The Longest Path in a Random Graph: Directed Case

Miklós Ajtai, János Komlós, Endre Szemerédi

Presented by Zouhaier Ferchiou, Harry Sivasubramaniam

September 21, 2020

Random Directed Graphs

- $D_{n,p}$ all directed edges chosen independently with probability p

Random Directed Graphs

- $D_{n,p}$ all directed edges chosen independently with probability p
- The expected number of edges in $D_{n.p}$ is $n(n-1)p$

Random Directed Graphs

- $D_{n,p}$ all directed edges chosen independently with probability p

- The expected number of edges in $D_{n.p}$ is $n(n-1)p$
- $D'_{n,N}$ uniform random digraph with n vertices and N edges

Longest Directed Paths

- Since $\binom{n}{k} k! \left(\frac{\alpha}{n}\right)^{k-1} \sim n\alpha^{k-1} e^{-k^2/2n}$

Longest Directed Paths

- Since $\binom{n}{k} k! \left(\frac{\alpha}{n}\right)^{k-1} \sim n\alpha^{k-1} e^{-k^2/2n}$
- random digraph with $(1-\epsilon)n$ edges has longest directed path of length $O(\log n)$ a.a.s

Longest Directed Paths

- Since $\binom{n}{k} k! \left(\frac{\alpha}{n}\right)^{k-1} \sim n\alpha^{k-1} e^{-k^2/2n}$
- random digraph with $(1-\epsilon)n$ edges has longest directed path of length $O(\log n)$ a.a.s
- random digraph with n edges has longest directed path of length $O(\sqrt{n \log n})$ a.a.s

Longest Directed Paths

- Since $\binom{n}{k} k! \left(\frac{\alpha}{n}\right)^{k-1} \sim n\alpha^{k-1} e^{-k^2/2n}$
- random digraph with $(1-\epsilon)n$ edges has longest directed path of length $O(\log n)$ a.a.s
- random digraph with n edges has longest directed path of length $O(\sqrt{n \log n})$ a.a.s
- Note: We can just work with $D_{n,p}$ instead of $D'_{n,N}$ by coupling

Main Result

Main Theorem ($D'_{n,N}$)

$D'_{n,\alpha n}$ contains a path of length cn a.a.s, where $\alpha > 1$ and $c = c(\alpha)$ is constant.

Main Result

Main Theorem ($D'_{n,N}$)

$D'_{n,\alpha n}$ contains a directed path of length cn a.a.s, where $\alpha > 1$ and $c = c(\alpha)$ is constant.

Main Theorem ($D_{n,p}$)

For $\alpha > 1$ there are positive numbers c, K and $\theta < 1$ such that

$$\Pr[D_{n,\frac{\alpha}{n}} \text{ contains a directed path of length } cn] > 1 - K\theta^n.$$

Here and onwards we let $p = \frac{\alpha}{n}$

Proof Overview

1. We define a graph process that generates $D_{n,p}$ and the desired long directed path

Proof Overview

1. We define a graph process that generates $D_{n,p}$ and the desired long directed path
2. The process constructs the graph in a Depth First Search like manner

Proof Overview

1. We define a graph process that generates $D_{n,p}$ and the desired long directed path
2. The process constructs the graph in a Depth First Search like manner
3. Use a branching process to analyze it (can have infinitely many points)

Proof Overview

1. We define a graph process that generates $D_{n,p}$ and the desired long directed path
2. The process constructs the graph in a Depth First Search like manner
3. Use a branching process to analyze it (can have infinitely many points)
4. Show that *cn* generations occur in the branching process (i.e. a cn dipath in the first process) well before we have used up all the points in the first process

Directed Graph Process: Attempt 1

Fix an ordering $v_1, \ldots, v_n$ of the vertices

Directed Graph Process: Attempt 1

Fix an ordering $v_1, ..., v_n$ of the vertices

1. Set v_1 as root vertex, then select the children of v_1

Directed Graph Process: Attempt 1

Fix an ordering $v_1, ..., v_n$ of the vertices

1. Set v_1 as root vertex, then select the children of v_1
2. The probability that the children form a given set of k vertices is $p^k(1-p)^{n-1-k}$

