Order Bases : Computation and uses in Computer Algebra

George Labahn

Symbolic Computation Group
Cheriton School of Computer Science
University of Waterloo, Canada

Sardinia, Oct 14, 2011

Purpose

We give a bit of information on the topic of **Order Bases**:

Specifically:

- What are Order Bases?
- How are order bases used
  - (particularly in field of Computer Algebra)?
- How to compute order bases quickly?

Outline

1. Introduction
   - General Setting
2. Rational Approximation
   - Linear Systems
3. Order Bases
   - Background
   - Computation
   - Fraction-Free Computation
   - Recursive Computation
   - Matrix Normal Forms
   - Current Activities

Hermite-Padé Approximation

Given power series $A_1(z), \ldots, A_m(z)$ and integers $n_0, \ldots, n_m$

Find $P_1(z), ..., P_m(z)$ with deg $P_i(z) \leq n_i$ and

$$A_1(z)P_1(z) + \cdots + A_m(z)P_m(z) \approx 0 \ .$$

Hermite-Padé Approximation

Given power series $A_1(z), \ldots, A_m(z)$ and integers $n_0, \ldots, n_m$

Find $P_1(z), ..., P_m(z)$ with deg $P_i(z) \leq n_i$ and

$$A_1(z)P_1(z) + \cdots + A_m(z)P_m(z) \approx 0 \; .$$

Formally:

$$A_1(z)P_1(z) + \cdots + A_m(z)P_m(z) = r_0 z^{N+1} + r_1 z^{N+2} + \cdots$$

with $N = n_1 + \cdots + n_m + m - 1$.

($m = 2$ and $A_1(z) = -1$ gives Padé approximation)

Examples

Given power series $y(z)$ find $P_0(z), P_1(z), P_2(z)$ such that

- $$P_0(z)y(z) + P_1(z)y'(z) + P_2(z)y''(z) \approx 0$$

- $$P_0(z) \quad + \quad P_1(z)y(z) \quad + \quad P_2(z)y^2(z) \approx 0$$

  (generalized rational reconstruction)

Outline

1. Introduction
   - General Setting
2. Rational Approximation
   - Linear Systems
3. Order Bases
   - Background
   - Computation
   - Fraction-Free Computation
   - Recursive Computation
   - Matrix Normal Forms
   - Current Activities

Rational Approximation Problems

Given $m \times m$ matrix $\mathbf{A}(z)$, orders, $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$, find basis of solutions of

$$\mathbf{A}(z) \cdot \mathbf{Q}(z) = z^{\vec{\sigma}} \mathbf{R}(z).$$

with some added degree constraints $\vec{n} = (n_1, \ldots, n_m)$

$$\deg Q_i(z) \leq n_i.$$

$\mathbf{R}(z)$ called residual.

Rational Approximation Problems

Given $m \times m$ matrix $\mathbf{A}(z)$, orders, $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$, find basis of solutions of

$$\mathbf{A}(z) \cdot \mathbf{Q}(z) = z^{\vec{\sigma}} \mathbf{R}(z).$$

with some added degree constraints $\vec{n} = (n_1, \ldots, n_m)$

$$\deg Q_i(z) \leq n_i.$$

$\mathbf{R}(z)$ called residual.

Question : Basis in what sense?

Applications

Rational approximation problems appear in:

- Transcendence of e and other famous numbers
- Inversion formulae for structured matrices (scalar and block)
- Linear diophantine equations (and hence to GCDs)
- Guessing recurrence formulae (e.g. Gfun)
- Reconstruction of power series to polynomial problems (e.g. DFactor)
- Matrix normal forms (Popov, etc)
- Fast polynomial matrix arithmetic
- $\dots$

Outline

1. Introduction
   - General Setting
2. Rational Approximation
   - Linear Systems
3. Order Bases
   - Background
   - Computation
   - Fraction-Free Computation
   - Recursive Computation
   - Matrix Normal Forms
   - Current Activities

