Creative Telescoping via Abramov's Reduction

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Outline

- Rational summability
- Abramov's reduction for rational functions
- Telescoping via Abramov's reduction
- Hypergeometric case

Rational functions

Definition. A function $f(y)$ is rational over $\mathbb{C}$ if

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Notations. For $f \in \mathbb{C}(y)$,

- $\mathrm{num}(f)$: the numerator of f.
- $\mathrm{den}(f)$: the denominator of f.
- $S_y(f(y)) = f(y+1)$ and $\Delta_y(f) = S_y(f) - f$.

Summability

Definition. $f \in \mathbb{C}(y)$ is said to be (rational) summable if

$$f = \Delta_y(g) \text{ for some } g \in \mathbb{C}(y).$$

Examples.

- All polynomials in $\mathbb{C}[y]$ are summable
- $1/y$ is not summable

Shift-freeness

Definition. $p \in \mathbb{C}[y]$ is shift-free if

$$\gcd\left(p,\ S_y^{\ell}(p)\right) = 1 \quad \text{for all } \ell \in \mathbb{Z} \setminus \{0\}.$$

Fact. Given $r \in \mathbb{F}(y)$ with $\operatorname{den}(r)$ shift-free, then r is not summable.

Recall. $1/y$ is not summable.

Abramov's reduction (1995)

Definition. $r \in \mathbb{C}(y)$ is call a residual form if

$$\deg_y(\text{num}(r)) < \deg_y(\text{den}(r)) \text{ and } \text{den}(r) \text{ is shift-free.}$$

Theorem. Let $f \in \mathbb{C}(y)$. Then $\exists\ g \in \mathbb{C}(y)$ and a residual form r s.t.

$$f = \Delta_y(g) + r$$

Moreover, f is summable $\iff r = 0$.

Remark. r is not unique.

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   - Reduction: $f = \Delta_y(g) + 0$ with $\deg_y(g) = \deg_y(f) + 1$.
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2. $f = 1/y$.
   - Reduction: $\frac{1}{y} = \Delta_y(0) + \frac{1}{y}$
   - f is not summable.

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   - Reduction: $\frac{1}{y} = \Delta_y(0) + \frac{1}{y}$
   - f is not summable.
   - $\frac{1}{y} = \frac{1}{y} - \frac{1}{y+1} + \frac{1}{y+1} = \Delta_y\left(-\frac{1}{y}\right) + \frac{1}{y+1}$

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   - $\frac{1}{y} = \frac{1}{y} - \frac{1}{y-1} + \frac{1}{y-1} = \Delta_y\left(\frac{1}{y-1}\right) + \frac{1}{y-1}$

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Example.

$$\frac{1}{y} + \frac{1}{y+1} = \frac{2y+1}{y(y+1)} \text{ is not a residual form.}$$

$$\frac{1}{y} + \frac{1}{y+1} = \frac{1}{y} + \left(\Delta_y\left(\frac{1}{y}\right) + \frac{1}{y}\right) = \Delta_y\left(\frac{1}{y}\right) + \frac{2}{y}$$

and

$$\frac{2}{y} \quad \text{is a residual form.}$$

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and

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Theorem. Let $r, s \in \mathbb{C}(y)$ be two residual forms. Then $\exists\, g \in \mathbb{C}(y)$ and a residual form t with

$$s = \Delta_y(g) + t \quad \text{and} \quad r + t \quad \text{is a residual form.}$$

Bivariate rational functions

Let $\mathbb{C}(x, y)$ be the field of bivariate rational functions.

Telescoping problem. Given $f \in \mathbb{C}(x, y)$, find a nonzero operator $L(x, S_x) \in \mathbb{C}(x)\langle S_x \rangle$ s.t.

$$L(f) = \Delta_y(g) \text{ for some } g \in \mathbb{C}(x, y).$$

Call

- L: a telescoper for f;
- g: the corresponding certificate.

Existence of telescopers

Definition. $p \in \mathbb{C}[x, y]$ is integer-linear if

$$p = \prod_i (m_i x + n_i y + k_i)$$

where $m_i, n_i \in \mathbb{Z}$ and $k_i \in \mathbb{C}$.

