Succinct Data Structures

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General Motivation

- In Many Computations ...
  - Storage Costs of Pointers and Other Structures Dominate that of Real Data
- Often this information is not “just random pointers”
- How do we encode a combinatorial object (e.g. a tree) of specialized information … even a static one

  in a small amount of space & still perform queries in constant time ???

Succinct Data Structure

Representation of a combinatorial object:

Space requirement of representation "close to" information theoretic lower bound

and

Time for operations required of the data type comparable to that of representation without such space constraints ($O(1)$)

Example : Static Bounded Subset

Given: Universe $[m]= 0,\ldots,m-1$ and n arbitrary elements from this universe

Create: Static data structure to support "member?" in constant time in the lg m bit RAM model

Using: Close to information theory lower bound space, i.e. about $\lg \binom{m}{n}$ bits

(Brodnik & M)

Careful .. Lower Bounds

Beame-Fich: Find largest less than i is tough in some ranges of m(e.g. $m \approx 2^{\sqrt{\lg n}}$)

But OK if i is present this can be added (Raman, Raman, Rao etc)

Focus on Trees

.. Because Computer Science is .. Arborphilic

Directories (Unix, all the rest)

Search trees (B-trees, binary search trees, digital trees or tries)

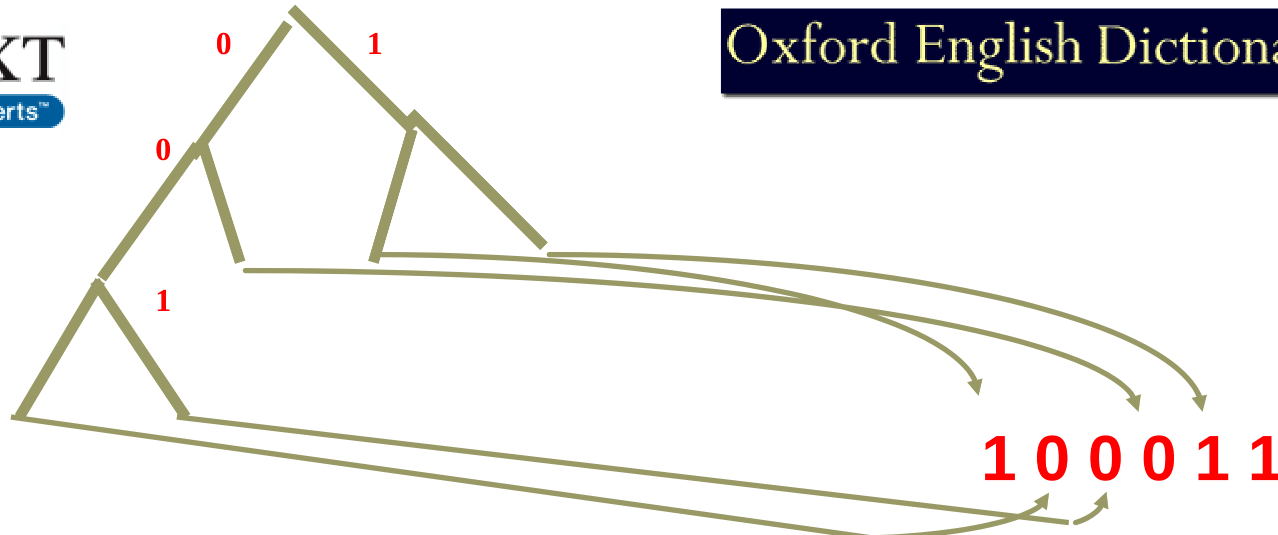
Graph structures (we do a tree based search)

and a key application

Search indices for text (including DNA)

Preprocess Text for Search
A Big Patricia Trie/Suffix Trie

OPEN TEXT
The Content Experts™



Oxford English Dictionary

- Given a large text file; treat it as bit vector
- Construct a trie with leaves pointing to unique locations in text that “match” path in trie (paths must start at character boundaries)
- Skip the nodes where there is no branching (n-1 internal nodes)

So the *basic* story on text search

A suffix tree (*40 years old last year*) permits search for any arbitrary query string in time proportional to the query string. But the *usual* space for the tree can be prohibitive

Most users, especially in Bioinformatics as well as Open Text and Manber & Myers went to suffix arrays instead.

Suffix array: reference to each index point in order by what is pointed to

The Issue

Suffix tree/ array methods remain extremely effective, especially for single user, single machine searches.