Associated Linear System

$$A(z)V_n(z) - U_m(z) = z^{m+n+1}W(z)$$

$$(a_0 + a_1 z + \cdots)(v_0 + \cdots + v_n z^n) - (u_0 + \cdots + u_m z^m) = z^{m+n+1}w_0 + z^{m+n+2}w_1 + \cdots$$

Same as

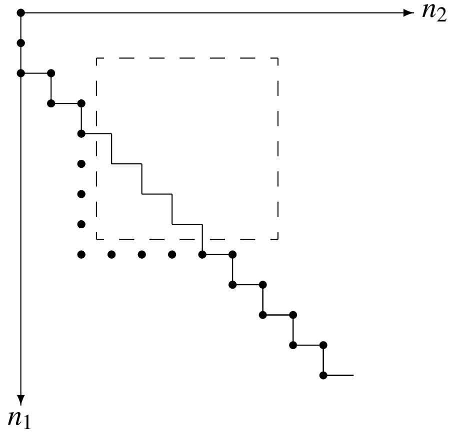
$$\begin{bmatrix} a_{m-n+1} & \cdots & \cdots & a_{n-1} & a_n \\ a_{m-n+2} & \cdots & \cdots & a_n & a_{n+1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n-1} & \cdots & \cdots & & a_{m+n-2} \\ a_n & \cdots & \cdots & a_{m+n-2} & a_{m+n-1} \end{bmatrix} \cdot \begin{bmatrix} v_n \\ v_{n-1} \\ \vdots \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -v_0 \begin{bmatrix} a_{m+1} \\ a_{m+2} \\ \vdots \\ \vdots \\ a_{m+n-1} \\ a_{m+n} \end{bmatrix}$$

Similarly for a_i square matrices.
Similarly have structured linear system for other approx problems.
Nice when coefficient matrix is nonsingular.

Additional Information

- All Padé approximants known in scalar case
  - Padé table in scalar case has a type of block structure
- Padé approximants related to diophantine equations
  - There are algorithms corresponding to Euclidean algorithm
  - Fast way to compute Padé approximants in scalar case
- Nothing known about structure of matrix Padé case or Hermite-Padé case by 1990
- Use in inversion formulas for Hankel and Toeplitz matrices gives rise to efficient numerically stable algorithms.

Structure of scalar Padé table helpful for algorithms. For example, a staircase path of computation in a Padé table:



Outline

1. Introduction
   - General Setting
2. Rational Approximation
   - Linear Systems
3. Order Bases
   - Background
   - Computation
   - Fraction-Free Computation
   - Recursive Computation
   - Matrix Normal Forms
   - Current Activities

People

- K. Mahler(1925-1969), J. Coates(1965) , J. Della Dora (1980),
  - (strong conditions always assumed)
- B. Beckermann and G. Labahn; A. Bultheel and M. van Barel
- B. Salvy and P. Zimmermann (gfun); M. Rubey (Extended Rate)
- M. van Hoeij (use in differential factorization)
- G. Villard; Beckermann, Labahn, Villard (matrix normal forms)
- P. Giorgi, C-P. Jeannerod, G. Villard (fast polynomial matrix arithmetic)
- B. Beckermann, H. Cheng, G. Labahn (Noncommutative domains); $\cdots$

Order Bases

Idea : look at order condition independently of degree bounds,

$$R_\sigma = \{\mathbf{Q}(z) \in F^{(m)}[z] \ \mid \ \mathbf{A}(z) \cdot \mathbf{Q}(z) = O(z^\sigma)\}$$

Find basis of R_σ as a *module* over $F[z]$.

- Basis always has m elements
- Write as columns of an $m \times m$ matrix polynomial $\mathbf{M}(z)$.