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Existence criterion (Wilf&Zeilberger1992, Abramov2003).

Given $f \in \mathbb{C}(x, y)$. Let $r \in \mathbb{C}(x, y)$ be a residual form with

$$f = \Delta_y(g) + r \text{ for some } g \in \mathbb{C}(x, y).$$

Then

$$f \text{ has a telescoper } \iff \operatorname{den}(r) \text{ is integer-linear.}$$

Creative Telescoping

Classical: Zeilberger's algorithm (1990)

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Example. Consider

$$f = \frac{y^{10}}{x+y}$$

- The minimal telescoper L for f is

$$L = S_x - \frac{1}{x^{10}}(x^{10} + 10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1)$$

Certificate for the example

$$
\begin{aligned}
G = \frac{1}{10} \Bigg(& -1/21 \, \frac{x^3 \left(175\,x^7+700\,x^6+1234\,x^5+1252\,x^4+790\,x^3+310\,x^2+70\,x+7\right)}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& - \frac{1}{42}\,\frac{x\left(1750\,x^7+5950\,x^6+9558\,x^5+9186\,x^4+5630\,x^3+2180\,x^2+490\,x+49\right)y^2}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& - \frac{1}{18}\,\frac{\left(990\,x^9+3960\,x^8+7890\,x^7+10260\,x^6+9654\,x^5+6780\,x^4+3490\,x^3+1240\,x^2+270\,x+27\right)y^3}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& + \frac{5}{36}\,\frac{x\left(792\,x^7+2574\,x^6+4020\,x^5+3801\,x^4+2310\,x^3+891\,x^2+200\,x+20\right)y^4}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& + \frac{1}{12}\,\frac{\left(1320\,x^9+5280\,x^8+11352\,x^7+16566\,x^6+17540\,x^5+13535\,x^4+7410\,x^3+2721\,x^2+600\,x+60\right)y^5}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& - \frac{1}{6}\,\frac{x\left(660\,x^7+1980\,x^6+2948\,x^5+2717\,x^4+1630\,x^3+625\,x^2+140\,x+14\right)y^6}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& - \frac{1}{42}\,\frac{\left(4620\,x^9+18480\,x^8+42900\,x^7+68640\,x^6+78188\,x^5+63305\,x^4+35630\,x^3+13265\,x^2+2940\,x+294\right)y^7}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& + \frac{5}{84}\,\frac{x\left(924\,x^7+2310\,x^6+3168\,x^5+2805\,x^4+1650\,x^3+627\,x^2+140\,x+14\right)y^8}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& + \frac{5}{36}\,\frac{\left(660\,x^9+2640\,x^8+6732\,x^7+11550\,x^6+13728\,x^5+11385\,x^4+6490\,x^3+2431\,x^2+540\,x+54\right)y^9}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} \\
& - \frac{1}{9}\,\frac{\left(495\,x^9+2145\,x^8+5610\,x^7+9702\,x^6+11550\,x^5+9570\,x^4+5445\,x^3+2035\,x^2+451\,x+45\right)y^{10}}{10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1} + y^{11} \Bigg) \\
& \cdot \left(10\,x^9+45\,x^8+120\,x^7+210\,x^6+252\,x^5+210\,x^4+120\,x^3+45\,x^2+10\,x+1\right)(x+y)^{-1}
\end{aligned}
$$

Creative telescoping (cont.)

New: Reduction-based telescoping (2015)

- a difference variant of Hermite telescoping (2010)
- separate the computation of telescopers from certs

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Notation. For $f, h \in \mathbb{C}(x, y)$, denote

$$f \equiv_y h$$

if $f - h$ is summable w.r.t. y.

Telescoping via reduction

Goal. Given $f \in \mathbb{C}(x, y)$, find a telescoper L for f w.r.t. y.

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$$L = \sum_{j=0}^{\rho} c_j S_x^j$$

with $\rho \in \mathbb{N}$ and $c_j \in \mathbb{C}(x)$, not all zero.

