Efficient Rational Creative Telescoping

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Joint work with Mark Giesbrecht, George Labahn and Eugene Zima

Outline

- Technique of creative telescoping
- New approach for rational functions

The creative telescoping problem

GIVEN $f(n,k)$, FIND $g(n,k)$ s.t.

$$f(n,k) \ = \ g(n,k+1) - g(n,k).$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$F(n) \ = \sum_{k=0}^{n} \big(g(n,k+1) - g(n,k)\big).$$

The creative telescoping problem

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Then $F(n)=\sum_{k=0}^{n} f(n,k)$ satisfies

$$F(n) \;=\; g(n,n+1)-g(n,0).$$

The creative telescoping problem

GIVEN $k \cdot k!$, FIND $k!$ s.t.

$$k \cdot k! \; = \; (k+1)! - k!.$$

Then $F(n) = \sum_{k=0}^{n} k \cdot k!$ satisfies

$$F(n) \; = (n+1)! - 1.$$

The creative telescoping problem

GIVEN $\binom{n}{k}$, FIND $g(n,k)$ s.t.

$$\binom{n}{k} = g(n,k+1) - g(n,k).$$

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FAIL!

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The creative telescoping problem

GIVEN $\binom{n}{k}$, FIND $-2, 1$ and $-\binom{n}{k-1}$ s.t.

$$-2\binom{n}{k} + \binom{n+1}{k} = -\binom{n}{k} - \left(-\binom{n}{k-1}\right).$$

Then $F(n) = \sum_{k=0}^{n} \binom{n}{k}$ satisfies

$$-2F(n) + F(n+1) = 0.$$

The creative telescoping problem

GIVEN $f(n,k)$, FIND $c_0(n),\ldots,c_\rho(n)$ and $g(n,k)$ s.t.

$$c_0(n)f(n,k)+\cdots+c_\rho(n)f(n+\rho,k) \;=\; g(n,k+1)-g(n,k).$$

Then $F(n)=\sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n)+\cdots+c_\rho(n)F(n+\rho)=\text{explicit}(n).$$

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GIVEN $f(n,k)$, FIND $c_0(n),\ldots,c_\rho(n)$ and $g(n,k)$ s.t.

$$\big(c_0(n)+\cdots+c_\rho(n)S_n^\rho\big)\big(f(n,k)\big) \;=\; (S_k-1)\big(g(n,k)\big)$$

Then $F(n)=\sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n)+\cdots+c_\rho(n)F(n+\rho)=\text{explicit}(n).$$

Notation. $S_n(f(n,k))=f(n+1,k)$ and $S_k(f(n,k))=f(n,k+1)$.

The creative telescoping problem

GIVEN $f(n,k)$, FIND $c_0(n),\ldots,c_\rho(n)$ and $g(n,k)$ s.t.

$$\underbrace{\left(c_0(n)+\cdots+c_\rho(n)S_n^\rho\right)}_{\text{telescoper}}\big(f(n,k)\big) \;=\; (S_k-1)\underbrace{\big(g(n,k)\big)}_{\text{certificate}}$$

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Generations of creative telescoping algorithms

1. Elimination in operator algebras / Sister Celine's algorithm (since $\approx$ 1947)
2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)
3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)
4. The reduction-based approach (since $\approx$ 2010)

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Integer-linear decompositions

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Definition. $p \in \mathbb{C}[n, k]$ is integer-linear over $\mathbb{C}$ if

$$p = \prod_{i=1}^{m} P_i(\lambda_i n + \mu_i k)^{e_i}$$

- $P_i(z) \in \mathbb{C}[z]$ irreducible;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$;
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Abramov-Le's criterion. $f \in \mathbb{C}(n,k)$ with $f = (S_k - 1)\big(\cdots\big) + \frac{a}{b}$.

$$f \text{ has a telescoper} \quad \Longleftrightarrow \quad b \text{ is integer-linear.}$$

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$$P_i(\lambda_i n + \mu_i k) \sim_{n,k} P_j(\lambda_j n + \mu_j k),\ i \neq j$$

$$\Updownarrow$$

$$(\lambda_i, \mu_i) = (\lambda_j, \mu_j)\ \&\ P_i(z) = P_j(z + \nu),\ \nu \in \mathbb{Z}$$