Directed Graph Process: Attempt 1

Fix an ordering $v_1, ..., v_n$ of the vertices

1. Set v_1 as root vertex, then select the children of v_1
2. The probability that the children form a given set of k vertices is $p^k(1-p)^{n-1-k}$
3. Then select the child of v_1 with smallest index and determine its children from the unused vertices

Directed Graph Process: Attempt 1

Fix an ordering $v_1, \ldots, v_n$ of the vertices

1. Set v_1 as root vertex, then select the children of v_1
2. The probability that the children form a given set of k vertices is $p^k(1-p)^{n-1-k}$
3. Then select the child of v_1 with smallest index and determine its children from the unused vertices
4. ...

Issue: The distributions keep changing. Ex. the probability that v_1 has k children is $\binom{n-1}{k}p^k(1-p)^{n-1-k}$ while the probability that its first child has l is $\binom{n-1-k}{l}p^l(1-p)^{n-1-k-l}$ and so on...

Directed Graph Process

Fix an ordering $v_1, ..., v_n$ of the vertices, let $m = (1 - \delta)n$ for some $\delta = \delta(\alpha) > 0$ chosen later

Directed Graph Process

Fix an ordering $v_1, ..., v_n$ of the vertices, let $m = (1 - \delta)n$ for some $\delta = \delta(\alpha) > 0$ chosen later

1. Set v_1 as root vertex and initialize m element set $M = \{v_2, ..., v_{m+1}\}$

Directed Graph Process

Fix an ordering $v_1, \dots, v_n$ of the vertices, let $m = (1-\delta)n$ for some $\delta = \delta(\alpha) > 0$ chosen later

1. Set v_1 as root vertex and initialize m element set $M = \{v_2, \dots, v_{m+1}\}$
2. Determine the random number **Ch** of children of v_1 in M according to

$$\lambda_k = \Pr[\mathbf{Ch} = k] = \binom{m}{k} p^k (1-p)^{m-k}$$

Directed Graph Process

Fix an ordering $v_1, ..., v_n$ of the vertices, let $m = (1 - \delta)n$ for some $\delta = \delta(\alpha) > 0$ chosen later

1. Set v_1 as root vertex and initialize m element set $M = \{v_2, ..., v_{m+1}\}$
2. Determine the random number **Ch** of children of v_1 in M according to

$$\lambda_k = \Pr[\mathbf{Ch} = k] = \binom{m}{k} p^k (1-p)^{m-k}$$

3. Select k points at random from M for the children of v_1 and then replace them in M with the first k unused vertices

Directed Graph Process

Fix an ordering $v_1, \ldots, v_n$ of the vertices, let $m = (1 - \delta)n$ for some $\delta = \delta(\alpha) > 0$ chosen later

1. Set v_1 as root vertex and initialize m element set $M = \{v_2, \ldots, v_{m+1}\}$
2. Determine the random number **Ch** of children of v_1 in M according to

$$\lambda_k = \Pr[\mathbf{Ch} = k] = \binom{m}{k} p^k (1 - p)^{m-k}$$

3. Select k points at random from M for the children of v_1 and then replace them in M with the first k unused vertices
4. Recursively repeat on the first child until branch dies, then repeat on the sibling of the node that went extinct (Essentially a depth first search!)

Directed Graph Process

Fix an ordering $v_1, ..., v_n$ of the vertices, let $m = (1-\delta)n$ for some $\delta = \delta(\alpha) > 0$ chosen later

1. Set v_1 as root vertex and initialize m element set $M = \{v_2, ..., v_{m+1}\}$
2. Determine the random number **Ch** of children of v_1 in M according to

$$\lambda_k = \Pr[\mathbf{Ch} = k] = \binom{m}{k} p^k (1-p)^{m-k}$$

3. Select k points at random from M for the children of v_1 and then replace them in M with the first k unused vertices
4. Recursively repeat on the first child until branch dies, then repeat on the sibling of the node that went extinct (Essentially a depth first search!)
5. If tree dies out, pick unused vertex with smallest index and restart the process until no vertices remain unused

Directed Graph Process Cont'd

- The previous procedure stops when we have used up all δn surplus points (recall $m = (1 - \delta)n$)

Directed Graph Process Cont'd

- The previous procedure stops when we have used up all δn surplus points (recall $m = (1 - \delta)n$)
- Do we use up all δn points before cn generations?

