrm{Syl}(\mathrm{rev}_{\mathbf{d}'}(\mathbf{f}))$ *has full rank.*

Approximate SNF via Sylvester Matrices

Theorem

A nearest rank at most e Sylvester matrix always exists.

Theorem

Suppose that $\mathbf{d}' = (\gamma, \gamma \ldots, \gamma)$ and $\mathsf{Syl}_{\mathbf{d}'}(\mathrm{Adj}(\mathcal{A}))$ has rank e.

$$\frac{\sigma_e(\mathsf{Syl}_{\mathbf{d}'}(\mathrm{Adj}(\mathcal{A})))}{\gamma n^3(d+1)^{3/2}\|\mathcal{A}\|_\infty^n n^{n/2}} \leq \|\mathcal{A} - \widehat{\mathcal{A}}\|_F, \text{ where } \mathrm{SNF}(\widehat{\mathcal{A}}) \text{ is non-trivial.}$$

- $\sigma_e(\mathsf{Syl}_{\mathbf{d}'}(\mathrm{Adj}(\mathcal{A})))$ *is the distance to a nearest singular matrix*

Example (Same $\mathcal{A}$ as the First Example)

A lower bound on the distance to non-triviality is $4.3556e-4$.

Nearest Matrix Polynomial with an Interesting SNF

Constrained Optimization Approach

$$\min \|\underbrace{\mathcal{A}-\widehat{\mathcal{A}}}_{\Delta\mathcal{A}}\|_F^2 \quad \text{such that} \quad \begin{cases} \mathrm{Adj}(\widehat{\mathcal{A}}) = \mathcal{F} h, \\ \quad \mathcal{F} \in \mathbb{R}[t]^{n\times n}, \\ \quad h = h_0 + h_1 t + h_2 t^2, \\ \quad h_2^2 + h_1^2 - 1 = 0. \end{cases}$$

- Assume the adjoint has a finite approximate GCD
  - Otherwise the reversal has a non-trivial GCD
- Generically, the approximate GCD has degree 1 or 2
- $h_2^2 + h_1^2 - 1 = 0 \implies h$ has degree at least 1
- Solve with Lagrange Multipliers and Levenberg-Marquardt

Levenberg-Marquardt Iteration

Theorem

The Levenberg-Marquardt iteration converges quadratically to the minimum value with a suitable initial guess.

Corollary

Under small perturbations:

- *Well-posed approximate SNF instances remain well-posed.*
- *Ill-posed instances cannot be regularized to be well-posed.*

- Theory applies by induction to arbitrary McCoy rank
- Applies to infinite eigenvalues: consider $t^d\mathcal{A}(t^{-1})$
- This is why existing algorithms fail and cannot be saved

Algorithm and Implementation in `Maple 2017`

- Compute derivatives quickly
  - Partial two variable ansatz and evaluation
- $\mathrm{Adj}(\mathcal{A} + \Delta\mathcal{A})$ has exponentially many coefficients
- Compute derivatives of $\mathrm{Adj}(\cdot)$ quickly
  - Details in an upcoming paper!
- LM iteration cost is polynomial $O(n^9 d^3)$ flops for $r = 2$
  - Grows exponentially in r, the specified McCoy Rank deficiency

Initial Guess

- Compute approximate GCD of the adjoint matrix
- Approximate Lagrange multipliers with linear least squares

Lower McCoy Rank Approximations

- Assume that $\mathcal{A} \in \mathbb{R}[t]^{nd\times nd}$ has degree 1

McCoy Rank At-Most $n - r$ Approximation

$$\min \|\Delta\mathcal{A}\|_F^2 \quad \text{such that} \quad \begin{cases} ((\mathcal{A} + \Delta\mathcal{A})(\omega))\, K = 0, \\ \omega \in \mathbb{C}, K \in \mathbb{C}^{nd\times r}, \\ K^* K = I_r. \end{cases}$$

- We use LM; gradient methods are acceptable
- Per-iteration cost is $O(n^6 d^6)$ (does not depend on r)

Initial Guess: Tri-linear alternating projections.

