

# New Bounds for Hypergeometric Creative Telescoping

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# Outline

- ▶ Modified Abramov–Petkovšek reduction
- ▶ Reduction-based creative telescoping
- ▶ Upper and lower order bounds for minimal telescopers

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**Notation.** For  $f(\mathbf{y})$ ,

$$\sigma_{\mathbf{y}}(f) := f(\mathbf{y} + \mathbf{1}) \quad \text{and} \quad \Delta_{\mathbf{y}}(f) := f(\mathbf{y} + \mathbf{1}) - f(\mathbf{y}).$$

# Hypergeometric summability

**Definition.** A nonzero term  $T(y)$  is **hypergeometric** over  $\mathbb{C}(y)$  if

$$\frac{\sigma_y(T)}{T} \in \mathbb{C}(y).$$

**Example.**  $T \in \mathbb{C}(y) \setminus \{0\}$ ,  $y!$ ,  $\binom{y}{4}$ ,  $\dots$

**Definition.** A hypergeom. term  $T(y)$  is **summable** if

$$T(y) = \Delta_y(\text{hypergeom.}).$$

**Example.**  $y \cdot y! = (y + 1)! - y!$  is summable; but  $y!$  is not.

# Multiplicative decomposition

**Definition.**  $u/v \in \mathbb{C}(y)$  is **shift-reduced** if

$$\forall \ell \in \mathbb{Z}, \quad \gcd\left(v, \sigma_y^\ell(u)\right) = 1.$$

For a hypergeom. term  $T(y)$ ,  $\exists K, S \in \mathbb{C}(y)$  with  $K$  shift-reduced s.t.

$$T = SH, \quad \text{where } \frac{\sigma_y(H)}{H} = K.$$

Call

- ▶  $K$ , a **kernel** of  $T$
- ▶  $S$ , the corresponding **shell** of  $T$

## Modified A.-P. reduction (CHKL2015)

**Theorem.** Let  $T(y)$  be hypergeom. with a multi. decomp.

$$T = SH \quad \text{and} \quad \frac{u}{v} := \frac{\sigma_y(H)}{H}.$$

Then  $\exists \alpha, b, q \in \mathbb{C}[y]$  with  $\deg_y(\alpha) < \deg_y(b)$  s.t.

$$T = \underbrace{\Delta_y(\dots)}_{\text{summable part}} + \underbrace{\left(\frac{\alpha}{b} + \frac{q}{v}\right)H}_{\text{non-summable part}}$$

Moreover,

$$T \text{ is summable} \iff \alpha = q = 0.$$

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$$T = \underbrace{\Delta_y(\dots)}_{\text{summable part}} + \underbrace{\left(\frac{a}{b} + \frac{q}{v}\right)H}_{\text{remainder}}$$

Moreover,

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**Note.**  $b, q$  satisfy shift-free, strongly-prime and other conditions.

# Term bound for $q$

Notation. (Iverson bracket)

$$\llbracket \dots \rrbracket = \begin{cases} 1 & \text{if } \dots \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

Proposition.

$$\# \text{ terms of } q \leq \max(\deg_y(\mathbf{u}), \deg_y(\mathbf{v})) - \llbracket \deg_y(\mathbf{u} - \mathbf{v}) \leq \deg_y(\mathbf{u}) - 1 \rrbracket.$$



# Bivariate hypergeometric terms

**Definition.** A nonzero term  $T(x, y)$  is **hypergeometric** over  $\mathbb{C}(x, y)$  if

$$\frac{\sigma_x(T)}{T}, \frac{\sigma_y(T)}{T} \in \mathbb{C}(x, y).$$

**Creative-telescoping problem.** Given  $T(x, y)$  hypergeom. , find a nonzero operator  $L \in \mathbb{C}(x)\langle\sigma_x\rangle$  s.t.

$$L(T) = \Delta_y(G) \text{ for some hypergeom. term } G(x, y)$$

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telescoper                      certificate

# Existence of telescopers

**Definition.**  $p \in \mathbb{C}[x, y]$  is **integer-linear** if

$$p = \prod_i (\alpha_i x + \beta_i y + \gamma_i)$$

where  $\alpha_i, \beta_i \in \mathbb{Z}$  and  $\gamma_i \in \mathbb{C}$ .

