# New Bounds for Hypergeometric Creative Telescoping

Hui Huang

KLMM, AMSS,
Chinese Academy of Sciences

Institute for Algebra,
Johannes Kepler University

# Outline

- Modified Abramov–Petkovšek reduction
- Reduction-based creative telescoping
- Upper and lower order bounds for minimal telescopers

# Outline

- Modified Abramov–Petkovšek reduction
- Reduction-based creative telescoping
- Upper and lower order bounds for minimal telescopers

Notation. For $f(y)$,

$$\sigma_y(f) := f(y+1) \quad \text{and} \quad \Delta_y(f) := f(y+1) - f(y).$$

# Hypergeometric summability

Definition. A nonzero term $T(y)$ is hypergeometric over $\mathbb{C}(y)$ if

$$\frac{\sigma_y(T)}{T} \in \mathbb{C}(y).$$

Example. $T \in \mathbb{C}(y) \setminus \{0\}$, $y!$, $\binom{y}{4}$, ...

Definition. A hypergeom. term $T(y)$ is summable if

$$T(y) = \Delta_y(\text{hypergeom.}).$$

Example. $y \cdot y! = (y+1)! - y!$ is summable; but $y!$ is not.

# Multiplicative decomposition

Definition. $u/v \in \mathbb{C}(y)$ is shift-reduced if

$$\forall \ell \in \mathbb{Z}, \quad \gcd\left(v, \sigma_y^{\ell}(u)\right) = 1.$$

For a hypergeom. term $T(y)$, $\exists\ K, S \in \mathbb{C}(y)$ with $K$ shift-reduced s.t.

$$T = SH, \quad \text{where } \frac{\sigma_y(H)}{H} = K.$$

Call

- $K$, a kernel of $T$
- $S$, the corresponding shell of $T$

# Modified A.-P. reduction (CHKL2015)

Theorem. Let $T(y)$ be hypergeom. with a multi. decomp.

$$T = SH \quad \text{and} \quad \frac{u}{v} := \frac{\sigma_y(H)}{H}.$$

Then $\exists\ a, b, q \in \mathbb{C}[y]$ with $\deg_y(a) < \deg_y(b)$ s.t.

$$T = \underbrace{\Delta_y\left(\cdots\right)}_{\text{summable part}} + \underbrace{\left(\frac{a}{b} + \frac{q}{v}\right) H}_{\text{non-summable part}}$$

Moreover,

$$T \text{ is summable } \iff a = q = 0.$$

# Modified A.-P. reduction (CHKL2015)

Theorem. Let $T(y)$ be hypergeom. with a multi. decomp.

$$T = \mathcal{S}H \quad \text{and} \quad \frac{u}{v} := \frac{\sigma_y(H)}{H}.$$

Then $\exists\ a, b, q \in \mathbb{C}[y]$ with $\deg_y(a) < \deg_y(b)$ s.t.

$$T = \underbrace{\Delta_y\Big(\cdots\Big)}_{\text{summable part}} + \underbrace{\left(\frac{a}{b} + \frac{q}{v}\right)H}_{\text{remainder}}$$

Moreover,

$$T \text{ is summable} \iff a = q = 0.$$

Note. $b, q$ satisfy shift-free, strongly-prime and other conditions.

# Term bound for q

Notation. (Iverson bracket)

$$[\![\cdots]\!] = \begin{cases} 1 & \text{if } \cdots \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

Proposition.

$$\#\text{ terms of } q \quad \leq \quad \begin{array}{l} \max\left(\deg_y(u), \deg_y(v)\right) \\ -[\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]. \end{array}$$

# Bivariate hypergeometric terms

**Definition.** A nonzero term $T(x, y)$ is hypergeometric over $\mathbb{C}(x, y)$ if

$$\frac{\sigma_x(T)}{T}, \ \frac{\sigma_y(T)}{T} \ \in \mathbb{C}(x, y).$$

**Creative-telescoping problem.** Given $T(x, y)$ hypergeom. , find a nonzero operator $L \in C(x)\langle\sigma_x\rangle$ s.t.

$$L(T) = \Delta_y(G) \text{ for some hypergeom. term } G(x, y)$$

# Bivariate hypergeometric terms

Definition. A nonzero term $T(x, y)$ is hypergeometric over $\mathbb{C}(x, y)$ if

$$\frac{\sigma_x(T)}{T}, \frac{\sigma_y(T)}{T} \in \mathbb{C}(x, y).$$

Creative-telescoping problem. Given $T(x, y)$ hypergeom. , find a nonzero operator $L \in C(x)\langle\sigma_x\rangle$ s.t.

