A Modified Abramov-Petkovšek Reduction and Creative Telescoping for Hypergeometric Terms

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Outline

- Hypergeometric summability
- Abramov–Petkovšek (AP) reduction
- Modified AP reduction
- Reduction-based creative telescoping

Hypergeometric summability

Definition. A nonzero term $T(y)$ is hypergeometric over $\mathbb{C}(y)$ if $T(y+1)/T(y) \in \mathbb{C}(y)$.

Examples.

$f(y) \in \mathbb{C}(y) \setminus \{0\}$, c^y with $c \in \mathbb{C} \setminus \{0\}$, $y!$, and binomial coefficients, etc.

Definition. A hypergeom. term $T(y)$ is summable if

$$T(y) = G(y+1) - G(y) \text{ for some hypergeom. term } G(y).$$

Example. $y \cdot y! = (y+1)! - y!$ is summable; but $y!$ is not.

Multiplicative decomposition

Notation.

- $\sigma_y(T(y)) = T(y+1)$, $\Delta_y(T) = \sigma_y(T) - T$ for a term $T(y)$.
- f_d and f_n: the denominator and numerator of $f \in \mathbb{C}(y)$.

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For a hypergeom. term $T(y)$, $\exists\ S \in \mathbb{C}(y)$ and H hypergeom. s.t.

- $T(y) = S(y) \cdot H(y)$;
- $K := \sigma_y(H)/H$ is shift-reduced, i.e.

$$\gcd\left(K_d, \sigma_y^{\ell}(K_n)\right) = 1 \quad \text{for all } \ell \in \mathbb{Z}.$$

Call K a kernel of T, and S the corr. shell.

AP reduction (2001)

Let $T(y)$ be hypergeom. with a kernel K and shell S. Then

$$T = \Delta_y(\dots) + \left(\frac{a}{b} + \frac{p}{K_d}\right) H,$$

where $H = T/S$, and $a, b, p \in \mathbb{C}[y]$ satisfy proper, shift-free, and strongly-prime conditions.

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Proposition. T is summable iff

- $a = 0$,
- $K_n z(y+1) - K_d z(y) = p$ has a solution in $\mathbb{C}[y]$.

Question

Can one determine hypergeometric summability directly without solving any equations?

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Known results:

- Hyperexponentional Hermite reduction (Bostan et. al 2013)
- Rational Abramov reduction (1995)

Polynomial reduction (new)

Let $K \in \mathbb{C}(y)$ be shift-reduced, define

- polynomial reduction map (w.r.t. K):

$$\begin{array}{rccc} \phi_K: & \mathbb{C}[y] & \longrightarrow & \mathbb{C}[y] \\ & a & \longmapsto & K_n\sigma_y(a) - K_d a. \end{array}$$

- standard complement of $\operatorname{im}(\phi_K)$:

$$\mathcal{N}_K = \operatorname{span}_{\mathbb{C}} \left\{ y^i \mid i \neq \deg(a) \text{ for all } a \in \operatorname{im}(\phi_K) \right\}.$$

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Proposition. $\mathbb{C}[y] = \mathrm{im}(\phi_K) \oplus \mathcal{N}_K$.

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$$T = \Delta_y(\dots) + \left(\frac{a}{b} + \frac{p_2}{K_d}\right) H$$

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Proposition.

- T is summable iff $a = p_2 = 0$.
- $\#$ nonzero terms of $p_2 \leq \max\left(\deg_y(K_n), \deg_y(K_d)\right)$.

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$$T = \Delta_y(\dots) + \underbrace{\left(\frac{a}{b} + \frac{p_2}{K_d}\right)}_{\text{a residual form (w.r.t. } K\text{)}} H$$

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Example

$$T = (y^3 + 1) \cdot y!,\ K = y + 2 \text{ and } H = (y + 1)y!.$$

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$$K_n z(y+1) - K_d z(y) = p$$

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$$z(y) \notin \mathbb{C}[y]$$

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$$K_n z(y+1) - K_d z(y) = p$$
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$$z(y) \notin \mathbb{C}[y]$$

$$p = 2 \mod \operatorname{im}(\phi_K)$$
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$$T = \Delta_y(\cdots) + 2\cdot H$$

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$$T = \Delta_y(\cdots) + 2 \cdot H$$

⇘ ⇙

$(y^3 + 1) \cdot y!$ is not summable

Bivariate hypergeometric terms

Definition. A nonzero term $T(x, y)$ is hypergeometric over $\mathbb{C}(x, y)$ if $\sigma_x(T)/T,\ \sigma_y(T)/T \ \in \mathbb{C}(x, y)$.

Telescoping problem. Given $T(x, y)$ hypergeom. , find a nonzero operator $L \in \mathbb{C}(x)\langle\sigma_x\rangle$ s.t.

$$L(T) = \Delta_y(G) \text{ for some } G(x, y) \text{ hypergeom.}$$

Call

- L a telescoper for T w.r.t. y;
- G the corr. certificate.