Existence criterion (Wilf&Zeilberger1992, Abramov2003).

Assume applying modified A.-P. reduction yields

$$T = \Delta_y(\dots) + \left(\frac{a}{b} + \frac{q}{v}\right) H.$$

Then

$T$  has a telescoper  $\Leftrightarrow b$  is integer-linear.

# Reduction-based telescoping (CHKL2015)

**Goal.** Given  $\rho \in \mathbb{N}$ , find a telescoper for  $T$  w.r.t.  $y$  with order  $\rho$ .

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$$\begin{aligned}T &= \Delta_y(\dots) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \\ \sigma_x(T) &= \Delta_y(\dots) + \left(\frac{a_1}{b_1} + \frac{q_1}{v}\right) H \\ &\vdots \\ \sigma_x^\rho(T) &= \Delta_y(\dots) + \left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) H\end{aligned}$$



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**Idea.** Set  $T = SH$ , a multi. decomp. and  $u/v = \sigma_y(H)/H$

$$c_0(x) T = \Delta_y(\dots) + c_0(x) \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H$$

$$c_1(x) \sigma_x(T) = \Delta_y(\dots) + c_1(x) \left( \frac{a_1}{b_1} + \frac{q_1}{v} \right) H$$

$\vdots$

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---

$$\left( \sum_{i=0}^{\rho} c_i(x) \sigma_x^i \right) (T) = \Delta_y(\dots) + \left( \sum_{j=0}^{\rho} c_j(x) \left( \frac{a_j}{b_j} + \frac{q_j}{v} \right) \right) H$$

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$$\begin{aligned} & c_0(x) T = \Delta_y(\dots) + c_0(x) \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H \\ + \left\{ \begin{aligned} & c_1(x) \sigma_x(T) = \Delta_y(\dots) + c_1(x) \left( \frac{a_1}{b_1} + \frac{q_1}{v} \right) H \\ & \vdots \\ & c_\rho(x) \sigma_x^\rho(T) = \Delta_y(\dots) + c_\rho(x) \left( \frac{a_\rho}{b_\rho} + \frac{q_\rho}{v} \right) H \stackrel{?}{=} 0 \end{aligned} \right. \\ \text{telescoper?} \end{aligned}$$

---

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$$\sum_{i=0}^{\rho} \mathbf{c}_i(\mathbf{x}) \sigma_x^i \text{ is a telescoper for } T$$

$\Updownarrow$  MAP reduction

$$\sum_{j=0}^{\rho} \mathbf{c}_j(\mathbf{x}) \left( \frac{a_j}{b_j} + \frac{q_j}{v} \right) = 0$$

$\Updownarrow$   $\gcd(b_j, v) = 1$

$$\begin{cases} \mathbf{c}_0(\mathbf{x}) \frac{a_0(\mathbf{x}, y)}{b_0(\mathbf{x}, y)} + \cdots + \mathbf{c}_\rho(\mathbf{x}) \frac{a_\rho(\mathbf{x}, y)}{b_\rho(\mathbf{x}, y)} = 0 \\ \mathbf{c}_0(\mathbf{x}) q_0(\mathbf{x}, y) + \cdots + \mathbf{c}_\rho(\mathbf{x}) q_\rho(\mathbf{x}, y) = 0 \end{cases}$$

## Example

Consider

$$T = \frac{1}{x + 2y} \cdot y!$$

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- ▶ A multi. decomp.  $T = SH$  where

$$S = \frac{1}{x + 2y}$$

and

$$H = y! \quad \text{with} \quad \frac{\sigma_y(H)}{H} = y + 1.$$

- ▶  $u := y + 1$  and  $v := 1$ .

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$$c_0(x) \cdot \left( \frac{2}{x + 2y} + \frac{0}{v} \right) \\ + c_1(x) \cdot \left( \frac{2}{x + 2y + 1} + \frac{0}{v} \right)$$

$$= 0$$

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No solution in  $\mathbb{C}(x)$ !

$$= 0$$

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$$= 0$$

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$$\begin{aligned}T &= \frac{1}{x+2y} \cdot y! \\ &- 2 \cdot \left( \frac{2}{x+2y} + \frac{0}{v} \right) \\ &+ 2 \cdot \left( \frac{2}{x+2y+1} + \frac{0}{v} \right) \\ &- x \cdot \left( -\frac{4/x}{x+2y} + \frac{2/x}{v} \right) \\ &+ (x+1) \cdot \left( -\frac{4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v} \right) \\ &= 0\end{aligned}$$