So, can we represent a tree (e.g. a binary tree) in substantially less space?

Space for Trees

Abstract data type: binary tree

Size: n-1 internal nodes, n leaves

Operations: child, parent, subtree size, leaf data

Motivation: "Obvious" representation of an n node tree takes about 6 n lg n bit words (up, left, right, size, memory manager, leaf reference)

i.e. full suffix tree takes about 5 or 6 times the space of suffix array (i.e. leaf references only)

Succinct Representations of Trees

Start with Jacobson, then others:

Catalan number

= # ordered rooted forests

Or # binary trees

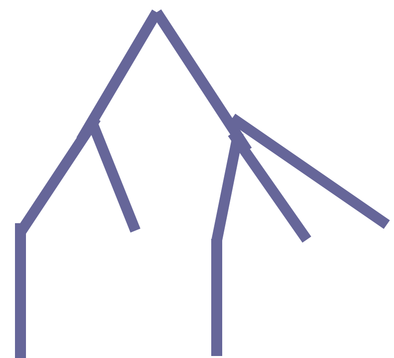
$$=\frac{1}{n+1}\binom{2n}{n} \approx 4^n/(\pi n)^{3/2}$$

So lower bound on specifying is about $2n$ bits

What are natural representations?

Arbitrary Order Trees

- Use parenthesis notation
- Represent the tree



- As the binary string (((())())((())()())): traverse tree as "(" for node, then subtrees, then ")"
- Each node takes 2 bits … but operations?

What you learned about Heaps

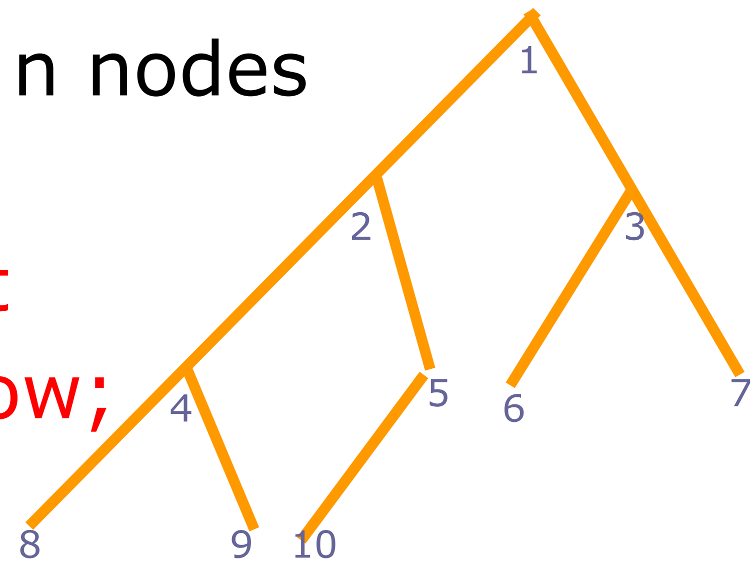
Only 1 heap (shape) on n nodes

Balanced tree,

bottom level pushed left

number nodes row by row;

lchild(i)=2i; rchild(i)=2i+1



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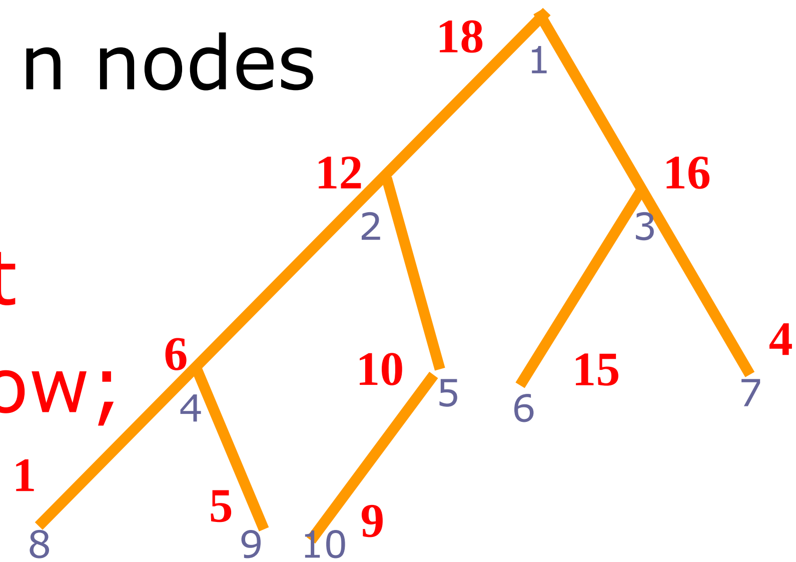