$$\boxed{L}(T) = \Delta_y(G) \text{ for some hypergeom. term } G(x, y)$$

telescoper

# Bivariate hypergeometric terms

**Definition.** A nonzero term $T(x, y)$ is hypergeometric over $\mathbb{C}(x, y)$ if

$$\frac{\sigma_x(T)}{T}, \frac{\sigma_y(T)}{T} \in \mathbb{C}(x, y).$$

**Creative-telescoping problem.** Given $T(x, y)$ hypergeom. , find a nonzero operator $L \in C(x)\langle\sigma_x\rangle$ s.t.

$$\underbrace{\boxed{L}}_{\text{telescoper}}(T) = \Delta_y(\underbrace{\boxed{G}}_{\text{certificate}}) \text{ for some hypergeom. term } G(x, y)$$

# Existence of telescopers

Definition. $p \in \mathbb{C}[x, y]$ is integer-linear if

$$p = \prod_i (\alpha_i x + \beta_i y + \gamma_i)$$

where $\alpha_i, \beta_i \in \mathbb{Z}$ and $\gamma_i \in \mathbb{C}$.

Existence criterion (Wilf&Zeilberger1992, Abramov2003).

Assume applying modified A.-P. reduction yields

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a}{b} + \frac{q}{v}\right) H.$$

Then

$$T \text{ has a telescoper} \iff b \text{ is integer-linear.}$$

# Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T$ w.r.t. $y$ with order $\rho$.

# Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T$ w.r.t. $y$ with order $\rho$.

Idea. Set $T = SH$, a multi. decomp. and $u/v = \sigma_y(H)/H$

$$T = \Delta_y\left(\cdots\right) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H$$

# Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T$ w.r.t. $y$ with order $\rho$.

Idea. Set $T = SH$, a multi. decomp. and $u/v = \sigma_y(H)/H$

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a_0}{\boxed{b_0}} + \frac{q_0}{v}\right) H$$

integer-linear

# Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T$ w.r.t. $y$ with order $\rho$.

Idea. Set $T = \mathcal{S}H$, a multi. decomp. and $u/v = \sigma_y(H)/H$

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H$$

$$\sigma_x(T) = \Delta_y\Big(\cdots\Big) + \left(\frac{a_1}{b_1} + \frac{q_1}{v}\right) H$$

$$\vdots$$

$$\sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + \left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) H$$

# Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T$ w.r.t. $y$ with order $\rho$.

Idea. Set $T = SH$, a multi. decomp. and $u/v = \sigma_y(H)/H$

$$c_0(x)\, T = \Delta_y\Big(\cdots\Big) + c_0(x)\, \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H$$

$$c_1(x)\, \sigma_x(T) = \Delta_y\Big(\cdots\Big) + c_1(x)\, \left(\frac{a_1}{b_1} + \frac{q_1}{v}\right) H$$

$$\vdots$$

$$c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + c_\rho(x)\, \left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) H$$

# Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T$ w.r.t. $y$ with order $\rho$.

Idea. Set $T = \mathcal{S}H$, a multi. decomp. and $u/v = \sigma_y(H)/H$

$$
+\begin{cases}
c_0(x)\, T = \Delta_y\left(\cdots\right) + c_0(x) \left(\dfrac{a_0}{b_0} + \dfrac{q_0}{v}\right) H \\[2ex]
c_1(x)\, \sigma_x(T) = \Delta_y\left(\cdots\right) + c_1(x) \left(\dfrac{a_1}{b_1} + \dfrac{q_1}{v}\right) H \\[2ex]
\qquad \vdots \\[2ex]
c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\left(\cdots\right) + c_\rho(x) \left(\dfrac{a_\rho}{b_\rho} + \dfrac{q_\rho}{v}\right) H
\end{cases}
$$

---

$$
\left(\sum_{i=0}^{\rho} c_i(x)\sigma_x^i\right)(T) = \Delta_y\left(\cdots\right) + \left(\sum_{j=0}^{\rho} c_j(x) \left(\frac{a_j}{b_j} + \frac{q_j}{v}\right)\right) H
$$

# Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T$ w.r.t. $y$ with order $\rho$.

Idea. Set $T = SH$, a multi. decomp. and $u/v = \sigma_y(H)/H$

$$
+\left\{
\begin{array}{rl}
c_0(x)\,T &= \Delta_y\left(\cdots\right) + c_0(x)\left(\dfrac{a_0}{b_0} + \dfrac{q_0}{v}\right)H \\
c_1(x)\,\sigma_x(T) &= \Delta_y\left(\cdots\right) + c_1(x)\left(\dfrac{a_1}{b_1} + \dfrac{q_1}{v}\right)H \\
&\vdots \\
c_\rho(x)\,\sigma_x^\rho(T) &= \Delta_y\left(\cdots\right) + c_\rho(x)\left(\dfrac{a_\rho}{b_\rho} + \dfrac{q_\rho}{v}\right)H
\end{array}
\right.
$$

telescoper?