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$$p = \prod_{i=1}^{m} \prod_{j=1}^{n_i} P_i(\lambda_i n + \mu_i k + \nu_{ij})^{e_{ij}}$$

- $P_i(z) \in \mathbb{C}[z]$ irreducible;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$ coprime, $\mu_i \geq 0$;
- $e_{ij} \in \mathbb{Z}^+$; $0 = \nu_{i1} < \cdots < \nu_{in_i}$ in $\mathbb{Z}$;
- $P_i(\lambda_i n + \mu_i k) \not\sim_{n,k} P_j(\lambda_j n + \mu_j k), \;\; i \neq j$.

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- $P_i(z) \in \mathbb{C}[z]$ squarefree, $\gcd(P_i, P_i(z+\ell)) = 1$, $\forall\, \ell \in \mathbb{Z}\backslash\{0\}$;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$ coprime, $\mu_i \geq 0$;
- $e_{ij} \in \mathbb{Z}^+$; $0 = \nu_{i1} < \cdots < \nu_{in_i}$ in $\mathbb{Z}$;
- $(\lambda_i, \mu_i) \neq (\lambda_j, \mu_j)$ or $\gcd(P_i(z), P_j(z+\ell)) = 1$, $\forall\, \ell \in \mathbb{Z}$, $i \neq j$.

Integer-linear decompositions

Definition. $p \in \mathbb{C}[n,k]$ admits the integer-linear decomposition

$$p = P_0(n,k) \cdot \prod_{i=1}^{m} \prod_{j=1}^{n_i} P_i(\lambda_i n + \mu_i k + \nu_{ij})^{e_{ij}}$$

- $P_0 \in \mathbb{C}[n,k]$ merely having non-integer-linear factors except for constants;
- $P_i(z) \in \mathbb{C}[z] \setminus \mathbb{C}$ squarefree, $\gcd(P_i, P_i(z+\ell)) = 1$, $\forall\, \ell \in \mathbb{Z}\setminus\{0\}$;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$ coprime, $\mu_i \geq 0$;
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$$S_{\lambda,\mu}(P(\lambda n + \mu k)) = P(\lambda n + \mu k + 1)$$

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Integer-linear operators of type (λ, μ)

Integer-linear operators

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$$\phi_{\lambda,\mu} : \quad \underbrace{\mathbb{C}(n,k)[S_n, S_k, S_n^{-1}, S_k^{-1}]}_{\mathcal{A}} \quad \longrightarrow \quad \underbrace{\mathbb{C}(n,k)[S_{\lambda,\mu}, S_{\lambda,\mu}^{-1}]}_{\mathcal{A}_{\lambda,\mu}}$$

$$\sum_{i,j\in\mathbb{Z}} a_{ij} S_n^i S_k^j \quad \longmapsto \quad \sum_{i,j\in\mathbb{Z}} a_{ij} S_{\lambda,\mu}^{i\lambda+j\mu}$$

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- Division with remainder: $\forall\, M \in \mathcal{A}_{\lambda,\mu}, \exists!\, Q, R \in \mathcal{A}_{\lambda,\mu}$ s.t.

$$M = (S_k - 1) \odot Q + R,$$

and either $R = 0$ or $0 \le \mathrm{ldeg}_{S_{\lambda,\mu}}(R) \le \deg_{S_{\lambda,\mu}}(R) < \mu - 1$.

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and either $R = 0$ or $0 \le \operatorname{ldeg}_{S_{\lambda,\mu}}(R) \le \deg_{S_{\lambda,\mu}}(R) < \mu - 1$.

Our new approach

Example. $\frac{-n}{(nk+1)(nk+n+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}$

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$$f = \frac{*}{(nk+1)(nk+n+1)((n+2k)^2+2)((n+2k+2)^2+2)((n+2k+22)^2+2)}$$

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$$f = \frac{*}{\underbrace{(nk+1)(nk+n+1)}_{P_0(n,k)}\underbrace{((n+2k)^2+2)}_{P_1(n+2k)}\underbrace{((n+2k+2)^2+2)}_{P_1(n+2k+2)}\underbrace{((n+2k+22)^2+2)}_{P_1(n+2k+22)}}$$