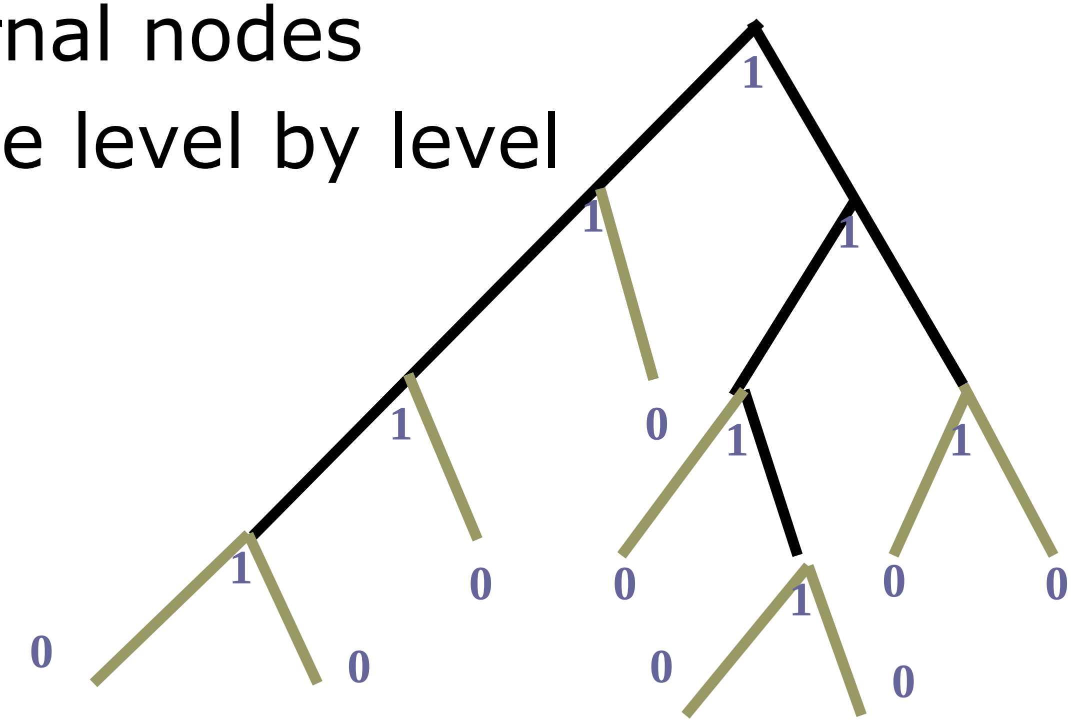
Data: Parent value > child

This gives an implicit data structure for priority queue

Generalizing: Heap-like Notation for ANY Binary Tree

Add external nodes

Enumerate level by level



Store vector 11110111001000000 length 2n+1

(Here we don’t know size of subtrees; can be overcome. Could use isomorphism to flip between notations)

How do we Navigate?

Jacobson's key suggestion:
  Operations on a bit vector

rank(x) = # 1's up to & including x

select(x) = position of x^{th} 1

So in the binary tree

leftchild(x) = 2 rank(x)