$\stackrel{?}{=} 0$

$$
\boxed{\left(\sum_{i=0}^{\rho} c_i(x)\sigma_x^i\right)}(T) = \Delta_y\left(\cdots\right) + \boxed{\left(\sum_{j=0}^{\rho} c_j(x)\left(\frac{a_j}{b_j} + \frac{q_j}{v}\right)\right)}H
$$

# Reduction-based telescoping (CHKL2015)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for $T$ w.r.t. $y$ with order $\rho$.

Idea. Set $T = SH$, a multi. decomp. and $u/v = \sigma_y(H)/H$

$$\sum_{i=0}^{\rho} c_i(x)\sigma_x^i \text{ is a telescoper for } T$$

$$\Updownarrow \text{ MAP reduction}$$

$$\sum_{j=0}^{\rho} c_j(x)\left(\frac{a_j}{b_j} + \frac{q_j}{v}\right) = 0$$

$$\Updownarrow \ \gcd(b_j, v) = 1$$

$$\begin{cases} c_0(x)\dfrac{a_0(x,y)}{b_0(x,y)} + \cdots + c_\rho(x)\dfrac{a_\rho(x,y)}{b_\rho(x,y)} = 0 \\ c_0(x)q_0(x,y) + \cdots + c_\rho(x)q_\rho(x,y) = 0 \end{cases}$$

# Example

Consider

$$T = \frac{1}{x + 2y} \cdot y!$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

- A multi. decomp. $T = SH$ where

  $$S = \frac{1}{x+2y}$$

  and

  $$H = y! \quad \text{with} \quad \frac{\sigma_y(H)}{H} = y+1.$$

- $u := y+1$ and $v := 1$.

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right)H$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right) H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{v}\right) H$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{v}\right)H$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$\left(\frac{2}{x+2y} + \frac{0}{v}\right)$$

$$\left(\frac{2}{x+2y+1} + \frac{0}{v}\right)$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$c_0(x) \cdot \left( \frac{2}{x+2y} + \frac{0}{v} \right)$$

$$+ c_1(x) \cdot \left( \frac{2}{x+2y+1} + \frac{0}{v} \right)$$

$$= 0$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$c_0(x) \cdot \left( \frac{2}{x+2y} + \frac{0}{v} \right)$$

$$+ c_1(x) \cdot \left( \frac{2}{x+2y+1} + \frac{0}{v} \right)$$

No solution in $\mathbb{C}(x)$!

$$= 0$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{v}\right)H$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{v}\right)H$$

$$\sigma_x^2(T) = \Delta_y\left(g_2\right) + \left(-\frac{-4/x}{x+2y} + \frac{2/x}{v}\right)H$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right)H$$

$$\sigma_x(T) = \Delta_y\left(g_1\right) + \left(\frac{2}{x+2y+1} + \frac{0}{v}\right)H$$

$$\sigma_x^2(T) = \Delta_y\left(g_2\right) + \left(-\frac{-4/x}{x+2y} + \frac{2/x}{v}\right)H$$

$$\sigma_x^3(T) = \Delta_y\left(g_3\right) + \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v}\right)H$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$T = \Delta_y(g_0) + \left(\frac{2}{x+2y} + \frac{0}{v}\right)H$$

$$\sigma_x(T) = \Delta_y(g_1) + \left(\frac{2}{x+2y+1} + \frac{0}{v}\right)H$$

$$\sigma_x^2(T) = \Delta_y(g_2) + \left(-\frac{-4/x}{x+2y} + \frac{2/x}{v}\right)H$$

$$\sigma_x^3(T) = \Delta_y(g_3) + \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v}\right)H$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$\left(\frac{2}{x+2y} + \frac{0}{v}\right)$$

$$\left(\frac{2}{x+2y+1} + \frac{0}{v}\right)$$

$$\left(-\frac{-4/x}{x+2y} + \frac{2/x}{v}\right)$$

$$\left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v}\right)$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$
\begin{aligned}
& c_0(x) \cdot \left( \frac{2}{x+2y} + \frac{0}{v} \right) \\
+ & c_1(x) \cdot \left( \frac{2}{x+2y+1} + \frac{0}{v} \right) \\
+ & c_2(x) \cdot \left( -\frac{-4/x}{x+2y} + \frac{2/x}{v} \right) \\
+ & c_3(x) \cdot \left( -\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v} \right) \\
= & \, 0
\end{aligned}
$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

$$
\begin{aligned}
& -2\cdot \left(\frac{2}{x+2y} + \frac{0}{v}\right) \\
& +2\cdot \left(\frac{2}{x+2y+1} + \frac{0}{v}\right) \\
& -x\cdot \left(-\frac{-4/x}{x+2y} + \frac{2/x}{v}\right) \\
& +(x+1)\cdot \left(-\frac{-4/(x+1)}{x+2y+1} + \frac{2/(x+1)}{v}\right) \\
& = 0
\end{aligned}
$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

