D-finite Numbers

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joint work with Manuel Kauers

Motivation

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D-finite functions

$$p_\rho(z)f^{(\rho)}(z) + \cdots + p_0(z)f(z) = 0$$

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$\cup\!|$ Abel's theorem

Algebraic functions

roots $y(z)$ of $P(z, y) \in \mathbb{Q}[z, y]$

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quotients of polynomials

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$p_\rho(z)f^{(\rho)}(z) + \cdots + p_0(z)f(z) = 0$

D-finite numbers

Limits of P-recursive seqs

$\subseteq$ Abel's theorem

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Formal power series vs. analytic complex functions

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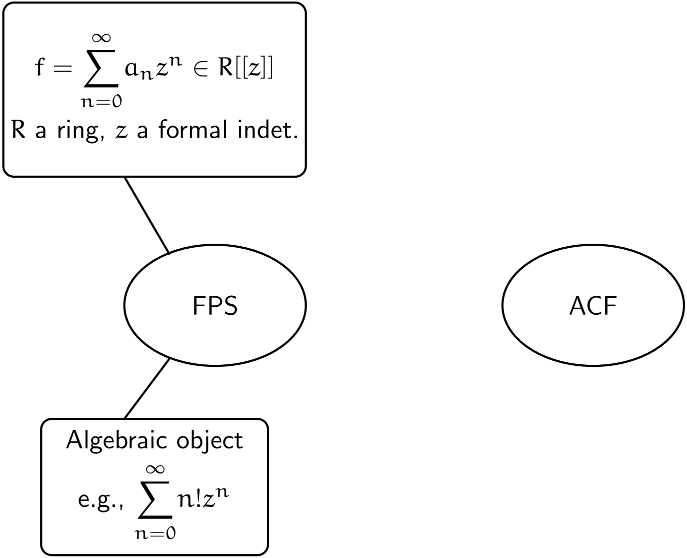
R a ring, z a formal indet.

FPS

ACF

Formal power series vs. analytic complex functions



Formal power series vs. analytic complex functions

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$f\colon U \to \mathbb{C}$, $U \subseteq \mathbb{C}$ open,

$\forall \zeta \in U$, $\forall z$ near ζ,

$f = \sum_{n=0}^{\infty} a_n (z-\zeta)^n$ converg.

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Algebraic object

e.g., $\sum_{n=0}^{\infty} n! z^n$

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Analytic object

e.g., $\exp(z)$

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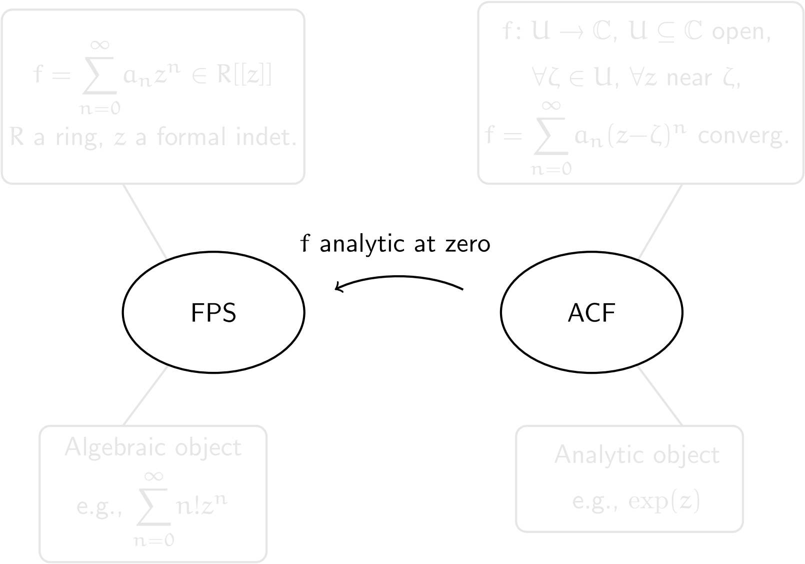