Directed Graph Process Cont'd

- The previous procedure stops when we have used up all δn surplus points (recall $m = (1 - \delta)n$)
- Do we use up all δn points before cn generations?
- The result of the previous process is a collection of disjoint trees (We have generated a subgraph of $D_{n,p}$ so far)

Directed Graph Process Cont'd

- The previous procedure stops when we have used up all δn surplus points (recall $m = (1-\delta)n$)
- Do we use up all δn points before cn generations?
- The result of the previous process is a collection of disjoint trees (We have generated a subgraph of $D_{n,p}$ so far)
- To guarantee a $D_{n,p}$ we appropriately randomize the remaining directed edges (i.e. edges between children, from one tree to another, from child to ancestor, etc.)

Directed Graph Process Cont'd

- The previous procedure stops when we have used up all δn surplus points (recall $m = (1 - \delta)n$)
- Do we use up all δn points before cn generations?
- The result of the previous process is a collection of disjoint trees (We have generated a subgraph of $D_{n,p}$ so far)
- To guarantee a $D_{n,p}$ we appropriately randomize the remaining directed edges (i.e. edges between children, from one tree to another, from child to ancestor, etc.)

Note: BFS can only give us at most a $\log n$ path but *DFS* can potentially give us much longer paths

Galton-Watson Branching Process

- Let $\lambda = mp = (1 - \delta)\alpha$. We will choose δ small so that $\lambda > 1$

Galton-Watson Branching Process

- Let $\lambda = mp = (1-\delta)\alpha$. We will choose δ small so that $\lambda > 1$
- Consider branching process with branching distribution

$$\lambda_k = \Pr[\mathbf{Ch} = k] = \binom{m}{k} p^k (1-p)^{m-k} \sim \frac{(mp)^k}{k!} e^{-mp} = \frac{\lambda^k}{k!} e^{-\lambda}$$

Galton-Watson Branching Process

- Let $\lambda = mp = (1-\delta)\alpha$. We will choose δ small so that $\lambda > 1$
- Consider branching process with branching distribution

$$\lambda_k = \Pr[\mathbf{Ch} = k] = \binom{m}{k} p^k (1-p)^{m-k} \sim \frac{(mp)^k}{k!} e^{-mp} = \frac{\lambda^k}{k!} e^{-\lambda}$$

- Recall: at time $t = 0$ a single particle is born and gives birth to **Ch** many children and then dies. At each $t \geq 1$ each existing particle independently gives birth to **Ch** children and dies.

Galton-Watson Branching Process

- Let $\lambda = mp = (1-\delta)\alpha$. We will choose δ small so that $\lambda > 1$
- Consider branching process with branching distribution

$$\lambda_k = \Pr[\mathbf{Ch} = k] = \binom{m}{k} p^k (1-p)^{m-k} \sim \frac{(mp)^k}{k!} e^{-mp} = \frac{\lambda^k}{k!} e^{-\lambda}$$

- Recall: at time $t = 0$ a single particle is born and gives birth to **Ch** many children and then dies. At each $t \geq 1$ each existing particle independently gives birth to **Ch** children and dies.
- At each level assign a random order to the branches so the "leftmost infinite path" (**LIP**) is well defined

Galton-Watson Branching Process

- Let $\lambda = mp = (1-\delta)\alpha$. We will choose δ small so that $\lambda > 1$
- Consider branching process with branching distribution

$$\lambda_k = \Pr[\mathbf{Ch} = k] = \binom{m}{k} p^k (1-p)^{m-k} \sim \frac{(mp)^k}{k!} e^{-mp} = \frac{\lambda^k}{k!} e^{-\lambda}$$