## Example

Consider

$$T = \frac{1}{x + 2y} \cdot y!$$

Therefore,

- ▶ a minimal telescoper for  $T$  w.r.t.  $y$  is

$$L = (x + 1) \cdot \sigma_x^3 - x \cdot \sigma_x^2 + 2 \cdot \sigma_x - 2$$

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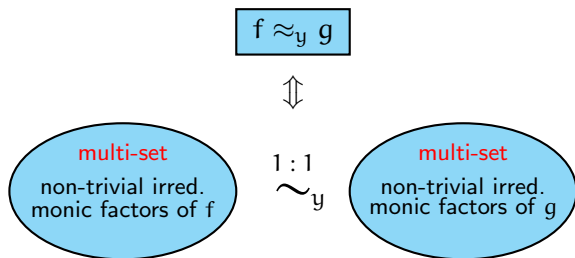
$$L = (x + 1) \cdot \sigma_x^3 - x \cdot \sigma_x^2 + 2 \cdot \sigma_x - 2$$

- ▶ the corresponding certificate is

$$\begin{aligned} G &= (x + 1) \cdot g_3 - x \cdot g_2 + 2 \cdot g_1 - 2 \cdot g_0 \\ &= \frac{2y!}{(x + 2y)(x + 2y + 1)} \end{aligned}$$

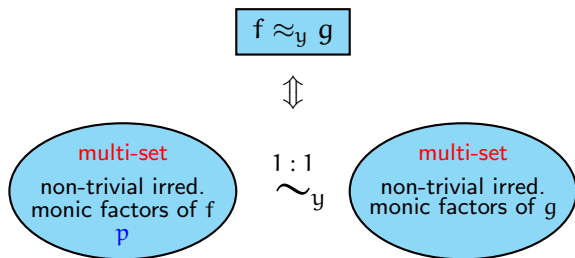
## Relationship among remainders

**Definition (shift-related).** Let  $f, g \in \mathcal{C}(x)[y]$  be shift-free w.r.t.  $y$ .



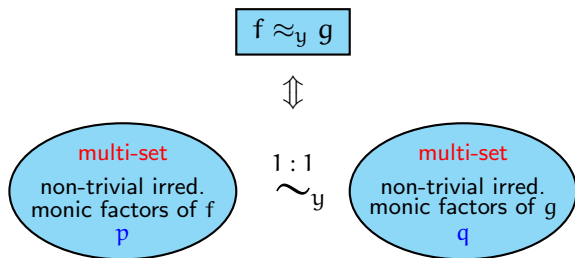
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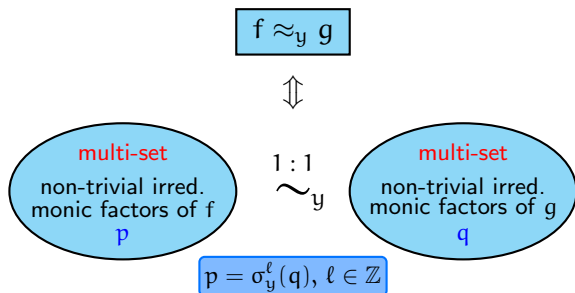
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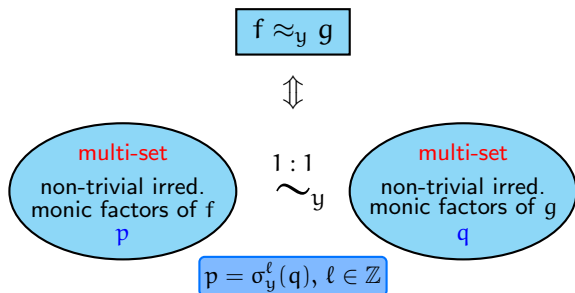
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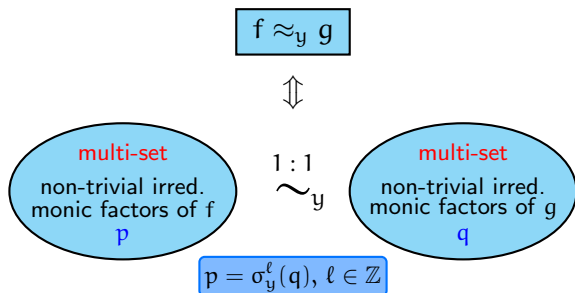
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$$\sigma_x^i(T) = \Delta_y(\dots) + \left( \frac{a_i}{b_i} + \frac{q_i}{v} \right) H, \quad \forall i \in \mathbb{N}.$$

Then  $b_i \approx_y \sigma_x^i(b_0)$ ,  $\forall i \in \mathbb{N}$ .