Example

Let

$$\mathbf{A}(z) = \begin{bmatrix} \frac{1}{2} + z^2 - z^4 & 1 + \sin(z^2)^4 & \frac{1}{\sqrt{1+z^2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\vec{\sigma} = (z^8, 1, 1)$. Then a basis for all solutions given by

$$\mathbf{M}(z) = \begin{bmatrix} z^4 + \frac{11\,z^2}{2} & -\frac{10\,z^2}{19} + \frac{2}{19} & \frac{9\,z^2}{19} - \frac{11}{76} \\ -\frac{59\,z^2}{4} & z^2 - \frac{33}{19} & -\frac{5\,z^2}{4} + \frac{59}{152} \\ 12\,z^2 & \frac{32}{19} & z^2 - \frac{6}{19} \end{bmatrix}$$

with $\det \mathbf{M}(z) = z^8$. In this case the first 4 terms of the order residual $\mathbf{R}$ of $\mathbf{M}$ are given by

$$\mathbf{R}(z) = \begin{bmatrix} -\frac{19}{4} - \frac{367}{32}z^2 - \frac{189}{64}z^4 + O\left(z^6\right) & -\frac{97}{76} + \frac{89}{152}z^2 + O\left(z^4\right) & -\frac{13}{1216} - \frac{1093}{1216}z^2 + O\left(z^4\right) \\ -\frac{59\,z^2}{4} & -\frac{33}{19} + z^2 & \frac{59}{152} - \frac{5\,z^2}{4} \\ 12\,z^2 & \frac{32}{19} & -\frac{6}{19} + z^2 \end{bmatrix}.$$

Order Bases

Degree bounds? Given $\vec{n} = (n_1, \ldots, n_m)$:

Then

$$\mathbf{Q}(z) \quad = \quad \alpha_1(z)\mathbf{M}_1(z) + \cdots + \alpha_m(z)\mathbf{M}_m(z)$$

with

$$\text{deg } \alpha_i(z) \leq \text{ defect } \mathbf{Q}(z) - \text{ defect } \mathbf{M}_i(z)$$

Here defect is a measure of the difference between degrees and bounds n_i.

Implies $\mathbf{M}(z)$ describes all solutions of $\mathbf{A}(z)\mathbf{Q}(z) = O(z^\sigma)$

Outline

1 Introduction
- General Setting

2 Rational Approximation
- Linear Systems

3 Order Bases
- Background
- Computation
- Fraction-Free Computation
- Recursive Computation
- Matrix Normal Forms
- Current Activities

Computation Complexity

Hermite-Padé approx. with degree bounds $(n_1, \dots n_m)$

Set $N = n_1 + \cdots + n_m$. Then :

- Linear algebra : $O(N^3)$
- Sigma basis : $O(N \log^2 N)$ i.e. - $O(N^{1+\epsilon})$ in scalar case (BL - 1994)
- MBasis : $O(m^\omega N^{1+\epsilon})$ - in case of matrix input (GJV - 2003)
- Generating set : $O(m^\omega (N/m)^{1+\epsilon})$ (Storjohann. 2006)
- Order basis : $O(m^\omega (N/m)^{1+\epsilon})$ Zhou (2008)

Sigma Basis Algorithm [SIMAX - BL]

1. Start with Order basis = $\mathbf{I}$ and order = 0.
2. Of all the columns that need to have order increased:
   - pick one with minimal defect.
   - use to eliminate other columns needing order increase.
3. Multiply pivot column by z. Continue. Quadratic complexity.

Double order everytime : obtain superfast version.

Alternatively (GJV)

To get $(\sigma + 1)$-basis from a σ-basis do:

1. we compute the terms in z^{σ} in the residue $\mathbf{R}(z)$. This gives us a matrix Δ
2. we compute a row echelon form of Δ
3. we apply some transformations according to the row echelon form. These transformations are of two types :
   - Either $\mathbf{M}_i$ is replaced by a linear combination of some $\mathbf{M}_j$
   - Or all the polynomials in $\mathbf{M}_i$ are multiplied by z

Complexity : Sigma Bases - MBasis

If $\mathbf{A}(z)$ is $m \times n$ matrix and we want approximation of order σ, then the complexity is :

- $O(n^2\ m\ \sigma^2)$ if we apply all the transformations needed at one step one by one.
- $O(n^\omega\ \sigma^2)$ if we use a matrix multiplication instead. (where $O(n^\omega)$ is the complexity of the matrix multiplication).