- Recall: at time $t = 0$ a single particle is born and gives birth to **Ch** many children and then dies. At each $t \geq 1$ each existing particle independently gives birth to **Ch** children and dies.
- At each level assign a random order to the branches so the "leftmost infinite path" (**LIP**) is well defined
- Recall: Since $\lambda > 1$ there exists $Q = Q(\lambda, m)$ such that the probability of extinction is $Q < 1$

Let T denote the total population until extinction conditioned on process dying. Let $T_1, T_2, ...$ be a sequence of independent RVs distributed according to T and set $S_k = T_1 + .. + T_k$

Let T denote the total population until extinction conditioned on process dying. Let $T_1, T_2, \ldots$ be a sequence of independent RVs distributed according to T and set $S_k = T_1 + .. + T_k$

Claim 1

$\Pr[S_{\epsilon n} > \frac{\delta}{2} n]$ is exponentially small, where $\epsilon = \epsilon(\alpha, \delta)$

Let T denote the total population until extinction conditioned on process dying. Let $T_1, T_2, ...$ be a sequence of independent RVs distributed according to T and set $S_k = T_1 + .. + T_k$

Claim 1

$\Pr[S_{\epsilon n} > \frac{\delta}{2} n]$ is exponentially small, where $\epsilon = \epsilon(\alpha, \delta)$

$L_k = \#$ of points to the left of **LIP** (inclusive) at level k.

Claim 2

There exists $c = c(\alpha) > 0$ such that $\Pr[L_{cn} > \frac{\delta}{2} n]$ is exponentially small.

Results on Branching Processes

Let T denote the total population until extinction conditioned on process dying. Let $T_1, T_2, \ldots$ be a sequence of independent RVs distributed according to T and set $S_k = T_1 + .. + T_k$

Claim 1

$\Pr[S_{\epsilon n} > \frac{\delta}{2} n]$ is exponentially small, where $\epsilon = \epsilon(\alpha, \delta)$

$L_k = \#$ of points to the left of **LIP** (inclusive) at level k.

Claim 2

There exists $c = c(\alpha) > 0$ such that $\Pr[L_{cn} > \frac{\delta}{2} n]$ is exponentially small.

Claim 3

For all m, extinction probability $Q = Q(\lambda, m) < Q_0 = Q_0(\lambda) < 1$.

Proof of Main Result

- Claim 3 and independence $\implies$ first ϵn processes go extinct with probability at most $Q_0^{\epsilon n}$

Proof of Main Result

- Claim 3 and independence $\implies$ first ϵn processes go extinct with probability at most $Q_0^{\epsilon n}$
- Claim 1 and 2 $\implies$ $\Pr[S_{\epsilon n} > \frac{\delta}{2} n]$ and $\Pr[L_{cn} > \frac{\delta}{2} n]$ are also exponentially small

Proof of Main Result

- Claim 3 and independence $\implies$ first ϵn processes go extinct with probability at most $Q_0^{\epsilon n}$
- Claim 1 and 2 $\implies$ $\Pr[S_{\epsilon n} > \frac{\delta}{2}n]$ and $\Pr[L_{cn} > \frac{\delta}{2}n]$ are also exponentially small
- By union bound none of these events occur $\implies$ there is an index $j \leq \epsilon n$ such that $T_1, ..., T_{j-1}$ are finite and T_j in infinite

Proof of Main Result

- Claim 3 and independence $\implies$ first ϵn processes go extinct with probability at most $Q_0^{\epsilon n}$
- Claim 1 and 2 $\implies$ $\Pr[S_{\epsilon n} > \frac{\delta}{2}n]$ and $\Pr[L_{cn} > \frac{\delta}{2}n]$ are also exponentially small
- By union bound none of these events occur $\implies$ there is an index $j \leq \epsilon n$ such that $T_1, ..., T_{j-1}$ are finite and T_j in infinite
- At most $\frac{\delta}{2}n$ vertices used in our corresponding graph process so far