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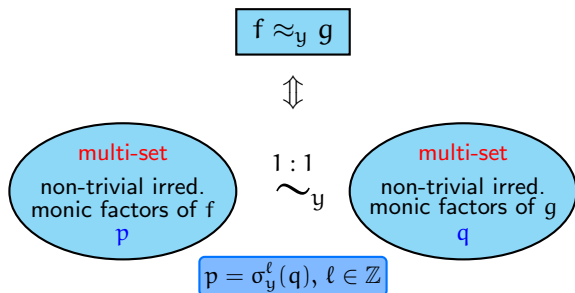
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**Proposition.**  $T = SH$  a multi. decomp. and  $u/v = \sigma_y(H)/H$ . Let

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Then  $b_i \approx_y \sigma_x^i(b_0)$ ,  $\forall i \in \mathbb{N}$ . **integer-linear**

**$b_0$  integer-linear**

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

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- ▶  $\exists \rho \in \mathbb{N}^*$  s.t.

$$c_0(x) \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) + \cdots + c_\rho(x) \left( \frac{a_\rho}{b_\rho} + \frac{q_\rho}{v} \right) = 0$$

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$$\begin{cases} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$



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$$\begin{cases} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

- ▶ **Property.**  $b_i = x + 2y$  or  $x + 2y + 1$ .

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

$$\begin{cases} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

$$\#vars = \rho + 1 \quad \left\{ \begin{array}{l} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{array} \right.$$

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

$$\begin{array}{l} \#vars = \rho + 1 \\ \uparrow \text{ Prop. } b_i = x + 2y \text{ or } x + 2y + 1 \\ \left\{ \begin{array}{l} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{array} \right. \end{array}$$

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

common denom.  $B = (x + 2y)(x + 2y + 1)$

↑↑ Prop.  $b_i = x + 2y$  or  $x + 2y + 1$

$$\#vars = \rho + 1 \quad \left\{ \begin{array}{l} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{array} \right.$$

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

$$\Uparrow \deg_y(a_i) < \deg_y(b_i)$$

common denom.  $B = (x + 2y)(x + 2y + 1)$

$$\Uparrow \text{Prop. } b_i = x + 2y \text{ or } x + 2y + 1$$

#vars =  $\rho + 1$

$$\begin{cases} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

#eqns over  $\mathbb{C}(x) \leq 2$

$\Uparrow \deg_y(a_i) < \deg_y(b_i)$

common denom.  $B = (x + 2y)(x + 2y + 1)$

$\Uparrow$  Prop.  $b_i = x + 2y$  or  $x + 2y + 1$

$$\#vars = \rho + 1 \quad \left\{ \begin{array}{l} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{array} \right.$$

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

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$$\begin{cases} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

$\Downarrow$  Prop. # terms of  $q \leq \max(\deg_y(u), \deg_y(v)) - \llbracket \deg_y(u - v) \leq \deg_y(u) - 1 \rrbracket$



# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

#eqns over  $\mathbb{C}(x) \leq 2$

$\Uparrow \deg_y(a_i) < \deg_y(b_i)$

common denom.  $B = (x + 2y)(x + 2y + 1)$

$\Uparrow$  Prop.  $b_i = x + 2y$  or  $x + 2y + 1$

$$\#vars = \rho + 1 \quad \left\{ \begin{array}{l} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{array} \right.$$

$\Downarrow$  Prop. # terms of  $q \leq \max(\deg_y(u), \deg_y(v)) - \lfloor \deg_y(u - v) \leq \deg_y(u) - 1 \rfloor$