Therefore,

- a minimal telescoper for $T$ w.r.t. $y$ is

$$L = (x+1) \cdot \sigma_x^3 - x \cdot \sigma_x^2 + 2 \cdot \sigma_x - 2$$

# Example

Consider

$$T = \frac{1}{x+2y} \cdot y!$$

Therefore,

- a minimal telescoper for $T$ w.r.t. $y$ is

$$L = (x+1) \cdot \sigma_x^3 - x \cdot \sigma_x^2 + 2 \cdot \sigma_x - 2$$

- the corresponding certificate is

$$\begin{aligned} G &= (x+1) \cdot g_3 - x \cdot g_2 + 2 \cdot g_1 - 2 \cdot g_0 \\ &= \frac{2y!}{(x+2y)(x+2y+1)} \end{aligned}$$

# Relationship among remainders

Definition (shift-related). Let $f, g \in \mathcal{C}(x)[y]$ be shift-free w.r.t. $y$.

$$\boxed{f \approx_y g}$$

$$\Updownarrow$$

| | | |
|---|---|---|
| multi-set<br>non-trivial irred. monic factors of $f$ | $\overset{1:1}{\sim_y}$ | multi-set<br>non-trivial irred. monic factors of $g$ |

# Relationship among remainders

**Definition (shift-related).** Let $f, g \in \mathcal{C}(x)[y]$ be shift-free w.r.t. $y$.

$$\boxed{f \approx_y g}$$

$$\Updownarrow$$

multi-set
non-trivial irred.
monic factors of $f$
$p$

$\overset{1:1}{\sim_y}$

multi-set
non-trivial irred.
monic factors of $g$

# Relationship among remainders

**Definition (shift-related).** Let $f, g \in \mathcal{C}(x)[y]$ be shift-free w.r.t. $y$.

$$\boxed{f \approx_y g}$$

$$\Updownarrow$$

multi-set
non-trivial irred. monic factors of $f$
$p$

$\overset{1:1}{\sim_y}$

multi-set
non-trivial irred. monic factors of $g$
$q$

# Relationship among remainders

**Definition (shift-related).** Let $f, g \in \mathcal{C}(x)[y]$ be shift-free w.r.t. $y$.

$$\boxed{f \approx_y g}$$

$$\Updownarrow$$

multi-set
non-trivial irred. monic factors of $f$
$p$

$1:1$
$\sim_y$

multi-set
non-trivial irred. monic factors of $g$
$q$

$$\boxed{p = \sigma_y^{\ell}(q),\ \ell \in \mathbb{Z}}$$

# Relationship among remainders

**Definition (shift-related).** Let $f, g \in \mathcal{C}(x)[y]$ be shift-free w.r.t. $y$.

$$\boxed{f \approx_y g}$$

$$\Updownarrow$$

multi-set: non-trivial irred. monic factors of $f$ — $p$ $\quad \overset{1:1}{\sim_y} \quad$ multi-set: non-trivial irred. monic factors of $g$ — $q$

$$\boxed{p = \sigma_y^{\ell}(q),\ \ell \in \mathbb{Z}}$$

**Proposition.** $T = \mathcal{S}H$ a multi. decomp. and $u/v = \sigma_y(H)/H$. Let

$$\sigma_x^i(T) = \Delta_y\Big(\cdots\Big) + \left(\frac{a_i}{b_i} + \frac{q_i}{v}\right)H, \quad \forall i \in \mathbb{N}.$$

Then $b_i \approx_y \sigma_x^i(b_0), \quad \forall i \in \mathbb{N}.$

# Relationship among remainders

**Definition (shift-related).** Let $f, g \in \mathcal{C}(x)[y]$ be shift-free w.r.t. $y$.

$$\boxed{f \approx_y g}$$

$$\Updownarrow$$

multi-set

non-trivial irred. monic factors of $f$

$p$

$1:1$

$\sim_y$

multi-set

non-trivial irred. monic factors of $g$

$q$

$\boxed{p = \sigma_y^\ell(q),\ \ell \in \mathbb{Z}}$

**Proposition.** $T = \mathcal{S}H$ a multi. decomp. and $u/v = \sigma_y(H)/H$. Let

$$\sigma_x^i(T) = \Delta_y\left(\cdots\right) + \left(\frac{a_i}{b_i} + \frac{q_i}{v}\right) H, \quad \forall i \in \mathbb{N}.$$