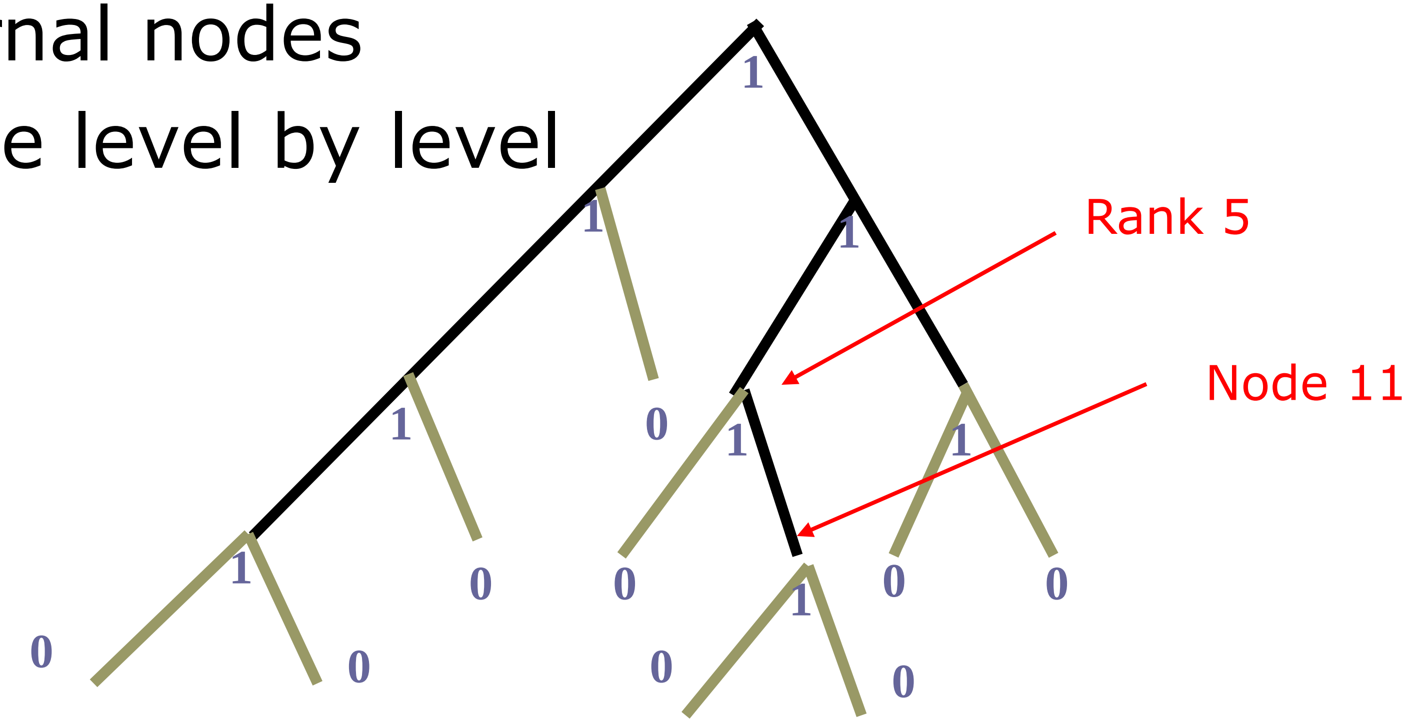
rightchild(x) = 2 rank(x) + 1

parent(x) = select($\lfloor x/2 \rfloor$)

Heap-like Notation for a Binary Tree

Add external nodes

Enumerate level by level



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Rank & Select

Rank: Auxiliary storage ~ 2nlglg n / lg n bits

#1's up to each $(\lg n)^2$ rd bit

#1's within these too each ½ lg nth bit

Table lookup after that

Select: More complicated (especially to get this lower order term) but similar notions

Key issue: Rank & Select take O(1) time with lg n bit word (M. et al)… as detailed on the board

Lower Bound: for Rank & for Select

Theorem (Golynski): Given a bit vector of length n and an "index" (extra data) of size r bits, let t be the number of bits probed to perform rank (or select) then: $r=\Omega(n (\lg t)/t)$.

Proof idea: Argue to reconstructing the entire string with too few rank queries (similarly for select)

Corollary (Golynski): Under the $\lg n$ bit RAM model, an index of size $\Theta(n \lg\lg n / \lg n)$ is necessary and sufficient to perform the rank and the select operations in $O(\lg n)$ bit probes, so in) $O(1)$ time.

Other Combinatorial Objects

Planar Graphs (Jacobson; Lu et al; Barbay et al)

Subset of [n] (Brodnik & M)

Permutations $[n] \rightarrow [n]$

Or more generally

Functions $[n] \rightarrow [n]$ But what operations?

Clearly $\pi(i)$, but also $\pi^{-1}(i)$

And then $\pi^{k}(i)$ and $\pi^{-k}(i)$

More Data Types

Suffix Arrays (special permutations; references to positions in text sorted lexicographically) in linear space … after all writing the string takes only linear space.

"Arbitrary" Classes of Trees

Consider classes of trees where "all small subtrees" are members of the class.

(e.g. ordinal trees of degree at most 2)

We can represent such trees in "near optimal space" and navigate in constant time. Even if we don't know the space lower bound!

(Arash and M)

Partial Orders

Partial order … the transitive closure of a directed graph.

What is the ITLB?