- Take ω_{init} as a local extrema of $|\det(\mathcal{A})|^2$
- Take K_{init} from the r smallest singular vectors of $\mathcal{A}(\omega_{init})$
- Approximate Lagrange multipliers with linear least squares

Summary of Examples (Same $\mathcal{A}$ as First Example)

$n-r$	Struct	# Lower	# GCD	$\|\Delta\mathcal{A}_{opt}\|_F$	ω_{opt}	$\deg \mathcal{S}_\varepsilon$
0	Support	191	N/A	2.11383	-.36276	5
0	Entry	189	N/A	2.11135	-.36580	5
0	Degree	179	N/A	2.07278	-.37822	6
1	Support	91	9	1.06963	-.27999	6
1	Entry	89	9	1.06914	-.28044	6
1	Degree	61	11	0.96031	-.22957	7

Compare with the Sylvester Matrix Lower Bounds...

	Support	Entry	Degree
SVD Bound	4.3556e-4	4.080713e-4	1.999026e-4

- Adjoint method is robust; Requires fewer iterations
- Optimization is local; Reasonable initial guesses are needed
- The coefficient displacement structure is very important

Related and Future Work

What I have done...

- Numerically Robust and Fast Matrix Polynomial $\mathrm{Adj}(\cdot)$
- Backwards/Mixed Stability of $\mathrm{Adj}(\cdot)$ Computations
- Numerically Robust and Fast Derivative Computation of $\mathrm{Adj}(\cdot)$

What I am working on...

- Implementation of a fast approximate SNF algorithm
- Sparse Approximate Factorizations over

$$\mathbb{R}[t][\partial;'\,] \quad \text{and} \quad \mathbb{C}[x_1, x_2, \ldots, x_k].$$

- Finishing my thesis and looking for new opportunities!

Rank 0 McCoy Rank Approximation

$$\mathcal{A} = \begin{pmatrix} t^3+3t+1 & 1 & t+1 \\ 0 & t^2+2t+2 & 0 \\ t+1 & t+1 & t^3+5t+1 \end{pmatrix}$$

- $\mathcal{A}$ has trivial SNF
- Take $\mathcal{A}_{init} = \mathcal{A}$
- $\omega_{init} \approx 0.4120084$

- Consider perturbations that do not change the support

191 iterations $\implies$ 12 digits of accuracy, $\omega_{opt} \approx -0.362762767179$

$$\underbrace{\begin{pmatrix} .99427t^3+2.9565t+1.12 & 0 & 1.2043t+.43687 \\ 0 & .83895t^2+2.4440t+.77617 & 0 \\ 1.2043t+.43687 & 1.2043t+.43687 & .96373t^3+4.7244t+1.7598 \end{pmatrix}}_{\mathcal{A}_{opt}}$$

$$\widehat{\mathcal{S}} \approx \begin{pmatrix} t-\omega_{opt} & & \\ & t-\omega_{opt} & \\ & & (t-\omega_{opt})S_\varepsilon \end{pmatrix}$$ and $\|\Delta\mathcal{A}_{opt}\|_F \approx 2.11383$

$$S_\varepsilon \approx 0.80388t^5+1.46695t^4+5.16105t^3+14.58267t^2+5.29517t+28.94238$$

Rank 0 McCoy Rank Approximation : Entry Degree

$$\mathcal{A} = \begin{pmatrix} t^3+3t+1 & 1 & t+1 \\ 0 & t^2+2t+2 & 0 \\ t+1 & t+1 & t^3+5t+1 \end{pmatrix}$$

- $\mathcal{A}$ has trivial SNF
- Take $\mathcal{A}_{init} = \mathcal{A}$
- $\omega_{init} \approx 0.4120084$

- Consider perturbations that do not change the entry degree

189 iterations $\implies$ 12 digits of accuracy, $\omega_{opt} \approx -.365806171787$

$$\underbrace{\begin{pmatrix} .99379t^3+.016971t^2+2.9536t+1.1268 & 0 & 1.2046t+.44065 \\ 0 & .83708t^2+2.4454t+.78252 & 0 \\ 1.2046t+.44065 & 1.2046t+.44065 & .96276t^3+.10180t^2+4.7217t+1.7607 \end{pmatrix}}_{\mathcal{A}_{opt}}$$

$$\widehat{\mathcal{S}} \approx \begin{pmatrix} t-\omega_{opt} & & \\ & t-\omega_{opt} & \\ & & (t-\omega_{opt})S_\varepsilon \end{pmatrix}$$ and $\|\Delta\mathcal{A}_{opt}\|_F \approx 2.1113588$

$$S_\varepsilon \approx 0.80090t^5+1.55911t^4+5.31324t^3+14.72015t^2+5.97834t+28.61277$$