Proof of Main Result

- Claim 3 and independence $\implies$ first ϵn processes go extinct with probability at most $Q_0^{\epsilon n}$
- Claim 1 and 2 $\implies$ $\Pr[S_{\epsilon n} > \frac{\delta}{2}n]$ and $\Pr[L_{cn} > \frac{\delta}{2}n]$ are also exponentially small
- By union bound none of these events occur $\implies$ there is an index $j \leq \epsilon n$ such that $T_1, ..., T_{j-1}$ are finite and T_j in infinite
- At most $\frac{\delta}{2}n$ vertices used in our corresponding graph process so far
- At most $\frac{\delta}{2}n$ vertices used by taking the corresponding points to the left of **LIP** (inclusive) at level cn. This also gives us the cn directed path.

Proof of Main Result

- Claim 3 and independence $\implies$ first ϵn processes go extinct with probability at most $Q_0^{\epsilon n}$
- Claim 1 and 2 $\implies$ $\Pr[S_{\epsilon n} > \frac{\delta}{2}n]$ and $\Pr[L_{cn} > \frac{\delta}{2}n]$ are also exponentially small
- By union bound none of these events occur $\implies$ there is an index $j \leq \epsilon n$ such that $T_1, \ldots, T_{j-1}$ are finite and T_j in infinite
- At most $\frac{\delta}{2}n$ vertices used in our corresponding graph process so far
- At most $\frac{\delta}{2}n$ vertices used by taking the corresponding points to the left of **LIP** (inclusive) at level cn. This also gives us the cn directed path.
- We had a surplus of δn vertices in the original graph process so we are done!

Proof of Claim 3

Let $f(x) = \sum_{k \geq 0} \lambda_k x^k$. Since $\mathbb{E}[\mathbf{Ch}] = \lambda > 1$ it is well known that Q is the unique root of the equation $Q = f(Q)$ for $0 \leq Q < 1$.

Proof of Claim 3

Let $f(x) = \sum_{k \geq 0} \lambda_k x^k$. Since $\mathbb{E}[\mathbf{Ch}] = \lambda > 1$ it is well known that Q is the unique root of the equation $Q = f(Q)$ for $0 \leq Q < 1$.

Note that $f(x) = (1 - p + px)^m = (1 - \frac{\lambda(1-x)}{m})^m$ and is monotone increasing in m since for $C = \lambda(1-x)$ by AM-GM

$$1 \cdot \left(1 - \frac{C}{m}\right) \dots \left(1 - \frac{C}{m}\right) \leq \left(\frac{1 + m(1 - \frac{C}{m})}{m+1}\right)^{m+1} = \left(1 - \frac{C}{m+1}\right)^{m+1}.$$

Proof of Claim 3

Let $f(x) = \sum_{k\geq 0} \lambda_k x^k$. Since $\mathbb{E}[\mathbf{Ch}] = \lambda > 1$ it is well known that Q is the unique root of the equation $Q = f(Q)$ for $0 \leq Q < 1$.

Note that $f(x) = (1 - p + px)^m = (1 - \frac{\lambda(1-x)}{m})^m$ and is monotone increasing in m since for $C = \lambda(1-x)$ by AM-GM

$$1 \cdot \left(1 - \frac{C}{m}\right) \dots \left(1 - \frac{C}{m}\right) \leq \left(\frac{1 + m(1 - \frac{C}{m})}{m+1}\right)^{m+1} = \left(1 - \frac{C}{m+1}\right)^{m+1}.$$

Also, $f(x) \leq e^{-\lambda(1-x)}$ and thus $Q < Q_0 < 1$ for all m where Q_0 is the unique solution on $(0,1)$ to $Q_0 = e^{-\lambda(1-Q_0)}$. Equivalently, $Q_0 = \frac{x}{\lambda}$ where x is unique value in $(0,1)$ for which $xe^{-x} = \lambda e^{-\lambda}$.