Algorithms for creative telescoping

Classical: Zeilberger's algorithm (1990)

- based on Gosper's algorithm (1978)
- telescopers and certs are computed simultaneously

Motivating example 1

Consider

$$T = \frac{y^{10}}{x + y}$$

- The minimal telescoper for T w.r.t. y is

$$L = \sigma_x - \frac{1}{x^{10}}(x + 1)^{10}$$

Certificate for the example

$$
\begin{aligned}
G = \frac{1}{10} \Bigg(& -1/21 \, \frac{x^3 \left(175x^7 + 700x^6 + 1234x^5 + 1252x^4 + 790x^3 + 310x^2 + 70x + 7\right)}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& - \frac{1}{42} \, \frac{x \left(1750x^7 + 5950x^6 + 9558x^5 + 9186x^4 + 5630x^3 + 2180x^2 + 490x + 49\right) y^2}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& - \frac{1}{18} \, \frac{\left(990x^9 + 3960x^8 + 7890x^7 + 10260x^6 + 9654x^5 + 6780x^4 + 3490x^3 + 1240x^2 + 270x + 27\right) y^3}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& + \frac{5}{36} \, \frac{x \left(792x^7 + 2574x^6 + 4020x^5 + 3801x^4 + 2310x^3 + 891x^2 + 200x + 20\right) y^4}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& + \frac{1}{12} \, \frac{\left(1320x^9 + 5280x^8 + 11352x^7 + 16566x^6 + 17540x^5 + 13535x^4 + 7410x^3 + 2721x^2 + 600x + 60\right) y^5}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& - \frac{1}{6} \, \frac{x \left(660x^7 + 1980x^6 + 2948x^5 + 2717x^4 + 1630x^3 + 625x^2 + 140x + 14\right) y^6}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& - \frac{1}{42} \, \frac{\left(4620x^9 + 18480x^8 + 42900x^7 + 68640x^6 + 78188x^5 + 63305x^4 + 35630x^3 + 13265x^2 + 2940x + 294\right) y^7}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& + \frac{5}{84} \, \frac{x \left(924x^7 + 2310x^6 + 3168x^5 + 2805x^4 + 1650x^3 + 627x^2 + 140x + 14\right) y^8}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& + \frac{5}{36} \, \frac{\left(660x^9 + 2640x^8 + 6732x^7 + 11550x^6 + 13728x^5 + 11385x^4 + 6490x^3 + 2431x^2 + 540x + 54\right) y^9}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} \\
& - \frac{1}{9} \, \frac{\left(495x^9 + 2145x^8 + 5610x^7 + 9702x^6 + 11550x^5 + 9570x^4 + 5445x^3 + 2035x^2 + 451x + 45\right) y^{10}}{10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1} + y^{11} \Bigg) \\
& \cdot \left(10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1\right) (x + y)^{-1}
\end{aligned}
$$

Certificate for the example

$$G = \frac{1}{10}\Bigg(-1/21\,\frac{x^3\left(175x^7+700x^6+1234x^5+1252x^4+790x^3+310x^2+70x+7\right)}{10x^9+45x^8+120x^7+210x^6+252x^5+210x^4+120x^3+45x^2+10x+1}$$

$$-\frac{1}{42}\,\frac{x\left(1750x^7+5950x^6+9558x^5+9186x^4+5630x^3+2180x^2+490x+49\right)y^2}{10x^9+45x^8+120x^7+210x^6+252x^5+210x^4+120x^3+45x^2+10x+1}$$

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$$+\frac{5}{36}\,\frac{x\left(792x^7+2574x^6+4020x^5+3801x^4+2310x^3+891x^2+200x+20\right)y^4}{10x^9+45x^8+120x^7+210x^6+252x^5+210x^4+120x^3+45x^2+10x+1}$$

$$+\frac{1}{12}\,\frac{\left(1320x^9+5256x^8+11352x^7+16566x^6+17540x^5+13535x^4+7410x^3+2721x^2+600x+60\right)y^5}{10x^9+45x^8+120x^7+210x^6+252x^5+210x^4+120x^3+45x^2+10x+1}$$

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$$-\frac{1}{42}\,\frac{\left(4620x^9+18480x^8+42900x^7+68640x^6+78188x^5+63305x^4+35630x^3+13265x^2+2940x+294\right)y^7}{10x^9+45x^8+120x^7+210x^6+252x^5+210x^4+120x^3+45x^2+10x+1}$$

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$$\cdot\left(10x^9+45x^8+120x^7+210x^6+252x^5+210x^4+120x^3+45x^2+10x+1\right)(x+y)^{-1}$$

Very often, certificates are not needed!