$\boxed{b_0 \text{ integer-linear}}$

Then $b_i \approx_y \sigma_x^i(b_0), \quad \forall i \in \mathbb{N}.$

# Relationship among remainders

**Definition (shift-related).** Let $f, g \in \mathcal{C}(x)[y]$ be shift-free w.r.t. $y$.

$$\boxed{f \approx_y g}$$

$$\Updownarrow$$

multi-set

non-trivial irred. monic factors of $f$

$p$

$\overset{1:1}{\sim_y}$

multi-set

non-trivial irred. monic factors of $g$

$q$

$$\boxed{p = \sigma_y^{\ell}(q),\ \ell \in \mathbb{Z}}$$

**Proposition.** $T = SH$ a multi. decomp. and $u/v = \sigma_y(H)/H$. Let

$$\sigma_x^i(T) = \Delta_y\left(\cdots\right) + \left(\frac{a_i}{\boxed{b_i}} + \frac{q_i}{v}\right) H, \quad \forall i \in \mathbb{N}.$$

integer-linear

$\boxed{b_0 \text{ integer-linear}}$

Then $b_i \approx_y \sigma_x^i(b_0), \quad \forall i \in \mathbb{N}.$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

- Modified A.-P. reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

- Modified A.-P. reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$c_0(x)\left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) + \cdots + c_\rho(x)\left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) = 0$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

- Modified A.-P. reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$\begin{cases} c_0(x)\dfrac{a_0}{b_0} + \cdots + c_\rho(x)\dfrac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

- Modified A.-P. reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$\begin{cases} c_0(x)\dfrac{a_0}{b_0} + \cdots + c_\rho(x)\dfrac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

- Property. $b_i = x + 2y$ or $x + 2y + 1$.

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

$\#\text{vars} = \rho + 1$

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

$\Uparrow$ Prop. $b_i = x+2y$ or $x+2y+1$

$\#\text{vars} = \rho + 1$

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

common denom. $B = (x+2y)(x+2y+1)$

$\Uparrow$ Prop. $b_i = x+2y$ or $x+2y+1$

#vars $= \rho + 1$

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\Uparrow \quad \deg_y(a_i) < \deg_y(b_i)$$

common denom. $B = (x+2y)(x+2y+1)$

$$\Uparrow \quad \text{Prop. } b_i = x+2y \text{ or } x+2y+1$$

$\#\text{vars} = \rho + 1$

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

#eqns over $\mathbb{C}(x) \leq 2$

$\Uparrow$ $\deg_y(a_i) < \deg_y(b_i)$

common denom. $B = (x+2y)(x+2y+1)$

$\Uparrow$ Prop. $b_i = x+2y$ or $x+2y+1$

#vars $= \rho + 1$

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\#\text{eqns over } \mathbb{C}(x) \le 2$$

$$\Uparrow \quad \deg_y(a_i) < \deg_y(b_i)$$

$$\text{common denom. } B = (x+2y)(x+2y+1)$$

$$\Uparrow \quad \text{Prop. } b_i = x+2y \text{ or } x+2y+1$$

$\#\text{vars} = \rho + 1$

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

$$\Downarrow \quad \text{Prop. } \#\text{ terms of } q \le \begin{array}{l} \max\left(\deg_y(u), \deg_y(v)\right) \\ -[\![\deg_y(u-v) \le \deg_y(u) - 1]\!] \end{array}$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

#eqns over $\mathbb{C}(x) \leq 2$

$\Uparrow$ $\deg_y(a_i) < \deg_y(b_i)$

common denom. $B = (x+2y)(x+2y+1)$

$\Uparrow$ Prop. $b_i = x+2y$ or $x+2y+1$

#vars $= \rho + 1$

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

$\Downarrow$ Prop. # terms of $q \leq \max\left(\deg_y(u), \deg_y(v)\right) - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$

#eqns over $\mathbb{C}(x) \leq 1$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\#\text{eqns over } \mathbb{C}(x) \leq 2$$

$$\Uparrow \quad \deg_y(a_i) < \deg_y(b_i)$$

$$\text{common denom. } B = (x+2y)(x+2y+1)$$

$$\Uparrow \quad \text{Prop. } b_i = x+2y \text{ or } x+2y+1$$

$\#\text{vars} = \rho + 1$
$\#\text{eqns over } \mathbb{C}(x) \leq 3$

$$\begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

$$\Downarrow \quad \text{Prop. } \#\text{ terms of } q \leq \begin{array}{l} \max\left(\deg_y(u), \deg_y(v)\right) \\ -[\![\deg_y(u-v) \leq \deg_y(u) - 1]\!] \end{array}$$

$$\#\text{eqns over } \mathbb{C}(x) \leq 1$$

# Upper bound

Example (cont.). $T = y!/(x+2y)$.