(Sketchy) Proof of Claim 1

We want $\Pr[T_1 + ... + T_{\epsilon n} > \frac{\delta}{2} n]$ exponentially small

(Sketchy) Proof of Claim 1

We want $\Pr[T_1 + ... + T_{\epsilon n} > \frac{\delta}{2} n]$ exponentially small

$\Pr[S_{\epsilon n} > \frac{\delta}{2} n] = \Pr[\exp(tS_{\epsilon n}) > \exp(t\delta n/2)] \leq \frac{\mathbb{E}[\exp(tS_{\epsilon n})]}{\exp(t\delta n/2)}$ (Markov)

(Sketchy) Proof of Claim 1

We want $\Pr[T_1 + ... + T_{\epsilon n} > \frac{\delta}{2} n]$ exponentially small

$\Pr[S_{\epsilon n} > \frac{\delta}{2} n] = \Pr[\exp(tS_{\epsilon n}) > \exp(t\delta n/2)] \leq \frac{\mathbb{E}[\exp(tS_{\epsilon n})]}{\exp(t\delta n/2)}$ (Markov)

By independence of T_i, $\mathbb{E}[\exp(tS_{\epsilon n})] = \prod_{i=1}^{\epsilon n} \mathbb{E} \exp(tT) \leq K^{\epsilon n}$

(Sketchy) Proof of Claim 1

We want $\Pr[T_1 + ... + T_{\epsilon n} > \frac{\delta}{2} n]$ exponentially small

$\Pr[S_{\epsilon n} > \frac{\delta}{2} n] = \Pr[\exp(tS_{\epsilon n}) > \exp(t\delta n/2)] \leq \frac{\mathbb{E}[\exp(tS_{\epsilon n})]}{\exp(t\delta n/2)}$ (Markov)

By independence of T_i, $\mathbb{E}[\exp(tS_{\epsilon n})] = \prod_{i=1}^{\epsilon n} \mathbb{E} \exp(tT) \leq K^{\epsilon n}$

Pick ϵ small (in terms of α, δ) and pick $t = t_0$ to get exponentially small probability

(Sketchy) Proof of Claim 1

We want $\Pr[T_1 + ... + T_{\epsilon n} > \frac{\delta}{2} n]$ exponentially small

$\Pr[S_{\epsilon n} > \frac{\delta}{2} n] = \Pr[\exp(t S_{\epsilon n}) > \exp(t\delta n/2)] \leq \frac{\mathbb{E}[\exp(tS_{\epsilon n})]}{\exp(t\delta n/2)}$ (Markov)

By independence of T_i, $\mathbb{E}[\exp(tS_{\epsilon n})] = \prod_{i=1}^{\epsilon n} \mathbb{E} \exp(tT) \leq K^{\epsilon n}$

Pick ϵ small (in terms of α, δ) and pick $t = t_0$ to get exponentially small probability

Lastly show $\mathbb{E}[e^{t_0 T}] \leq K$ for $t_0 = t_0(\lambda) > 0$ and $K = K(\lambda) > 0$

(Sketchy) Proof of Claim 1 Cont'd

Let $\overline{T}$ be the total population and $\overline{t}_l = \Pr[\overline{T} = l]$, then we have $\overline{t}_0 = 0$ and $\overline{t}_\infty = 1 - Q$, and for $l \geq 1$

$$\overline{t}_l = \sum_{k=0} \lambda_k \sum_{l_1+\ldots+l_k=l-1: l_i \geq 1} \overline{t}_{l_1} \cdots \overline{t}_{l_k}$$

(Sketchy) Proof of Claim 1 Cont'd

Let $\overline{T}$ be the total population and $\overline{t}_l = \Pr[\overline{T} = l]$, then we have $\overline{t}_0 = 0$ and $\overline{t}_\infty = 1 - Q$, and for $l \geq 1$

$$\overline{t}_l = \sum_{k=0} \lambda_k \sum_{l_1+\ldots+l_k=l-1: l_i \geq 1} \overline{t}_{l_1} \cdots \overline{t}_{l_k}$$