Rank 0 McCoy Rank Approximation : Matrix Degree

$$\mathcal{A} = \begin{pmatrix} t^3+3t+1 & 1 & t+1 \\ 0 & t^2+2t+2 & 0 \\ t+1 & t+1 & t^3+5t+1 \end{pmatrix}$$

- $\mathcal{A}$ has trivial SNF
- Take $\mathcal{A}_{init} = \mathcal{A}$
- $\omega_{init} \approx 0.4120084$

- Consider perturbations that can change all degrees

179 iterations $\implies$ 12 digits of accuracy, $\omega_{opt} \approx -0.378229408431$

$$\underbrace{\begin{pmatrix} .99124t^3+.023155t^2+2.9388t+1.1619 & .046387t^3-.12264t^2+.32426t+.14270 & .028842t^3-.076256t^2+1.2016t+.46696 \\ 0 & .064321t^3+.82994t^2+2.4496t+.81127 & 0 \\ .028842t^3-.076256t^2+1.2016t+.46696 & .028842t^3-.076256t^2+1.2016t+.46696 & .95615t^3+.11593t^2+4.6935t+1.8104 \end{pmatrix}}_{\mathcal{A}_{opt}}$$

$$\widehat{\mathcal{S}} \approx \begin{pmatrix} t-\omega_{opt} & & \\ & t-\omega_{opt} & \\ & & (t-\omega_{opt})S_{\varepsilon} \end{pmatrix}$$ and $\|\Delta\mathcal{A}_{opt}\|_F \approx 2.07278948063$

$$S_{\varepsilon} \approx 0.06090t^6 + 0.72589t^5 + 2.06256t^4 + 4.81853t^3 + 15.54934t^2 + 5.84844t + 28.26751$$

Rank 1 McCoy Rank Approximation

Using the Adjoint/Approximate GCD Formulation

$$\mathcal{A} = \begin{pmatrix} t^3+3t+1 & 1 & t+1 \\ 0 & t^2+2t+2 & 0 \\ t+1 & t+1 & t^3+5t+1 \end{pmatrix}$$

- $\mathcal{A}$ has trivial SNF
- Take $\mathcal{A}_{init} = \mathcal{A}$
- $h_{init} = t$ ($\omega_{init} = 0$)

- Consider perturbations that do not change the support

9 iterations $\implies$ 15 digits of accuracy, $\omega_{opt} \approx -.27999154088436$

$$\underbrace{\begin{pmatrix} 1.0028t^3+3.0358t+.87202 & 1 & 1.1869t+.33233 \\ 0 & t^2+2t+2 & 0 \\ 1.1869t+.33233 & t+1 & .99142t^3+4.8905t+1.3911 \end{pmatrix}}_{\mathcal{A}_{opt}}$$

$$\widehat{\mathcal{S}} \approx \begin{pmatrix} 1 & & \\ & t-\omega_{opt} & \\ & & (t-\omega_{opt})S_\varepsilon \end{pmatrix}$$ and $\|\Delta\mathcal{A}_{opt}\|_F \approx 1.06963271820$

$$S_\varepsilon \approx 0.99420t^6 + 1.43166t^5 + 9.02277t^4 + 12.92270t^3 + 25.84113t^2 + 23.60992t + 28.12892$$

Rank 1 McCoy Rank Approximation

Using the Adjoint/Approximate GCD Formulation

$$\mathcal{A} = \begin{pmatrix} t^3+3t+1 & 1 & t+1 \\ 0 & t^2+2t+2 & 0 \\ t+1 & t+1 & t^3+5t+1 \end{pmatrix}$$