#eqns over  $\mathbb{C}(x) \leq 1$

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

#eqns over  $\mathbb{C}(x) \leq 2$

$\Uparrow \deg_y(a_i) < \deg_y(b_i)$

common denom.  $B = (x + 2y)(x + 2y + 1)$

$\Uparrow$  Prop.  $b_i = x + 2y$  or  $x + 2y + 1$

$$\begin{array}{l} \#vars = \rho + 1 \\ \#eqns \text{ over } \mathbb{C}(x) \leq 3 \end{array} \left\{ \begin{array}{l} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{array} \right.$$

$\Downarrow$  Prop. # terms of  $q \leq \max(\deg_y(u), \deg_y(v)) - \lfloor \deg_y(u - v) \leq \deg_y(u) - 1 \rfloor$

#eqns over  $\mathbb{C}(x) \leq 1$

# Upper bound

Example (cont.).  $T = y!/(x + 2y)$ .

#eqns over  $\mathbb{C}(x) \leq 2$

$\Uparrow \deg_y(a_i) < \deg_y(b_i)$

common denom.  $B = (x + 2y)(x + 2y + 1)$

$\Uparrow$  Prop.  $b_i = x + 2y$  or  $x + 2y + 1$

$$\begin{array}{l} \#vars = \rho + 1 \\ \#eqns \text{ over } \mathbb{C}(x) \leq 3 \end{array} \left\{ \begin{array}{l} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{array} \right.$$

$\Downarrow$  Prop. # terms of  $q \leq \max(\deg_y(u), \deg_y(v)) - \lfloor \deg_y(u - v) \rfloor$

#eqns over  $\mathbb{C}(x) \leq 1$

Conclusion. Upper bound is 3.

## New upper bound

**Theorem.** Assume applying modified A.-P. reduction yields

$$T = \Delta_y(\dots) + \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H,$$

where  $b_0 = c \prod_{i=1}^m \prod_{k=0}^{d_i} (\alpha_i x + \beta_i y + \gamma_i + k)^{e_{ik}}$ , and for each  $i \neq j$ , either

$$\alpha_i \neq \alpha_j \quad \text{or} \quad \beta_i \neq \beta_j \quad \text{or} \quad \gamma_i - \gamma_j \notin \mathbb{Z}.$$

Then the order of a minimal telescoper for  $T$  w.r.t.  $y$  is **no more than**

$$B_{\text{New}} := \max\{\deg_y(\mathbf{u}), \deg_y(\mathbf{v})\} - \llbracket \deg_y(\mathbf{u} - \mathbf{v}) \leq \deg_y(\mathbf{u}) - 1 \rrbracket \\ + \sum_{i=1}^m \beta_i \cdot \max_{0 \leq k \leq d_i} \{e_{ik}\}$$

## Apagodu-Zeilberger upper bound (2005)

**Definition.** A hypergeom. term  $T$  is said to be **proper** if it is of the form

$$T = p(x, y) \prod_{i=1}^m \frac{(\alpha_i x + \alpha'_i y + \alpha''_i)! (\beta_i x - \beta'_i y + \beta''_i)!}{(\mu_i x + \mu'_i y + \mu''_i)! (\nu_i x - \nu'_i y + \nu''_i)!} z^y$$

**Theorem.** Assume  $T$  is **generic** proper hypergeom. . Then the order of a minimal telescoper for  $T$  w.r.t.  $y$  is **no more than**

$$B_{AZ} = \max \left\{ \sum_{i=1}^m (\alpha'_i + \nu'_i), \sum_{i=1}^m (\beta'_i + \mu'_i) \right\}.$$

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$$\begin{array}{l} \exists 1 \leq i, j \leq m \text{ s.t.} \\ \{\alpha_i = \mu_j \quad \& \quad \alpha'_i = \mu'_j \quad \& \quad \alpha''_i - \mu''_j \in \mathbb{N}\} \\ \{\beta_i = \nu_j \quad \& \quad \beta'_i = \nu'_j \quad \& \quad \beta''_i - \nu''_j \in \mathbb{N}\}. \end{array}$$

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$$B_{AZ} = \max \left\{ \sum_{i=1}^m (\alpha'_i + \nu'_i), \sum_{i=1}^m (\beta'_i + \mu'_i) \right\}.$$

## New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	$B_{\text{New}}$	$B_{\text{AZ}}$	$\leq B_{\text{New}}$
non-proper			
example			