$$\#\text{eqns over } \mathbb{C}(x) \leq 2$$

$$\Uparrow \ \deg_y(a_i) < \deg_y(b_i)$$

$$\text{common denom. } B = (x+2y)(x+2y+1)$$

$$\Uparrow \ \text{Prop. } b_i = x+2y \text{ or } x+2y+1$$

$$\begin{array}{l} \#\text{vars} = \rho + 1 \\ \#\text{eqns over } \mathbb{C}(x) \leq 3 \end{array} \quad \begin{cases} c_0(x)\frac{a_0}{b_0} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

$$\Downarrow \ \text{Prop. } \#\text{ terms of } q \leq \begin{array}{l} \max\left(\deg_y(u), \deg_y(v)\right) \\ -[\![\deg_y(u-v) \leq \deg_y(u) - 1]\!] \end{array}$$

$$\#\text{eqns over } \mathbb{C}(x) \leq 1$$

Conclusion. Upper bound is 3.

# New upper bound

Theorem. Assume applying modified A.-P. reduction yields

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H,$$

where $b_0 = c \prod_{i=1}^{m} \prod_{k=0}^{d_i} (\alpha_i x + \beta_i y + \gamma_i + k)^{e_{ik}}$, and for each $i \neq j$, either

$$\alpha_i \neq \alpha_j \quad \text{or} \quad \beta_i \neq \beta_j \quad \text{or} \quad \gamma_i - \gamma_j \notin \mathbb{Z}.$$

Then the order of a minimal telescoper for $T$ w.r.t. $y$ is no more than

$$\begin{aligned} B_{New} := &\max\{\deg_y(u), \deg_y(v)\} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!] \\ &+ \sum_{i=1}^{m} \beta_i \cdot \max_{0 \leq k \leq d_i} \{e_{ik}\} \end{aligned}$$

# Apagodu-Zeilberger upper bound (2005)

Definition. A hypergeom. term $T$ is said to be proper if it is of the form

$$T = p(x,y)\prod_{i=1}^{m}\frac{(\alpha_i x+\alpha_i' y+\alpha_i'')!(\beta_i x-\beta_i' y+\beta_i'')!}{(\mu_i x+\mu_i' y+\mu_i'')!(\nu_i x-\nu_i' y+\nu_i'')!}z^y$$

Theorem. Assume $T$ is generic proper hypergeom. . Then the order of a minimal telescoper for $T$ w.r.t. $y$ is no more than

$$B_{AZ} = \max\left\{\sum_{i=1}^{m}(\alpha_i'+\nu_i'), \sum_{i=1}^{m}(\beta_i'+\mu_i')\right\}.$$

# Apagodu-Zeilberger upper bound (2005)

Definition. A hypergeom. term T is said to be proper if it is of the form

$$T = p(x,y)\prod_{i=1}^{m}\frac{(\alpha_i x+\alpha_i' y+\alpha_i'')!(\beta_i x-\beta_i' y+\beta_i'')!}{(\mu_i x+\mu_i' y+\mu_i'')!(\nu_i x-\nu_i' y+\nu_i'')!}z^y$$

$$\begin{array}{c}\nexists 1\leq i,j\leq m \text{ s.t.}\\ \{\alpha_i=\mu_j \quad \& \quad \alpha_i'=\mu_j' \quad \& \quad \alpha_i''-\mu_j''\in\mathbb{N}\}\\ \{\beta_i=\nu_j \quad \& \quad \beta_i'=\nu_j'' \quad \& \quad \beta_i''-\nu_j''\in\mathbb{N}\}.\end{array}$$

Theorem. Assume T is generic proper hypergeom. . Then the order of a minimal telescoper for T w.r.t. $y$ is no more than

$$B_{AZ}=\max\left\{\sum_{i=1}^{m}(\alpha_i'+\nu_i'),\sum_{i=1}^{m}(\beta_i'+\mu_i')\right\}.$$

# New upper bound v.s. Apagodu-Zeilberger's

| | New | AZ | Order |
|---|---|---|---|
| proper | $B_{New}$ | $B_{AZ}$ | $\leq B_{New}$ |
| non-proper | | | |
| example | | | |

# New upper bound v.s. Apagodu-Zeilberger's

| | New | AZ | Order |
|---|---|---|---|
| proper | $B_{New}$<br>$\|\|$<br>$B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$ | $B_{AZ}$ | $\leq B_{New}$ |
| non-proper | | | |
| example | | | |