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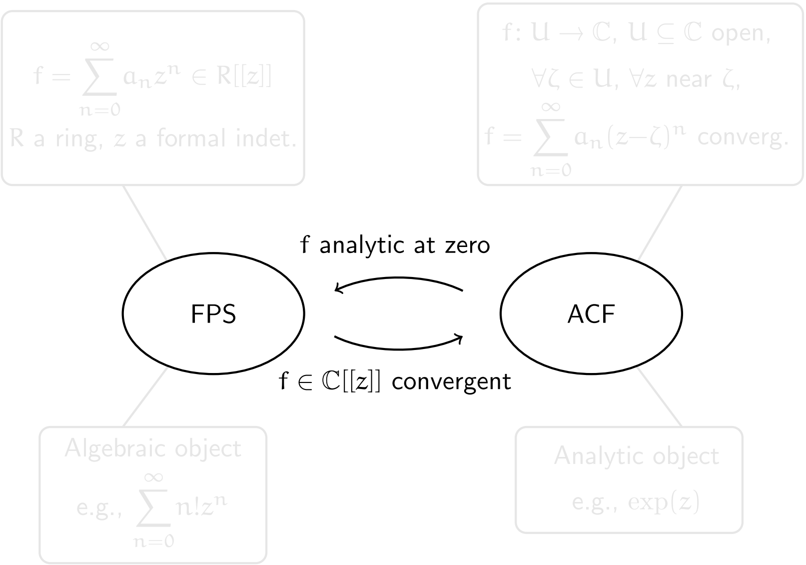

Algebraic functions and sequences

$\mathbb{F}$ a subfield of $\mathbb{C}$.

Definition. $f \in \mathbb{F}[[z]]$ is algebraic over $\mathbb{F}$ if there exists nonzero $P(z, y) \in \mathbb{F}[z, y]$ s.t. $P(z, f(z)) = 0$.

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Goal. Study

$$\mathcal{A}_{\mathbb{F}} = \left\{ \lim_{n \to \infty} a_n \,\middle|\, (a_n)_n \in \mathbb{F}^{\mathbb{N}} \text{ algebraic and convergent} \right\}.$$

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Answer: Not clear!

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- If $\mathbb{F} \subseteq \mathbb{R}$ then $\bar{\mathbb{F}} \cap \mathbb{R} \subseteq \mathcal{A}_{\mathbb{F}}$.
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$$\frac{\sqrt{z + 49}}{5(1 - z)} = \tfrac{7}{5} + \tfrac{99}{70} z + \tfrac{19403}{13720} z^2 + \tfrac{380299}{268912} z^3 + \tfrac{149077207}{105413504} z^4 + \cdots \in \mathbb{Q}[[z]]$$

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$$\approx 1.4 + 1.41428z + 1.414212z^2 + 1.41421357z^3 + 1.41421356z^4 + \cdots$$

Sum up

Theorem.

- If $\mathbb{F} \subseteq \mathbb{R}$ then $\mathcal{A}_{\mathbb{F}} = \bar{\mathbb{F}} \cap \mathbb{R}$.
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Sum up

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Conclusion. $\mathcal{A}_{\mathbb{F}}$ is a field!

D-finite functions and sequences

R a subring of $\mathbb{C}$ and $\mathbb{F}$ a subfield of $\mathbb{C}$.

Definition. $f \in R[[z]]$ is D-finite over $\mathbb{F}$ if there exists $p_0, \ldots, p_\rho \in \mathbb{F}[z]$, not all zero , s.t.

$$p_\rho(z) D_z^\rho f(z) + \cdots + p_0(z) f(z) = 0.$$

Definition. $(a_n)_{n=0}^{\infty} \in R^{\mathbb{N}}$ is P-recursive over $\mathbb{F}$ if there exists $p_0, \ldots, p_\rho \in \mathbb{F}[n]$, not all zero , s.t.

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Remark. $\sum_{n=0}^{\infty} a_n z^n$ D-finite $\iff$ $(a_n)_n$ P-recursive.

D-finite numbers

Definition. $\zeta \in \mathbb{C}$ is D-finite (w.r.t. R and $\mathbb{F}$) if there exists $(a_n)_n \in R^{\mathbb{N}}$ convergent and P-recursive over $\mathbb{F}$ s.t.

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Goal. Study

$$\mathcal{D}_{R,\mathbb{F}} = \{\text{all D-finite numbers w.r.t. R and } \mathbb{F}\}.$$

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- $\gamma = \sum_{n=1}^{\infty} (-1)^k \binom{n}{k} \frac{1}{k} \left(1 - \frac{1}{k!}\right)$

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- $\Gamma(\alpha) = \sum_{k=0}^{\infty} \binom{n+\alpha}{k+\alpha} \frac{(-1)^k}{k!(k+\alpha)}$ with $\alpha < 1$ and $\alpha \in \mathbb{Q}$
- $\gamma = \sum_{n=1}^{\infty} (-1)^k \binom{n}{k} \frac{1}{k} \left(1 - \frac{1}{k!}\right)$
- ......