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$$f = (S_k - 1)\left(\frac{1}{nk+1}\right) + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}$$

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$$f = (S_k - 1)\left(\frac{1}{nk+1}\right) + M\left(\frac{1}{(n+2k)^2+2}\right)$$

M: $nk - n(k+1)S_{1,2}^{2} + n(k+11)S_{1,2}^{22}$

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Example. $$\underbrace{\frac{-n}{(nk+1)(nk+n+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

$$L(f) \;=\; L \cdot (S_k - 1)\left(\frac{1}{nk+1}\right) + L \cdot M\left(\frac{1}{(n+2k)^2+2}\right)$$

$$L = c_0(n) + c_1(n)S_n + c_2(n)S_n^2 + c_3(n)S_n^3 + c_4(n)S_n^4$$

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$$L(f) = (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right)\right) + (L \odot M)\left(\frac{1}{(n+2k)^2+2}\right)$$

$$= (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right)\right) + ((S_k - 1) \odot Q + R)\left(\frac{1}{(n+2k)^2+2}\right)$$

$$R = \big(c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2)\big) + \big(c_1(n)(n+1)k + c_3(n)(n+3)(k-1)\big)S_n$$

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$$L(f) = (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right)\right) + (L \odot M)\left(\frac{1}{(n+2k)^2+2}\right)$$

$$= (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right) + Q\left(\frac{1}{(n+2k)^2+2}\right)\right) + R\left(\frac{1}{(n+2k)^2+2}\right)$$

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$$R = \big(c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2)\big) + \big(c_1(n)(n+1)k + c_3(n)(n+3)(k-1)\big)S_n \stackrel{!}{=} 0$$

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Example. $\underbrace{\frac{-n}{(nk+1)(nk+n+1)} + \frac{nk}{(n+2k)^2+2} - \frac{n(k+1)}{(n+2k+2)^2+2} + \frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

$$L(f) = (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right)\right) + (L \odot M)\left(\frac{1}{(n+2k)^2+2}\right)$$

$$= (S_k - 1)\left(L\left(\tfrac{1}{nk+1}\right) + Q\left(\frac{1}{(n+2k)^2+2}\right)\right) + R\left(\frac{1}{(n+2k)^2+2}\right)$$

$$\begin{cases} c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2) = 0 \\ c_1(n)(n+1)k + c_3(n)(n+3)(k-1) = 0 \end{cases}$$

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- A certificate: $g = L\left(\frac{1}{nk+1}\right) + LQ(L \odot M, S_k - 1)\left(\frac{1}{(n+2k)^2+2}\right)$

Worst-case complexity (field operations)

Given $f \in \mathbb{C}(n,k)$ with $\deg_n(f) \le d_n$ and $\deg_k(f) \le d_k$.

RCT	NCT
$O\tilde{}(\mu^{\omega+2} d_n d_k^{\omega+3})$	$O\tilde{}(\mu^{\omega+1} d_n d_k^{\omega+2})$

- $\mu \in \mathbb{Z}^+$, $2 \le \omega \le 3$
- Without expanding the certificate
- Order of a minimal telescoper: $O(\mu d_k)$
- Size of a minimal telescoper: $O(\mu^2 d_n d_k^3)$

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Timings (in seconds)

Test suite: $f(n,k) = (S_k - 1)\left(\frac{f_0(n,k)}{P_0(n,k)}\right) + \frac{a(n,k)}{P_1(2n+\mu k)\cdot P_2(4n+\mu k)}$.

- $P_i(z) = p_i(z)\cdot p_i(z+2^i)\cdot p_i(z+\mu)\cdot p_i(z+2^i+\mu)$,
- $\mu \in \mathbb{Z}$, $\deg_{n,k}(a) = d_1$, $\deg_{n,k}(P_0) = \deg_z(p_i) = d_2$.

(d_1, d_2, μ)	RCT	NCT	Order
(1, 1, 1)	0.28	0.19	3
(1, 2, 1)	5.86	2.15	7
(1, 3, 1)	283.84	30.94	11
(1, 4, 1)	5734.80	448.09	15
(10, 2, 1)	7.79	3.18	7
(20, 2, 1)	9.49	4.21	7
(30, 2, 1)	16.57	10.17	8
(30, 2, 3)	807.31	41.16	12
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