Represent as upper triangular 0-1 matrix. $n^2/2$

But ~~all~~ most of these not “transitive closures”

Right answer $n^2/4$

Can achieve this bound

(Nicholson & M)

Arbitrary Graphs/Digraphs

n vertices and m edges, support adjacency and degree queries

Lower bound: impossible to answer such queries in constant time (per node) …

In information theory lower bound (unless the graph is very sparse ($m=o(n^{\delta})$ for any constant $\delta>.0$) or (similarly) too dense.

But in space $(1+\varepsilon)$ITLB, we can do it.

(Farzan &M)

But first … how about integers

Of "arbitrary" size

Clearly lg n bits … if we take n as an upper bound

But what if we have "no idea"

Elias: lglg n 0's, lg n in lglg n bits, n in lg n bits

Can we do better?

A useful trick in many representations

Dictionary over n elements [m]

Brodnik & M

Fredman, Komlós & Szemerédi (FKS)

Hashing gives constant search using "keys" plus n lg m + o() bits

B&M approach: Information theory lower bound is $\lg\binom{m}{n}$

Spare and dense cases

Sparse: can afford n bits as initial index

… several cases for sparse and for dense

More on Trees

"Two" types of trees … ordinal and cardinal

i.e. 1st 2nd 3rd versus 1,2,3

Cardinal trees:  e.g. Binary trees are cardinal trees of degree 2, each location "taken or not". Number of *k*-ary trees

$$C_n^k = \binom{kn+1}{n} / (kn+1)$$

So ITLB $\approx \lg(k-1) + k\lg(k/(k-1)n$ bits

Ordinal Trees

- Children ordered, no bound on number of children, i^{th} cannot exist without $i\text{-}1^{st}$
- These correspond to balanced parentheses expressions, Catalan number of forests on n nodes
- A variety of representations …..

But first we need: Indexable Dictionaries

Getting that "n" down if there are few 1's

S = n elements for [m]

Rank(i,S) gives # elements $\leq$i

Select(i,S) gives ith smallest

in ITLB =*B* = $\lg \binom{m}{n}$ … or so

A problem … Atai lower bound Ω(lg lg n)

Sidestep by only asking for Rank(i,S) if i$\in$S

Raman, Raman & Rao

Trees

Key rule … nodes numbered 1 to n, but data structure gets to choose “names” of nodes

Would like ordinal operations:

parent, i^{th} child, degree, subtree size

Plus child i for cardinal

Ordinals

Many orderings: LevelOrderUnaryDegreeSequence

Node: d 1's (child birth announcements)

then a 0 (death of the node)

Write in level order: root has a "1 in the sky", then birth order = death order

Gives O(1) time for parent, i^{th} child, degree

Balanced parents gives others, DFUDS … all

- Easy approach: Each node gets k bits, saying which children are there
- So kn bits, say in LOUDS or DFS order

Problem, the space lower bound:

$$\lg\big(C(n,k)\big) \approx \lg(k-1) + k\lg\frac{k}{k-1})n$$

$$\approx (\lg k + \lg e)\, n \text{ bits}$$

(as k grows)

Another approach

- Ordinal for underlying structure (say DFUDS)
- Gives parent, ith child, degree, subtree size
- Now have to deal with child j
- Suppose a node has d of k children
- “just” need i= rank(j), use indexable dict.
- i.e. $d \lg k + o(d) + O(\lg \lg k)$ bits each
- Can be made $n \lg k + o(n) + O(\lg \lg k)$

  no n

More on Trees

Dynamic trees: Tough going, mainly memory management

M, Storm and Raman and Raman, Raman & Rao

Other classes: Decomposition into big tree (o(n) nodes); minitrees hanging off (again o(n) in total); and microtrees (most nodes here) microtrees small enough to be coded in table of size o(n)

If micotrees have "special feature", encoding can be optimal.. Even if you don't know what that means.

(Farzan & M)

Permutations and Functions

Permutation π, write in natural form:

$\pi(i)$ i = 1,…n: space n lg n bits, good!