Outline

1 Introduction
- General Setting

2 Rational Approximation
- Linear Systems

3 Order Bases
- Background
- Computation
- Fraction-Free Computation
- Recursive Computation
- Matrix Normal Forms
- Current Activities

Fraction-free Computation

Fraction-free computation for given $\vec{n}$.

- set up linear system (structured Krylov matrix) for each order
- find so-called Cramer's solutions
- eliminate known divisors by a type of Sylvester's identity
- Order basis called Mahler system

Also version for Ore case. Also modular versions.

Goal: Try to find Cramer solutions

- e.g. Hermite-Padé problem

$$a(x) \cdot p(x) + b(x) \cdot q(x) + c(x) \cdot r(x) = O(x^6)$$

with deg $p(x) \leq 2$, deg $q(x) \leq 1$, deg $r(x) \leq 1$

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \cdot \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ \hline q_0 \\ q_1 \\ \hline r_0 \\ r_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ d \end{array}\right]$$

where d is determinant of coefficient matrix.

- Solution has determinant representation in nonsingular case:

$$\text{e.g. } p(z) = \det \left[\begin{array}{ccc|ccc|c} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ 1 & z & z^2 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Unique in nonsingular case.
- Recursively build Cramer solutions from Cramer solutions of smaller problems along offdiagonal of associated table.

- Matrix $\mathbf{M}(z)$ of determinantal polynomials with degrees

$$\begin{bmatrix} n_1 & n_1-1 & \cdots & n_1-1 \\ n_2-1 & n_2 & \cdots & n_2-1 \\ \vdots & & \ddots & \vdots \\ n_m-1 & \cdots & \cdots & n_m \end{bmatrix}$$

  and lcoeff of diagonal = determinant of coeff matrix.
- Unique in nonsingular case
- Basic building block of recursions.
- Method 1: via modified Schur complements
  - nonsingular location to nonsingular location in table
  - similar to look ahead
- Method 2: via determinental identities
  - works in singular cases by computing at closest nonsingular locations (look around)

Outline

1 Introduction
- General Setting

2 Rational Approximation
- Linear Systems

3 Order Bases
- Background
- Computation
- Fraction-Free Computation
- Recursive Computation
- Matrix Normal Forms
- Current Activities

Recursive Computation

Order bases problems with degree and order having equal status.
Find $\mathbf{M}(z)$ satisfying:

$$\mathbf{A}(z) \cdot \mathbf{M}(z) = z^{\vec{\sigma}} \mathbf{R}(z)$$

$$\mathbf{H}(z^{-1}) \cdot \mathbf{M}(z) = O(z^0)_{z \to \infty}$$

- Order bases if $\mathbf{R}(0)$ nonsingular. Of $\mathbf{H}$-degree if the second residual starts with nonsingular matrix.
- B-L (1997) show how this can be solved recursively. Advantage : allows one to specify paths of computation via order or degrees.

Outline

1. Introduction
   - General Setting
2. Rational Approximation
   - Linear Systems
3. Order Bases
   - Background
   - Computation
   - Fraction-Free Computation
   - Recursive Computation
   - Matrix Normal Forms
   - Current Activities

Shifted Popov Normal Forms

Shifted Popov form problem : gives

$$\mathbf{A}(z) \cdot \mathbf{U}(z) = \mathbf{P}(z)$$

Embed normal form problem inside part of a Mahler system for

$$[\mathbf{A}(z), -\mathbf{I}] \left[\begin{array}{cc} \mathbf{V}(z) & \mathbf{U}(z) \\ \mathbf{Q}(z) & \mathbf{P}(z) \end{array} \right] = [O(z^{\vec{n}}), 0].$$

Fraction-free computation of normal forms.

Outline

1. Introduction
   - General Setting
2. Rational Approximation
   - Linear Systems
3. Order Bases
   - Background
   - Computation
   - Fraction-Free Computation
   - Recursive Computation
   - Matrix Normal Forms
   - Current Activities

Currently we are working on:

- Fast algorithms in differential case
- Better fraction-free algorithms
- Use with differential-algebraic problems
  - Popov forms
  - Invariants and algorithms
- Alternate bases
- Multivariate Order Bases