- $\mathcal{A}$ has trivial SNF
- Take $\mathcal{A}_{init} = \mathcal{A}$
- $h_{init} = t$ ($\omega_{init} = 0$)

- Consider perturbations that do not change the entry degree

9 iterations $\implies$ 15 digits of accuracy, $\omega_{opt} \approx -0.280440198593668$

$$\underbrace{\begin{pmatrix} 1.0028t^3 - .00990t^2 + 3.0353t + .87412 & 1 & 1.1871t + .33291 \\ 0 & t^2+2t+2 & 0 \\ 1.1871t + .33291 & t+1 & .99138t^3 + .030743t^2 + 4.8904t + 1.3909 \end{pmatrix}}_{\mathcal{A}_{opt}}$$

$$\widehat{\mathcal{S}} \approx \begin{pmatrix} 1 & & \\ & t-\omega_{opt} & \\ & & (t-\omega_{opt})S_\varepsilon \end{pmatrix} \quad \text{and} \quad \|\Delta\mathcal{A}_{opt}\|_F \approx 1.06914559551$$

$$S_\varepsilon \approx 0.99413t^6 + 1.45168t^5 + 9.050662t^4 + 12.98332t^3 + 25.8918t^2 + 23.67078t + 28.10003$$

Rank 1 McCoy Rank Approximation

Using the Adjoint/Approximate GCD Formulation

$$\mathcal{A} = \begin{pmatrix} t^3+3t+1 & 1 & t+1 \\ 0 & t^2+2t+2 & 0 \\ t+1 & t+1 & t^3+5t+1 \end{pmatrix}$$

- $\mathcal{A}$ has trivial SNF
- Take $\mathcal{A}_{init} = \mathcal{A}$
- $h_{init} = t$ $(\omega_{init} = 0)$

- Consider perturbations that can change all degrees

11 iterations $\implies$ 15 digits of accuracy, $\omega_{opt} \approx -0.22957727217562$

$$\underbrace{\begin{bmatrix} 1.0004t^3 - .00158t^2 + 3.0069t + .96993 & -.00124t^3 + .00542t^2 - .023616t + 1.1029 & .0066t^3 - .028771t^2 + 1.1253t + .45412 \\ -.00404t^3 + .017626t^2 - .076777t + .33443 & .00149t^3 + .99351t^2 + 2.0283t + 1.8768 & -.00293t^3 + .012798t^2 - .055748t + .24283 \\ .00647t^3 - .02819t^2 + 1.1228t + .46498 & -.00094t^3 + .00409t^2 + .98215t + 1.0777 & .99645t^3 + .015443t^2 + 4.9327t + 1.2930 \end{bmatrix}}_{\mathcal{A}_{opt}}$$

$$\widehat{\mathcal{S}} \approx \begin{pmatrix} 1 & & \\ & t - \omega_{opt} & \\ & & (t-\omega_{opt})S_{\varepsilon} \end{pmatrix} \quad \text{and} \quad \|\Delta\mathcal{A}_{opt}\|_F \approx 0.960310462257$$

$$S_{\varepsilon} \approx 0.0014t^7 + 0.9897t^6 + 1.59563t^5 + 8.9792t^4 + 14.2552t^3 + 26.07418t^2 + 26.2280t + 28.7424$$

Generalized Sylvester Matrices

Example (GCD at Infinity)

$\mathbf{f} = (2t + 1, 3t, 4)$, $\mathbf{d}' = (2, 2, 2)$ and $\mathrm{rev}_{\mathbf{d}'}(\mathbf{f}) = (1t^2 + 2t, 3t, 4t^2)$.

Is $\gcd(\mathbf{f})$ non-trivial with degree sequence $\mathbf{d}'$?

$$\mathrm{Syl}_{\mathbf{d}'}(\mathbf{f}) = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ \hline 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ \hline 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right) \text{ and } \mathrm{Syl}(\mathrm{rev}_{\mathbf{d}'}(\mathbf{f})) = \left(\begin{array}{c|ccc} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ \hline 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ \hline 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right).$$

- Approximate gcd of $\mathbf{f}$ with degrees $\mathbf{d}'$ is $(\varepsilon t + 1)$
- This is a GCD at infinity, of distance zero
- Obviously $\gcd(\mathbf{f}) = 1$

Generalized Sylvester Matrices

Example (No GCD at Infinity)

$\mathbf{f} = (2t+1, 3t, 4)$, $\mathbf{d}' = \underbrace{(2,1,2)}_{\text{No change}}$ and $\text{rev}_{\mathbf{d}'}(\mathbf{f}) = (1t^2+2t, 3, 4t^2)$.