# New upper bound v.s. Apagodu-Zeilberger's

| | New | AZ | Order |
|---|---|---|---|
| proper | $B_{New}$ <br> $\parallel$ <br> $B_{AZ} - [\![\deg_y(u-v) \le \deg_y(u) - 1]\!]$ | $B_{AZ}$ | $\le B_{New}$ |
| non-proper | $B_{New}$ | | |
| example | | | |

# New upper bound v.s. Apagodu-Zeilberger's

| | New | AZ | Order |
|---|---|---|---|
| proper | $B_{New}$ <br> $\parallel$ <br> $B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$ | $B_{AZ}$ | $\leq B_{New}$ |
| non-proper | $B_{New}$ | ? | |
| example | | | |

# New upper bound v.s. Apagodu-Zeilberger's

| | New | AZ | Order |
|---|---|---|---|
| proper | $B_{New}$ <br> $\Vert$ <br> $B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$ | $B_{AZ}$ | $\leq B_{New}$ |
| non-proper | $B_{New}$ | ? | $\leq B_{New}$ |
| $T_1$ | 9 | 10 | 9 |

Example.

$$T_1 = \frac{(x+3y)!(x-3y)!}{(5x+3y)(3x-y)(4x-3y)!(5x+3y)!}.$$

# New upper bound v.s. Apagodu-Zeilberger's

| | New | AZ | Order |
|---|---|---|---|
| proper | $B_{New}$ <br> $\Vert$ <br> $B_{AZ} - [\![\deg_y(u-v) \le \deg_y(u) - 1]\!]$ | $B_{AZ}$ | $\le B_{New}$ |
| non-proper | $B_{New}$ | ? | $\le B_{New}$ |
| $T_2$ | $\beta$ | $\alpha+\beta$ | $\beta$ |

Example.

$$T_2 = \frac{\alpha^2 y^2 + \alpha^2 y - \alpha\beta y + 2\alpha x y + x^2}{(x + \alpha y + \alpha)(x + \alpha y)(x + \beta y)}, \quad \alpha \neq \beta \text{ in } \mathbb{N} \setminus \{0\}.$$

# New upper bound v.s. Apagodu-Zeilberger's

| | New | AZ | Order |
|---|---|---|---|
| proper | $B_{New}$<br>$\parallel$<br>$B_{AZ} - [\![\deg_y(u-v) \leq \deg_y(u) - 1]\!]$ | $B_{AZ}$ | $\leq B_{New}$ |
| non-proper | $B_{New}$ | ? | $\leq B_{New}$ |
| $T_3$ | 3 | ? | 3 |

Example.

$$T_3 = \frac{x^4 + x^3y + 2x^2y^2 + 2x^2y + xy^2 + 2y^3 + x^2 + y^2 - x - y}{(x^2+y+1)(x^2+y)(x+2y)} y!$$

# Lower bound

Example (cont.). $T = y!/(x+2y)$.

# Lower bound

Example (cont.). $\mathrm{T} = y!/(x+2y)$.

- Modified A.-P. reduction:

$$\mathrm{T} = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right)\mathrm{H} \quad \text{with} \quad b_0 = x + 2y.$$

# Lower bound

Example (cont.). $T = y!/(x+2y)$.

- Modified A.-P. reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$c_0(x)\left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) + \cdots + c_\rho(x)\left(\frac{a_\rho}{b_\rho} + \frac{q_\rho}{v}\right) = 0$$

# Lower bound

Example (cont.). $T = y!/(x+2y)$.

- Modified A.-P. reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$\begin{cases} c_0(x)\dfrac{a_0}{b_0} + \cdots + c_\rho(x)\dfrac{a_\rho}{b_\rho} = 0 \\ c_0(x) q_0 + \cdots + c_\rho(x) q_\rho = 0 \end{cases}$$

# Lower bound

Example (cont.). $T = y!/(x+2y)$.

- Modified A.-P. reduction:

$$T = \Delta(g_0) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H \quad \text{with} \quad b_0 = x + 2y.$$

- $\exists\ \rho \in \mathbb{N}^*$ s.t.

$$\begin{cases} c_0(x)\dfrac{a_0}{b_0} + \cdots + c_\rho(x)\dfrac{a_\rho}{b_\rho} = 0 \\ c_0(x)q_0 + \cdots + c_\rho(x)q_\rho = 0 \end{cases}$$

- Property. $b_i = x + 2y$ or $x + 2y + 1$.

