A Worst Case, Constant Time Priority Queue: Beating a Lower Bound

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Joint work with

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Lower Bounds: What do they mean?

- If upper and lower bounds match, the problem is solved.
  - If lower bound exceeds the time you can take ... give up
Lower Bounds: What do they mean?

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- Lower bounds, and upper bounds, are proven under a model.
  - So if you have to “get under” a lower bound - focus on the operations the model does not permit
  - It’s time to become imaginative in terms of permitted operations
Beating Lower Bounds: Examples

- **Searching**: \( \lg n \) lower bound on comparisons, so hash
  - \( \lg n \) time becomes constant
- **Sorting**: \( n \lg n \) lower bound on comparisons, so use variants of bucketing
  - \( n \lg n \) time easily linear on average
  - \( n \lg n \) time becomes \( n \lg \lg n \) even in worst case
Beating the Lower Bound ... Another Case

Self organizing linear search...

- Move to front **heuristic** (⇒ **rheuistic**) is within a constant factor of offline optimal for linear search, amortized cost of searching is ~ $\frac{1}{\rho_i}$ under “exchange adjacent” model
  - But $1 \ 2 \ 3 \ 4 \ ... \ \frac{n}{2} \ \frac{n}{2}+1 \ ... \ n$ costs $\Theta(n^2)$
- Under “exchange any two” model offline cost is ~$\log(1/\rho_i)$ ... comparable to splay trees
The Problem at hand: Extended Priority Queue

• van Emde Boas (SWAT i.e. FOCS 1975)
• Universe integers \([1,..m]\) \{n of which are present\}
• Operations: insert / delete
  - find least value \(\geq x\) (or greatest \(\leq\))
• Bound: \(O(\lg \lg m)\) time
• Space: Improved to \(O(m)\) bits
• Model: Standard RAM, with bit twiddling
Some Subsequent Work

- Kurt Mehlhorn, Stephan Näher and Helmut Alt (SiComp ‘88): \( \text{vEB is optimal} \)
  \[ \Omega(\lg \lg m) \] on pointer machine

- Peter Miltersen (STOC ‘94):
  \[ \Omega(\sqrt{\lg \lg m}) \] on a RAM

- Paul Beame and Faith Fich (STOC ‘99):
  parameterization by number of values present
  matching upper & lower bounds- \( \Theta(\sqrt{\lg n/ \lg \lg n}) \)

- Ram model is rather powerful, how can we extend it for our problem?
Another Model: Rambo

- Random Access Machine with Byte Overlap
  Mike Fredman and Dan Willard
- Several words can share bits:
Can we do better under this model?

- Elements are at leaves; an internal node is flagged if it has a descendant.
- \( \lg n \) bit RAMBO word at leaf takes bit pattern of flags on path to root.
- Can find lowest ancestor in \( O(1) \) time.
The vEB Stratified Tree

Internal node keeps track of "outside" bottom elements
A Problem

- Any individual leaf may be referred to by many ancestors
- So one insertion/deletion can require up to $2\lg n$ reference changes
- Modify the approach
Split Tagged Tree

Keep track of left / right inside leaf … if different from parent’s

Each leaf can be referred to by at most two ancestors
So what do we have?

- Constant time “extended priority queue” … two memory accesses for search, three for update on our model
- How much space do we need?
  - $2m + O(lg m)$ bits of ordinary RAM
  - $m$ bits of RAMBO memory in a particular configuration we call Yggdrasil
- and it has been implemented in hardware