## New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	$B_{\text{New}}$    $B_{\text{AZ}} - \llbracket \deg_y(\mathbf{u} - \mathbf{v}) \leq \deg_y(\mathbf{u}) - 1 \rrbracket$	$B_{\text{AZ}}$	$\leq B_{\text{New}}$
non-proper			
example			



# New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	$B_{\text{New}}$    $B_{\text{AZ}} - \llbracket \deg_y(\mathbf{u} - \mathbf{v}) \leq \deg_y(\mathbf{u}) - 1 \rrbracket$	$B_{\text{AZ}}$	$\leq B_{\text{New}}$
non-proper	$B_{\text{New}}$		
example			

## New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	$B_{\text{New}}$    $B_{\text{AZ}} - \llbracket \deg_y(\mathbf{u} - \mathbf{v}) \leq \deg_y(\mathbf{u}) - 1 \rrbracket$	$B_{\text{AZ}}$	$\leq B_{\text{New}}$
non-proper	$B_{\text{New}}$	?	
example			

## New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	$B_{\text{New}}$ $\parallel$ $B_{\text{AZ}} - \llbracket \deg_y(u - v) \leq \deg_y(u) - 1 \rrbracket$	$B_{\text{AZ}}$	$\leq B_{\text{New}}$
non-proper	$B_{\text{New}}$	?	$\leq B_{\text{New}}$
$T_1$	9	10	9

Example.

$$T_1 = \frac{(x + 3y)!(x - 3y)!}{(5x + 3y)(3x - y)(4x - 3y)!(5x + 3y)!}$$

## New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	$B_{\text{New}}$ $\parallel$ $B_{\text{AZ}} - \llbracket \deg_y(u - v) \leq \deg_y(u) - 1 \rrbracket$	$B_{\text{AZ}}$	$\leq B_{\text{New}}$
non-proper	$B_{\text{New}}$	?	$\leq B_{\text{New}}$
$T_2$	$\beta$	$\alpha + \beta$	$\beta$

Example.

$$T_2 = \frac{\alpha^2 y^2 + \alpha^2 y - \alpha \beta y + 2\alpha x y + x^2}{(x + \alpha y + \alpha)(x + \alpha y)(x + \beta y)}, \quad \alpha \neq \beta \text{ in } \mathbb{N} \setminus \{0\}.$$

# New upper bound v.s. Apagodu-Zeilberger's

	New	AZ	Order
proper	$B_{\text{New}}$ $\parallel$ $B_{\text{AZ}} - \llbracket \deg_y(u - v) \leq \deg_y(u) - 1 \rrbracket$	$B_{\text{AZ}}$	$\leq B_{\text{New}}$
non-proper	$B_{\text{New}}$	?	$\leq B_{\text{New}}$
$T_3$	3	?	3

Example.

$$T_3 = \frac{x^4 + x^3y + 2x^2y^2 + 2x^2y + xy^2 + 2y^3 + x^2 + y^2 - x - y}{(x^2 + y + 1)(x^2 + y)(x + 2y)} y!$$

## Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

# Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

▶ Modified A.-P. reduction:

$$T = \Delta(g_0) + \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H \quad \text{with} \quad b_0 = x + 2y.$$

# Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

- ▶ Modified A.-P. reduction:

$$T = \Delta(g_0) + \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H \quad \text{with} \quad b_0 = x + 2y.$$

- ▶  $\exists \rho \in \mathbb{N}^*$  s.t.

$$c_0(x) \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) + \cdots + c_\rho(x) \left( \frac{a_\rho}{b_\rho} + \frac{q_\rho}{v} \right) = 0$$



# Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

- ▶ Modified A.-P. reduction:

$$T = \Delta(g_0) + \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H \quad \text{with} \quad b_0 = x + 2y.$$

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$$\begin{cases} c_0(x) \frac{a_0}{b_0} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

# Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

- ▶ Modified A.-P. reduction:

$$T = \Delta(g_0) + \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H \quad \text{with} \quad b_0 = x + 2y.$$

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- ▶ **Property.**  $b_i = x + 2y$  or  $x + 2y + 1$ .

## Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

$$c_0(x) \frac{a_0}{b_0} + \cdots + c_i(x) \frac{a_i}{b_i} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0$$

# Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

$$c_0(x) \frac{a_0}{b_0} + \cdots + c_i(x) \frac{a_i}{b_i} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0$$

↑ PFD &  $b_0 = x + 2y$  ↑

⇓

$\exists i, \text{ s.t. } b_i = b_0$

# Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

$$c_0(x) \frac{a_0}{b_0} + \cdots + c_i(x) \frac{a_i}{b_i} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0$$

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⇓

$\exists i$ , s.t.  $b_i = b_0$

⇓

Prop.  $b_i = x + 2y$  or  $x + 2y + 1$

minimal  $i = 2$

# Lower bound

Example (cont.).  $T = y!/(x + 2y)$ .

$$c_0(x) \frac{a_0}{b_0} + \cdots + c_i(x) \frac{a_i}{b_i} + \cdots + c_\rho(x) \frac{a_\rho}{b_\rho} = 0$$

↑ PFD &  $b_0 = x + 2y$  ↑

⇓

$\exists i$ , s.t.  $b_i = b_0$

⇓

Prop.  $b_i = x + 2y$  or  $x + 2y + 1$

minimal  $i = 2$

**Conclusion.** Lower bound is 2.

## New lower bound

**Theorem.** Assume applying modified A.-P. reduction yields

$$T = \Delta_y(\dots) + \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H,$$

with  $b_0$  integer-linear. Then the order of a telescoper for  $T$  w.r.t.  $y$  is **at least**

$$\max_{\substack{p \mid b_0 \text{ irred.} \\ \deg_y(p) \geq 1}} \min_{h \in \mathbb{Z}} \left\{ \rho \in \mathbb{N} \setminus \{0\} : \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \right\}$$

## Abramov-Le lower bound (2005)

**Theorem.** Assume applying modified A.-P. reduction yields

$$T = \Delta_y(\dots) + \left( \frac{a_0}{b_0} + \frac{q_0}{v} \right) H = \Delta_y(\dots) + \frac{a'_0}{b_0} H',$$

with  $b_0$  integer-linear,  $a'_0 = a_0 v + b_0 q_0$  and  $H' = H/v$ . Let

$$\frac{c'}{d'} := \frac{\sigma_x(H')}{H'}.$$

Then the order of a telescoper for  $T$  w.r.t.  $y$  is at least

$$\max_{\substack{p \mid b_0 \text{ irred.} \\ \deg_y(p) \geq 1}} \min_{h \in \mathbb{Z}} \left\{ \rho \in \mathbb{N} \setminus \{0\} : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$$



# New lower bound v.s. Abramov-Le's

	New	AL	Order
hypergeom.	$\max_p \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^p(b_0)\}$	$\max_p \min_h$ $\left\{ \rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^p(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{p-1}(d') \end{array} \right\}$	$\geq \dots$
example			

# New lower bound v.s. Abramov-Le's

	New	AL	Order
hypergeom.	$\max_p \min_h \{ \rho : \sigma_y^h(p) \mid \sigma_x^\rho(\mathbf{b}_0) \}$	$\max_p \min_h \left\{ \rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(\mathbf{b}_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$	$\geq \dots$
$T_1$	7	3	17

Example.

$$T_1 = \frac{1}{(x + 3y + 1)(5x - 7y)(5x - 7y + 14)!}.$$

# New lower bound v.s. Abramov-Le's

	New	AL	Order
hypergeom.	$\max_p \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^p(b_0)\}$	$\max_p \min_h$ $\left\{ \rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^p(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{p-1}(d') \end{array} \right\}$	$\geq \dots$
$T_2$	12	3	29

Example.

$$T_2 = \frac{1}{(x + 5y + 1)(5x - 12y)(5x - 12y + 24)!}$$

# New lower bound v.s. Abramov-Le's

	New	AL	Order
hypergeom.	$\max_p \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(\mathbf{b}_0)\}$	$\max_p \min_h$ $\left\{ \rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(\mathbf{b}_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$	$\geq \dots$
$T_3$	$\alpha$	2	$\alpha$

Example.

$$T_3 = \frac{1}{(x - \alpha y - \alpha)(x - \alpha y - 2)!}, \quad \alpha \geq 2 \text{ in } \mathbb{N}.$$

# Summary

## Result.

- ▶ Order bounds for minimal telescopers

## Future work.

- ▶ Creative telescoping for  $q$ -hypergeometric terms