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- Compute residual form r_j s.t.
$$S_x^j(f) \equiv_y r_j.$$

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- Compute residual form r_j s.t.
$$S_x^j(f) \equiv_y r_j.$$
- Apply L to f:
$$L(f) = \sum_{j=0}^{\rho} c_j S_x^j(f) \equiv_y \sum_{j=0}^{\rho} c_j r_j$$

Key problem

Consider

$$L(f) \equiv_y \sum_{j=0}^{\rho} c_j r_j$$

Is

L is a telescoper for f	$\overset{?}{\Longleftrightarrow}$	$\sum_{j=0}^{\rho} c_j r_j = 0$

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$\sum_{j=0}^{\rho} c_j r_j$ may not be a residual form $\leftarrow$

Key problem

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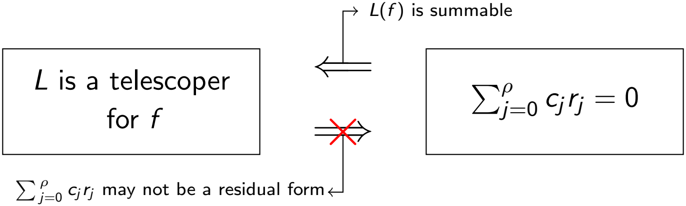
$$L(f) \equiv_y \sum_{j=0}^{\rho} c_j r_j$$



Idea. Choose appropriate r_j s.t. $\sum_{j=0}^{\rho} c_j r_j$ is a residual form.

Telescoping via reduction(cont.)

- Find residual forms r'_j for $j = 0, \ldots, \rho$ s.t.

$$r_j \equiv_y r'_j \text{ and } \sum_{j=0}^{\rho} c_j r'_j \text{ is a residual form.}$$

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Telescoping via reduction(cont.)

- Find residual forms r_j' for $j = 0, \ldots, \rho$ s.t.

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- A linear dependence among $\{r_j'\}_{j=0}^{\rho}$ gives a telescoper.

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Remarks.

- The first linear depend. leads to the minimal telescoper.
- One can leave the certificate as an un-normalized sum.

Example

Consider

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- $S_x^2(f) = \Delta_y\left(\frac{y-1}{(x+y+1)^2}\right) + \frac{y-1}{(x+y+1)^2}$

  $= \Delta_y\left(\frac{y-1}{(x+y+1)^2} + \frac{y-2}{(x+y)^2}\right) + \frac{y-2}{(x+y)^2}$
- $L = S_x^2 - 2S_x + 1$
- $G = \frac{y-1}{(x+y+1)^2} + \frac{y-2}{(x+y)^2} - 2\frac{y-1}{(x+y)^2} = -\frac{x^2+4xy+3y^2+y}{(x+y)^2(x+1+y)^2}$

Timings (in seconds)

Let

$$T = \frac{f(x,y)}{g_1(x+y)g_2(x+y)}$$

with

- $g_i(z) = p_i(z)p_i(z+\lambda)p_i(z+\mu)$, $\lambda, \mu \in \mathbb{N}$,
- $\deg(p_1) = \deg(p_2) = m$ and $\deg(f) = n$.

(m, n, λ, μ)	Zeilberger	TvR+cert	TvR	order
$(1, 0, 5, 5)$	1.27	0.44	0.24	3
$(1, 8, 5, 5)$	6.53	1.02	0.45	4
$(2, 0, 5, 10)$	20.99	4.01	1.03	3
$(2, 3, 5, 10)$	23.28	4.65	1.12	3
$(2, 0, 10, 15)$	84.26	12.76	1.66	3
$(2, 5, 10, 15)$	85.15	13.91	1.94	3
$(3, 0, 5, 10)$	587.84	29.88	5.61	5

Hypergeometric terms

A nonzero term $T(x, y)$ is hypergeometric if

$$\frac{S_x(T)}{T}, \frac{S_y(T)}{T} \in \mathbb{C}(x, y).$$

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Remark. Our paper in Proc. ISSAC 2015:

A modified Abramov-Petkovšek reduction and
creative telescoping for hypergeometric terms

Summary

Results.

- Telescoping via Abramov's reduction
- Generalization for hypergeometric terms

Future work.

- Creative telescoping for q-hypergeometric terms