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5. If $\mathrm{R} \subseteq \mathbb{F}$ then $\mathcal{D}_{\mathrm{R},\mathbb{F}} = \mathcal{D}_{\mathrm{R},\mathrm{Quot}(\mathrm{R})}$.
6. If R and $\mathbb{F}$ are closed under $(\ \bar{}\ )$, then so is $\mathcal{D}_{\mathrm{R},\mathbb{F}}$ and $\mathcal{D}_{\mathrm{R},\mathbb{F}} = \mathcal{D}_{\mathrm{R}\cap\mathbb{R},\mathbb{F}} + i\mathcal{D}_{\mathrm{R}\cap\mathbb{R},\mathbb{F}}$ (if $i \in \mathcal{D}_{\mathrm{R},\mathbb{F}}$).

D-finite numbers are "evaluations"

Theorem. For every $\xi \in \mathcal{D}_{R,\mathbb{F}}$, there exists $g(z) \in R[[z]]$ D-finite over $\mathbb{F}$ s.t. $\xi = \lim_{z \to 1^-} g(z)$.

Proof via example. $\zeta(3) = \sum_{n=0}^{\infty} \frac{1}{n^3} = \lim_{z \to 1^-} \mathrm{Li}_3(z) = \mathrm{Li}_3(1)$.

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$$+ \log(z-1) \cdot \Big(* \cdot (1-z) + * \cdot (1-z)^2 + \cdots \Big)$$

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Corollary. D-finite numbers are computable when R and $\mathbb{F}$ are.

Evaluations are D-finite numbers

Theorem.

Let $R \supseteq \mathbb{F}$ and $f \in \mathcal{D}_{R,\mathbb{F}}[[z]]$. If there exists $L \in \mathbb{F}[z][D_z] \setminus \{0\}$ with zero ordinary s.t. $L \cdot f = 0$, then $\forall \zeta \in \bar{\mathbb{F}}$ non-singular and $\forall k \in \mathbb{N}$,

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Proof. Algebraic case + analytic continuation.

Analytic continuation

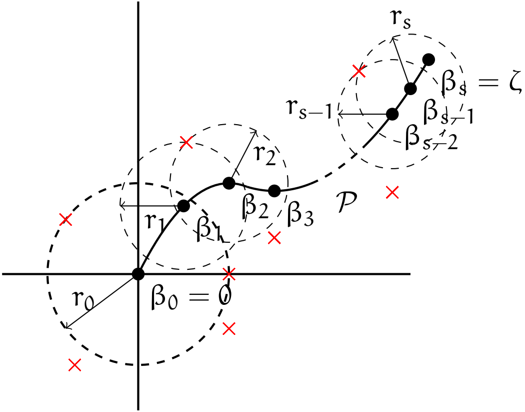

Figure: a simple path $\mathcal{P}$ with a finite cover $\bigcup_{j=0}^{s} \mathcal{B}_{r_j}(\beta_j)$, $\beta_j \in \mathbb{F}$ ($\times$ stands for singularities of L)

Examples of $\mathcal{D}_{\mathbb{Q},\mathbb{Q}}$ (cont.)

- $1/\mathrm{e}$, $\sqrt{\mathrm{e}}$, $\exp(\sqrt{2})$

- $\sqrt{2}^{\sqrt{2}}$

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- e^{π} : $\mathrm{f} = (z+1)^{-i} \in \mathbb{Q}(i)[[z]]$ annihilated by

$$\mathrm{L} = (z+1)\mathrm{D}_z + i$$

Open questions

- Evaluation at singularities
- Quotients of D-finite numbers

Summary

D-finite functions

$p_\rho(z)f^{(\rho)}(z) + \cdots + p_0(z)f(z) = 0$

D-finite numbers

Limits of P-recursive seqs

$\mid\bigcup$ Abel's theorem

Algebraic functions

roots $y(z)$ of $P(z, y) \in \mathbb{Q}[z, y]$

Algebraic numbers

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$e,\ \pi,\ \zeta(3),\ \gamma,\ e^{\pi},\ \ldots$

⊇ Abel's theorem (functions) ⊇ (numbers)

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