# Lower bound

Example (cont.). $T = y!/(x+2y)$.

$$c_0(x)\frac{a_0}{b_0} + \cdots + c_i(x)\frac{a_i}{b_i} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0$$

# Lower bound

Example (cont.). $T = y!/(x + 2y)$.

$$c_0(x)\frac{a_0}{b_0} + \cdots + c_i(x)\frac{a_i}{b_i} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0$$

PFD & $b_0 = x + 2y$

$$\Downarrow$$

$$\exists i, \text{s.t. } b_i = b_0$$

# Lower bound

Example (cont.). $T = y!/(x+2y)$.

$$c_0(x)\frac{a_0}{b_0} + \cdots + c_i(x)\frac{a_i}{b_i} + \cdots + c_\rho(x)\frac{a_\rho}{b_\rho} = 0$$

PFD & $b_0 = x + 2y$

$$\Downarrow$$

$$\exists i, \text{s.t. } b_i = b_0$$

$$\Downarrow \ \text{Prop. } b_i = x + 2y \text{ or } x + 2y + 1$$

minimal $i = 2$

# Lower bound

Example (cont.). $T = y!/(x+2y)$.

$$\textcolor{red}{c_0(x)}\frac{a_0}{b_0} + \cdots + \textcolor{red}{c_i(x)}\frac{a_i}{b_i} + \cdots + \textcolor{red}{c_\rho(x)}\frac{a_\rho}{b_\rho} = 0$$

PFD & $b_0 = x + 2y$

$$\Downarrow$$

$$\exists i, \text{s.t. } b_i = b_0$$

$$\Downarrow \text{ Prop. } b_i = x + 2y \text{ or } x + 2y + 1$$

minimal $i = 2$

Conclusion. Lower bound is $2$.

# New lower bound

Theorem. Assume applying modified A.-P. reduction yields

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H,$$

with $b_0$ integer-linear. Then the order of a telescoper for $T$ w.r.t. $y$ is at least

$$\max_{\substack{p \mid b_0 \text{ irred.} \\ \deg_y(p) \geq 1}} \quad \min_{h \in \mathbb{Z}} \left\{ \rho \in \mathbb{N} \setminus \{0\} : \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \right\}$$

# Abramov-Le lower bound (2005)

Theorem. Assume applying modified A.-P. reduction yields

$$T = \Delta_y\Big(\cdots\Big) + \left(\frac{a_0}{b_0} + \frac{q_0}{v}\right) H = \Delta_y\Big(\dots\Big) + \frac{a_0'}{b_0} H',$$

with $b_0$ integer-linear, $a_0' = a_0 v + b_0 q_0$ and $H' = H/v$. Let

$$\frac{c'}{d'} := \frac{\sigma_x(H')}{H'}.$$

Then the order of a telescoper for $T$ w.r.t. $y$ is at least

$$\max_{\substack{p \mid b_0 \text{ irred.} \\ \deg_y(p) \geq 1}} \min_{h \in \mathbb{Z}} \left\{ \rho \in \mathbb{N} \setminus \{0\} : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$$

# New lower bound v.s. Abramov-Le's

| | New | AL | Order |
|---|---|---|---|
| hypergeom. | $\max_p \ \min_h$ <br> $\left\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\right\}$ | $\max_p \ \min_h$ <br> $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$ | $\geq \ldots$ |
| example | | | |

# New lower bound v.s. Abramov-Le's

| | New | AL | Order |
|---|---|---|---|
| hypergeom. | $\max_p \ \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$ | $\max_p \ \min_h$ $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$ | $\geq \ldots$ |
| $T_1$ | 7 | 3 | 17 |

Example.

$$T_1 = \frac{1}{(x+3y+1)(5x-7y)(5x-7y+14)!}.$$

# New lower bound v.s. Abramov-Le's

| | New | AL | Order |
|---|---|---|---|
| hypergeom. | $\max_p \min_h \{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$ | $\max_p \min_h \left\{ \rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array} \right\}$ | $\geq \ldots$ |
| $T_2$ | 12 | 3 | 29 |

Example.

$$T_2 = \frac{1}{(x+5y+1)(5x-12y)(5x-12y+24)!}.$$

# New lower bound v.s. Abramov-Le's

| | New | AL | Order |
|---|---|---|---|
| hypergeom. | $\max_p \min_h$ $\{\rho : \sigma_y^h(p) \mid \sigma_x^\rho(b_0)\}$ | $\max_p \min_h$ $\left\{\rho : \begin{array}{c} \sigma_y^h(p) \mid \sigma_x^\rho(b_0) \\ \text{or} \\ \sigma_y^h(p) \mid \sigma_x^{\rho-1}(d') \end{array}\right\}$ | $\geq \ldots$ |
| $T_3$ | $\alpha$ | 2 | $\alpha$ |

Example.

$$T_3 = \frac{1}{(x - \alpha y - \alpha)(x - \alpha y - 2)!}, \quad \alpha \geq 2 \text{ in } \mathbb{N}.$$

# Summary

Result.

- Order bounds for minimal telescopers

Future work.

- Creative telescoping for $q$-hypergeometric terms