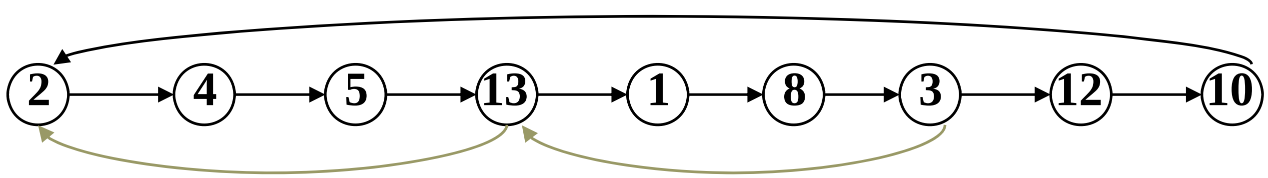
Great for computing π, but how about π^{-1} or π^k

Other option: write in cycles, mildly worse for space, much worse for any calculations above

Permutations: a Shortcut Notation

Let P be a simple array giving π; $P[i] = \pi[i]$

Also have B[i] as a pointer t positions back in (the cycle of) the permutation;
$B[i]= \pi^{-t}[i]$ .. But only define B for every t^{th} position in cycle. (t is a constant; ignore cycle length "round-off")



So array representation

P =	[8	4	12	5	13	x	x	3	x	2	x	10	1]
	1	2	3	4	5	6	7	8	9	10	11	12	13

Representing Shortcuts

In a cycle there is a B every t positions …

But these positions can be in "arbitrary" order

Which i's have a B, and how do we store it?

Keep a vector of all positions: 0 = no B 1 = B

Rank gives the position of B["i"] in B array

So: $\pi(i)$ & $\pi^{-1}(i)$ in O(1) time & $(1+\varepsilon)n \lg n$ bits

Theorem: Under a pointer machine model with space $(1+\varepsilon)\ n$ references, we need time $1/\varepsilon$ to answer π and π^{-1} queries; i.e. this is as good as it gets … in the pointer model.

Getting it n lg n Bits

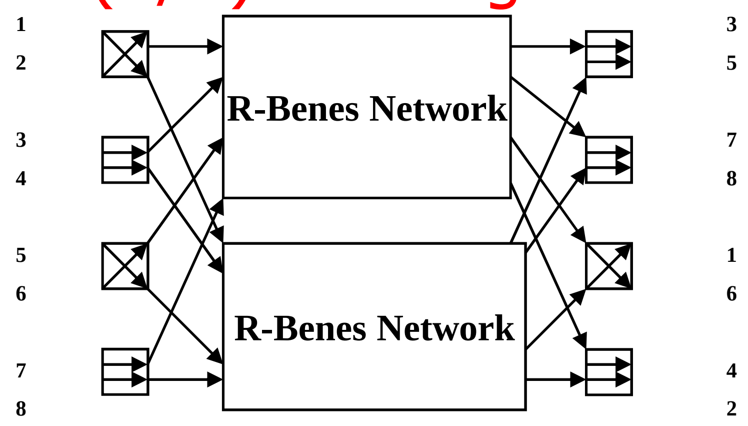
This is the best we can do for O(1) operations

But using Benes networks:

1-Benes network is a 2 input/2 output switch

r+1-Benes network … join tops to tops

#bits(n)=2#bits(n/2)+n=n lg n-n+1=min+Θ(n)

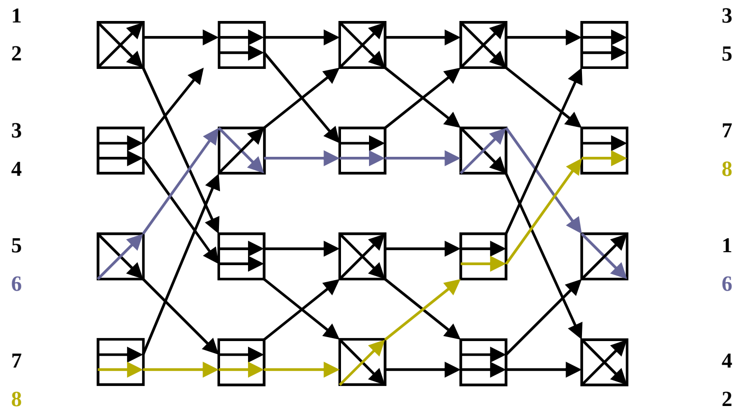


A Benes Network

Realizing the permutation (std $\pi(i)$ notation)

π = (5 8 1 7 2 6 3 4) ; π^{-1} = (3 5 7 8 1 6 4 2)

Note: $\Theta(n)$ bits more than "necessary"



What can we do with it?

Divide into blocks of $\lg \lg n$ gates … encode their actions in a word. Taking advantage of regularity of address mechanism

and also

Modify approach to avoid power of 2 issue

Can trace a path in time $O(\lg n/(\lg \lg n)$

This is the best time we are able get for π and π^{-1} in nearly minimum space.