Then $\overline{g}(x) = \sum_{l=1} \overline{t}_l x^l$ satisfies $\overline{g}(x) = xf(\overline{g}(x))$ so it is defined uniquely for $x < 1$ (By LIFT) and so $\overline{g}(x) < Q$

(Sketchy) Proof of Claim 1 Cont'd

Let $\overline{T}$ be the total population and $\overline{t}_l = \Pr[\overline{T} = l]$, then we have $\overline{t}_0 = 0$ and $\overline{t}_\infty = 1 - Q$, and for $l \geq 1$

$$\overline{t}_l = \sum_{k=0} \lambda_k \sum_{l_1 + \ldots + l_k = l-1 : l_i \geq 1} \overline{t}_{l_1} \cdots \overline{t}_{l_k}$$

Then $\overline{g}(x) = \sum_{l=1} \overline{t}_l x^l$ satisfies $\overline{g}(x) = xf(\overline{g}(x))$ so it is defined uniquely for $x < 1$ (By LIFT) and so $\overline{g}(x) < Q$

By a continuity argument $\overline{g}(x)$ can be defined up to $y_0 = \frac{x_0}{f(x_0)}$ where $x_0 = \frac{1-p}{(m-1)p}$ is the maximum of $\frac{x}{f(x)}$ under the assumption $\overline{g} < x_0$

(Sketchy) Proof of Claim 1 Cont'd

T is total population conditioned of the process going extinct. Thus

$$t_l = \Pr[T = l] = \Pr[\overline{T} = l | \overline{T} < \infty] = \frac{\overline{t}_l}{Q}$$

$\mathbb{E}\, T = g'(1) = \frac{1}{1-f'(Q)} \sim \frac{1}{1-\lambda Q}$ where $g(x) = \sum_{l=0} t_l x^l = \overline{g}(x)/Q$

(Sketchy) Proof of Claim 1 Cont'd

T is total population conditioned on the process going extinct. Thus

$$t_l = \Pr[T = l] = \Pr[\overline{T} = l | \overline{T} < \infty] = \frac{\overline{t}_l}{Q}$$

$\mathbb{E}T = g'(1) = \frac{1}{1 - f'(Q)} \sim \frac{1}{1 - \lambda Q}$ where $g(x) = \sum_{l=0} t_l x^l = \overline{g}(x)/Q$

(Sketchy) Proof of Claim 1 Cont'd

T is total population conditioned on the process going extinct. Thus

$$t_l = \Pr[T = l] = \Pr[\overline{T} = l | \overline{T} < \infty] = \frac{\overline{t}_l}{Q}$$

$\mathbb{E}T = g'(1) = \frac{1}{1 - f'(Q)} \sim \frac{1}{1 - \lambda Q}$ where $g(x) = \sum_{l=0} t_l x^l = \overline{g}(x)/Q$

Now $\mathbb{E}e^{t_0 T} = g(e^{t_0}) \leq g(y_0) = \frac{x_0}{Q} < \frac{1}{Q}$.

(Sketchy) Proof of Claim 1 Cont'd

T is total population conditioned on the process going extinct. Thus

$$t_l = \Pr[T = l] = \Pr[\overline{T} = l | \overline{T} < \infty] = \frac{\overline{t}_l}{Q}$$

$\mathbb{E}T = g'(1) = \frac{1}{1-f'(Q)} \sim \frac{1}{1-\lambda Q}$ where $g(x) = \sum_{l=0} t_l x^l = \overline{g}(x)/Q$

Now $\mathbb{E}e^{t_0 T} = g(e^{t_0}) \leq g(y_0) = \frac{x_0}{Q} < \frac{1}{Q}$.

We are done as Q is an increasing function in m so $\frac{1}{Q}$ is uniformly bounded in m as needed

Conclusion

The proof of claim 2 can be done by a similar approach to the proof of claim 1. That is, we define an appropriate RV, bound the MGF and use this to derive the concentration bound.

Thank you!