Motivating example 2

Dixon's sum. Consider $F(x) = \sum_y T(x, y)$ with

$$T = (-1)^y \binom{2x}{y}^3$$

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- The corresponding certificate is

$$G = -\frac{1}{2} \frac{(-1)^y y^3 \binom{2x}{y}^3}{(2x + 1 - y)^3 (x^2 + 2x + 1)(2x + 2 - y)^3} \Big(448x^5 - 624x^4 y + 348x^3 y^2 - 90x^2 y^3$$
$$+9xy^4 + 1760x^4 - 1932x^3 y + 792x^2 y^2 - 132xy^3 + 6y^4 + 2728x^3 - 2214x^2 y + 594xy^2$$
$$-48y^3 + 2084x^2 - 1113xy + 147y^2 + 784x - 207y + 116\Big)$$

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we could have known this without knowing G

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Telescoping via reduction

New: Reduction-based telescoping

- a difference variant of Hermite telescoping
  (Bostan et. al 2010, 2013)
- separate the computation of telescopers from certs

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Convention. Let $T(x, y)$ be hypergeom. with a kernel K and shell S. Set $H = T/S$.

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Goal. Given $\rho \in \mathbb{N}$, find a telescoper for T with order ρ.

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$$\sigma_x(T) = \Delta_y\Big(\cdots\Big) + r_1 H$$

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$$
\begin{aligned}
T &= \Delta_y\Big(\cdots\Big) + r_0 H \\
\sigma_x(T) &= \Delta_y\Big(\cdots\Big) + r_1 H \\
\sigma_x^2(T) &= \Delta_y\Big(\cdots\Big) + r_2 H \\
&\vdots \\
\sigma_x^\rho(T) &= \Delta_y\Big(\cdots\Big) + r_\rho H
\end{aligned}
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c_\rho(x)\, \sigma_x^\rho(T) = \Delta_y\Big(\cdots\Big) + c_\rho(x)\, r_\rho H
\end{cases}
$$

$$
\left(c_0(x) + \cdots + c_\rho(x)\sigma_x^\rho\right)(T) = \Delta_y\Big(\cdots\Big) + \left(\sum_{j=0}^{\rho} c_j(x) r_j\right) H
$$

Problem

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$$\overset{?}{\not\Longleftrightarrow}$$

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$$\Longleftarrow$$

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$$\not\Longrightarrow$$

Problem

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$\Longleftarrow$

$\not\Longrightarrow$

$\sum_{j=0}^{\rho} c_j r_j$
may not be a residual form

Idea

Example. Let H be hypergeom. with $K = \sigma_y(H)/H = 1/y$.

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Theorem. r, s residual forms w.r.t. K, $\exists$ a residual form t s.t.

$$sH = \Delta_y(\dots) + tH \quad \text{and} \quad r + t \quad \text{is a residual form.}$$

Reduction-based telescoping (cont.)

Goal. Given $\rho \in \mathbb{N}$, find a telescoper for T with order ρ.

Idea.

$$c_0(x)\, T = \Delta_y\Big(\cdots\Big) + c_0(x)\, r_0 H$$

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$$c_0(x) r_0' + c_1(x) r_1' + \cdots + c_\rho(x) r_\rho' \stackrel{?}{=} 0$$

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Remarks.

- The first linear depend. leads to the minimal telescoper.
- One can leave the certificate as an un-normalized sum.

Outline of reduction-based telescoping

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5. For $\rho = 1, 2, \ldots$ do

   find a telescoper L for T with order ρ and return L.

Example

Consider

$$T = \frac{1}{x+y} \cdot y!$$

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$$c_0(x) \cdot \frac{1}{x+y}$$

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$$+ c_2(x) \cdot \left(-\frac{1/(x(x+1))}{x+y} + \frac{x-1}{x(x+1)} \right)$$

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$$-\,1 \cdot \frac{1}{x+y}$$

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$$\begin{aligned} G &= (x+1) \cdot g_2 - (x-1) \cdot g_1 - 1 \cdot g_0 \\ &= \frac{y!}{(x+y)(x+y+1)} \end{aligned}$$

Timings (in seconds)

Let

$$T = \frac{f(x,y)}{g_1(x+y)g_2(2x+y)} \frac{\Gamma(2\alpha x + y)}{\Gamma(x+\alpha y)}$$

with

- $g_i(z) = p_i(z)p_i(z+\lambda)p_i(z+\mu)$, $\alpha, \lambda, \mu \in \mathbb{N}$,
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$(m, n, \alpha, \lambda, \mu)$	Zeilberger	RCT+cert	RCT	order
$(2, 0, 1, 5, 10)$	354.46	58.01	4.93	4
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Summary

Results.

- Modified AP reduction for hypergeometric terms
- A reduction-based telescoping method

Future work.

- Creative telescoping for q-hypergeometric terms