An Improved Abramov–Petkovšek Reduction for Hypergeometric Terms

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Outline

- Hypergeometric summability
- Gosper algorithm
- Abramov–Petkovšek reduction
- Improved Abramov–Petkovšek reduction

Hypergeometric terms

Let C be a field of characteristic zero and x an indeterminate.

Definition. A term $T(x)$ is hypergeometric over $C(x)$ if its shift quotient $T(x+1)/T(x)$ is in $C(x)$.

Examples.

$r(x) \in C(x) \setminus \{0\}$, c^x with $c \in C \setminus \{0\}$, $x!$ and binomial coefficients, etc.

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Notation.

- For $r \in C(x)$, r_d and r_n denote the denominator and numerator of r, respectively.
- $\Delta(r(x)) = r(x+1) - r(x)$.

Summability

A hypergeometric term $T(x)$ is said to be summable if there exists a hypergeometric term $G(x)$ s.t.

$$T(x) = \Delta(G(x)).$$

Example.

- $x!$ is not summable
- $x \cdot x!$ is summable

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Fact.

$$T(x) \text{ is summable}$$

$$\Updownarrow$$

$$\frac{T(x+1)}{T(x)} z(x+1) - z(x) = 1 \text{ has a solution in } C(x).$$

Gosper algorithm (1978)

Given a hypergeometric term $T(x)$, compute $a(x)$, $b(x)$, $c(x) \in C[x]$ s.t.

$$\frac{T(x+1)}{T(x)} = \frac{a(x)}{b(x)} \frac{c(x+1)}{c(x)}$$

and $\gcd(a(x), b(x+\ell)) = 1$ for all $\ell \in \mathbb{N}$. Then

$T(x)$ is summable

$$\Updownarrow$$

$$a(x)z(x+1) - b(x-1)z(x) = c(x)$$

has a polynomial solution in $C[x]$,

Kernels and shells

Definition. A rational function $r \in C(x)$ is shift-reduced if

$$\gcd\left(r_n,\, r_d(x+\ell)\right) = 1 \quad \text{for all } \ell \in \mathbb{Z}.$$

For $r \in C(x)$, $\exists\ K, S \in C(x)$ with K being shift-reduced s.t.

$$r(x) = K(x) \cdot \frac{S(x+1)}{S(x)}.$$

Let $T(x)$ be a hypergeometric term with $r = T(x+1)/T(x)$. Call

- K: a kernel of T;
- S: the shell of T w.r.t. K.

Multiplicative decomposition.

$$T(x) = S(x) \cdot H(x) \quad \text{with } H(x+1)/H(x) = K(x).$$

Shift-freeness and strong co-primeness

A polynomial $a(x) \in C[x]$ is shift-free if

$$\gcd\left(a(x), a(x+\ell)\right) = 1 \quad \text{for all } \ell \in \mathbb{Z} \setminus \{0\}.$$

Assume that $r \in C(x)$ is shift-reduced, an irreducible polynomial $q(x) \in C[x]$ is strongly co-prime with r if

- $q \nmid r_d r_n$,
- $q(x+m) \nmid r_n$ for all $m \in \mathbb{Z}^+$,
- $q(x-m) \nmid r_d$ for all $m \in \mathbb{Z}^+$.

A nonzero polynomial is strongly co-prime with r if all its irreducible factors are strongly co-prime with r.

Abramov–Petkovšek reduction (2001)

Let T be a hypergeometric term with a kernel K and shell S.

Set $H = T/S$.

- Decompose

$$T = \Delta(f\,H) + \underbrace{\frac{w}{K_d}H}_{\text{possibly summable}} + \underbrace{\frac{a}{b}H}_{\text{not summable}},$$

  where $f \in C(x)$ and $w, a, b \in C[x]$ with $\deg(a) < \deg(b)$.

- Moreover, b is shift-free and strongly co-prime with K.

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Proposition. "possibly summable" is summable iff

$$K_n z(x+1) - K_d z(x) = w$$

has a polynomial solution in $C[x]$.

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$$\frac{w}{K_d} H$$

further to

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Idea:

- Abramov and Petkovšek (2001): reduce the leading term of w;
- Bostan et. al. (2013): Hermite reduction for hyperexp functs. (Polynomial reduction: reducing the number of terms.)