Both are Best

Observe: This method "violates" the pointer machine lower bound by using "micropointers".

But …

More general Lower Bound (Golynski): Both methods are optimal for their respective extra space constraints

Back to the main track: Powers of π

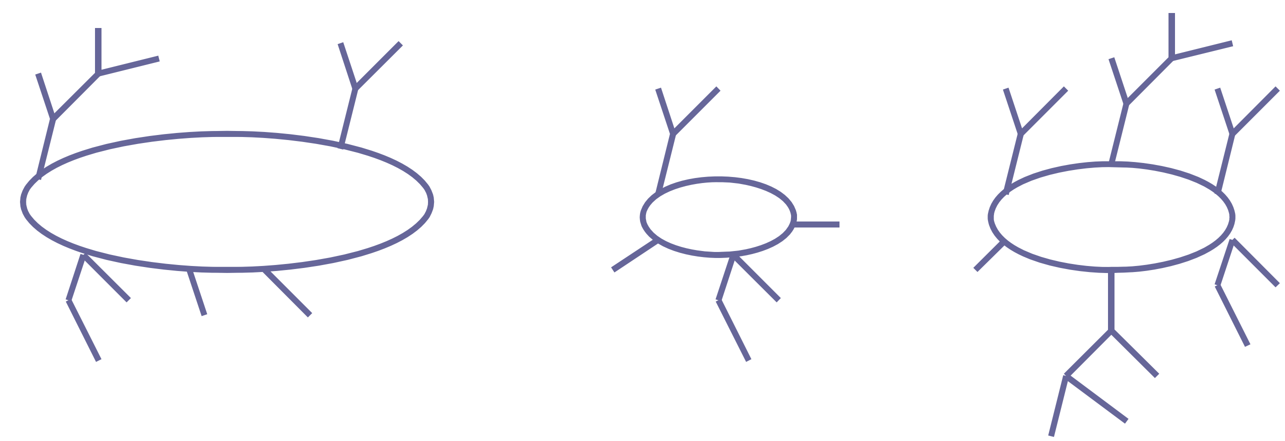
- Consider the cycles of π
- (2 6 8)(3 5 9 10)(4 1 7)
- Bit vector indicates start of each cycle
- (2 6 8 3 5 9 10 4 1 7)
- Ignore parens, view as new permutation, ψ.
- Note: $\psi^{-1}(i)$ is position containing i …
- So we have ψ and ψ^{-1} as before
- Use $\psi^{-1}(i)$ to find i, then n bit vector (rank, select) to find π^k or π^{-k}

Functions

Now consider arbitrary functions [n]→[n]

"A function is just a hairy permutation"

All tree edges lead to a cycle



Challenges here

Essentially write down the components in a convenient order and use the $n \lg n$ bits to describe the mapping (as per permutations)

To get $f^k(i)$:

Find the level ancestor (k levels up) in a tree

Or

Go up to root and apply f the remaining number of steps around a cycle

Level Ancestors

There are several level ancestor techniques using

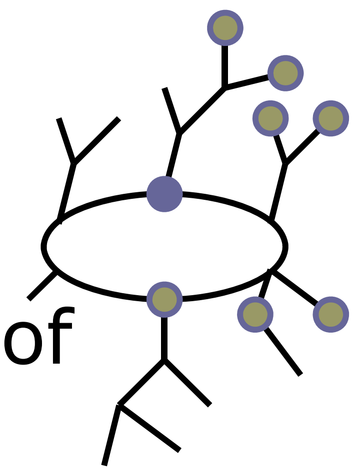
O(1) time and O(n) WORDS.

Adapt Bender & Farach-Colton to work in O(n) bits

But going the other way …

Level Ancestors

- Moving Down the tree (toward leaves) requires care
- f^{-3}(●) = (●)
- The trick:
- Report all nodes on a given level of a tree in time proportional to the number of nodes, and
- Don't waste time on trees with no answers



Final Function Result

Given an arbitrary function $f: [n] \rightarrow [n]$

With an $n \lg n + O(n)$ bit representation we can compute $f^k(i)$ in $O(1)$ time and $f^{-k}(i)$ in time $O(1 + \text{size of answer})$.

f & $f\text{-}1$ are very useful in several applications

… then on to binary relations (HTML markup)