Is $\gcd(\mathbf{f})$ non-trivial with degree sequence $\mathbf{d}'$?

$$\text{Syl}_{\mathbf{d}'}(\mathbf{f}) = \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ \hline 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ \hline 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right) \text{ and } \text{Syl}(\text{rev}_{\mathbf{d}'}(\mathbf{f})) = \left(\begin{array}{cccc} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ \hline 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right).$$

- $\text{Syl}_{\mathbf{d}'}(\mathbf{f})$ is over-padded with a column of zeros
- No GCD at infinity since $\text{Syl}(\text{rev}_{\mathbf{d}'}(\mathbf{f}))$ has full rank
- Both Sylvester matrices used to decide non-triviality

Lagrange Multipliers

Define the Lagrangian

$$L = \|\Delta\mathcal{A}\|_F^2 + \lambda^T \begin{pmatrix} \mathrm{Adj}(\mathcal{A}+\Delta\mathcal{A}) - f^* h \\ h_2^2 + h_1^2 - 1 \end{pmatrix} \text{ and } x = \begin{pmatrix} \mathrm{vec}(\Delta\mathcal{A}) \\ \mathrm{vec}(f^*) \\ \mathrm{vec}(h) \end{pmatrix}.$$

- The Gradient of L is ∇L
- The Jacobian of the constraints is J
- The Hessian of L (w.r.t. to x) is $H = \nabla^2 L$ ($H_{xx} = \nabla^2_{xx} L$)

$$J = \nabla \begin{pmatrix} \mathrm{Adj}(\mathcal{A}+\Delta\mathcal{A}) - f^* h \\ h_2^2 + h_1^2 - 1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} H_{xx} & J^T \\ J & \end{pmatrix}$$

Fact (First Order Necessary Condition)

It is **necessary** *that* $\nabla L = 0$ *at a local minimizer.*

Newton's Method and Variants

Let $L = L(x^k, \lambda^k)$ and $H = H(x^k, \lambda^k)$.

Newton's Method to Solve $\nabla L = 0$

$$\text{Compute} \begin{pmatrix} x^k + \Delta x \\ \lambda^k + \Delta\lambda \end{pmatrix} \quad \text{where} \quad H \begin{pmatrix} \Delta x \\ \Delta\lambda \end{pmatrix} = -\nabla L.$$

- If H is rank deficient then the iteration is ill-defined

A Quasi Newton Method: Levenberg-Marquardt

$$\left(H^T H + \mu_k I\right) \begin{pmatrix} \Delta x^k \\ \Delta\lambda^k \end{pmatrix} = -H^T \nabla L, \text{ for } \mu_k > 0.$$

- LM step is always well defined since $H^T H + \mu_k I$ has full rank
- $H^T H + \mu_k I$ is positive definite $\implies$ converges globally

Second-Order Optimality Conditions

Let $\nabla L = \nabla L(x^\star, \lambda^\star)$, $H = H(x^\star, \lambda^\star)$ and $J = J(x^\star, \lambda^\star)$.

Fact (Second Order Sufficiency Condition (**SOSC**))

If $\nabla L = 0$ *and* $\ker(J)^T H_{xx} \ker(J) \succ 0$

then $(x^\star, \lambda^\star)$ *is a local minimizer.*

Theorem (Second Order Sufficiency Holds)

If $\|\mathcal{A} - \widetilde{\mathcal{A}}\|$ *is sufficiently small, then under mild normalization assumptions we have that second-order sufficiency holds.*

- Solutions will be isolated
- $\kappa_2\left(\begin{pmatrix} H_{xx} \\ J \end{pmatrix}\right)$ acts as a condition number of the problem
- **SOSC** $\implies$ quasi-Newton methods are reliable