Polynomial reduction

Given $K \in C(x)$, define

$$\begin{array}{rccc} \phi_K: & C[x] & \longrightarrow & C[x] \\ & a & \mapsto & K_n a(x+1) - K_d a(x). \end{array}$$

Call ϕ_K the polynomial reduction map w.r.t. K.

Write

$$C[x] = \mathrm{im}\,(\phi_K) \oplus \mathcal{N}_K,$$

where

$$\mathcal{N}_K = \mathrm{span}_C \left\{ x^i \mid i \neq \deg(a) \text{ for all } a \in \mathrm{im}\,(\phi_K) \right\}.$$

Call $\mathcal{N}_K$ the standard complement of $\mathrm{im}(\phi_K)$.

Properties

Lemma. Assume that K is shift-reduced and $K \neq 1$. Then

- $\operatorname{im}(\phi_K)$ has a C-basis $\{\phi_K(x^i) \mid i \in \mathbb{N}\}$,
- $\dim_C(\mathcal{N}_K) \leq \max(\deg(K_n), \deg(K_d))$.

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Polynomial reduction.

Decompose $w = p_1 + p_2$ with $p_1 \in \mathrm{im}\left(\phi_K\right)$ and $p_2 \in \mathcal{N}_K$.

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Solution to the key problem.

$$\underbrace{\frac{w}{K_d} H}_{\text{possibly summable}} = \Delta\left(\phi_K^{-1}(p_1) H\right) + \underbrace{\frac{p_2}{K_d} H}_{\text{non-summable}} .$$

Improved Abramov-Petkovšek reduction

Theorem. Let T be hypergeometric with a kernel K and shell S. Set $H = T/S$. Then one can decompose

$$T = \underbrace{\Delta(gH)}_{\text{summable}} + \underbrace{\left(\frac{p}{K_d} + \frac{a}{b}\right) H}_{\text{non-summable}}, \quad (*)$$

where

- $g \in C(x)$
- $p \in \mathcal{N}_K$
- $\deg(a) < \deg(b)$
- b is shift-free and strongly co-prime with K.

Moreover, assume that $(*)$ holds. Then

$$T \text{ is summable} \iff p = a = 0.$$

Uniqueness

Two decompositions

$$\begin{aligned} T &= \Delta(gH) + \left(\frac{p}{K_d} + \frac{a}{b}\right) H \\ &= \Delta(g'H) + \left(\frac{p'}{K_d} + \frac{a'}{b'}\right) H, \end{aligned}$$

Then

$$\{\text{irr. factors of } b\} \overset{1:1}{\sim} \{\text{irr. factors of } b'\}.$$

Examples

1. $T = x!$.
   - $K = x + 1$ and $S = 1$.
   - AP reduction: $T = 0 + H$ where $H = x!$.
   - Polynomial reduction: $1 \in \mathcal{N}_K \implies x!$ is not summable.
2. $T = x \cdot x!$.
   - $K = x + 1$ and $S = x$.
   - AP reduction: $T = 0 + x \cdot H$ where $H = x!$.
   - Polynomial reduction: $x = \phi_K(1) \implies x \cdot x! = \Delta(x!)$.

Timings (in seconds)

Examples in the following form:

$$T := \underbrace{\frac{a(x)}{p_1(x)p_1(x+\lambda)p_1(x+\mu)p_2(x)p_2(x+\lambda)p_2(x+\mu)}}_{\text{shell}} \prod_{i=1}^{x} r(i),$$

where

- $\deg(a) = 20$ and $\deg(p_1) = \deg(p_2) = 10$,
- $\deg(r_d) = \deg(r_n) = 2$,
- $\lambda, \mu \in \mathbb{Z}^+$ and $\lambda < \mu$.

Results

Non-summable inputs: Gosper $<$ Improved AP $<$ AP.

Summable inputs: $\Delta(T)$.

(λ, μ)	Gosper in Maple	Improved AP	AP in Maple
(10, 15)	4.009	13.089	73.133
(10, 20)	7.488	13.058	103.694
(10, 25)	12.09	15.163	176.297
(10, 30)	18.439	15.148	323.702
(10, 40)	50.107	15.272	540.497
(10, 50)	121.182	20.015	984.148

- Improved AP $<$ AP;
- Improved AP is faster than Gosper for large dispersions.

Summary

Result.

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Future work.

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Thank you!