Solving Linear ODEs in Maple

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Outline

- History of Linear ODE solver in Maple
  - 1983-1991
  - 1992-present
- Solving linear ODEs with doubly-periodic coefficients
  - in terms of doubly-periodic functions
  - in terms of doubly-periodic functions of second kind

History of dsolve: 1983-1991

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First version written by undergraduate students at Waterloo:
Bruce Sutherland, Andre Trudel 1983.

- implements standard algorithms for both linear and nonlinear equations
- user options: solve via laplace transform; solve in terms of series; solve numerically

History of dsolve/linear: 1983-1991

- Standard algorithms for linear ODEs:
  - e.g. constant coefficients, Euler equations, etc.
- Kovacic's algorithm for 2nd order equations (Carolyn Smith 1983)
  - first decision procedure of any kind in Maple.
- Later some specialized routines for special linear ODEs
  - e.g. Bessel's equation.

Some Facts: 1983-1991

dsolve*(ode,y(x))*

- either returned a complete solution or nothing
- some algorithms (e.g. Kovacic), but mostly heuristics,
- output a bit clumsy to use

History: 1992-1993

Progress in 1992 and 1993.

- more decision procedures
  - *rational solver* : from Manuel Bronstein (1992)
  - *exponential solver* : breaking through the order 2 barrier implementation by S. Schwendimann (1993)
- ability to return *partial* solutions using the new *DESol* function.
- try to reduce to second order linear ODEs

Simple Examples

$$x^2y'''(x)-(3x^2-x)y''(x)+(4x^2-2x-n^2)y'(x)-(2x^2-x-n^2)y(x) = 0;$$

$$y(x) = \_C1\ e^x + \_C2\ e^x \int J_n(x)dx + \_C3\ e^x \int Y_n(x)dx$$

$$y'''(x) - 3y''(x) + (x^2+3)y'(x) + (x^3+7)y(x) = 0;$$

$$y(x) = \_C1\ e^{-x}+e^{-x} \int DESol(w''(x)-6w'(x)+(x^3-12)w(x), w(x))dx$$

Special Functions : 1994-1996

- Need for more methods to solve ODEs having special functions as solutions.

- Attempt to find a fast front-end for finding *special function* solutions of second order linear ODEs (via heuristics)

- Sometimes trivial

$$dsolve(x^2 y''(x) + xy'(x) + (x^2 - a^2)y(x) = 0, y(x));$$

$$y(x) = \_C1\ J_a(x) + \_C2\ Y_a(x)$$

Special Functions

But what about (from Abramowitz and Stegun: 9.1.49-9.1.56)?

$$
\begin{aligned}
&y''(x) + (\lambda^2 - \tfrac{a^2-\frac{1}{4}}{x^2})y(x) = 0 && \Rightarrow y(x) = \_C1\sqrt{x}J_a(\lambda x) + \cdots \\
&y''(x) + (\tfrac{\lambda^2}{4x} - \tfrac{a^2-1}{4x^2})y(x) = 0 && \Rightarrow y(x) = \_C1\sqrt{x}J_a(\lambda\sqrt{x}) + \cdots \\
&y''(x) + \lambda^2 x^{p-2} y(x) = 0 && \Rightarrow y(x) = \_C1\sqrt{x}J_{\frac{1}{p}}(\tfrac{2\lambda}{p}x^{\frac{p}{2}}) + \cdots \\
&x^2y''(x) + (1-2p)xy'(x) && \\
&+(\lambda^2q^2x^{2q} + p^2 - a^2q^2)y(x) = 0 && \Rightarrow y(x) = \_C1x^pJ_a(\lambda x^q) + \cdots \\
&y''(x) - \tfrac{2a-1}{x}y'(x) + \lambda^2 y(x) = 0 && \Rightarrow y(x) = \_C1x^aJ_a(\lambda x) + \cdots \\
&y^{(n)}(x) - (-1)^n x^{-n} y(x) = 0 &&
\end{aligned}
$$

etc. etc.

Heuristic Special Function Solver (Labahn: 1995-1996)

For common second order linear ODEs (e.g. Bessels, Legendre, Whittaker, Hypergeometric, ...) do:

- transform to new ODE via $x \to ax^b$
- convert new ODE to normal form: $y''(x) + I(x)y(x) = 0$
- obtain a set of "common" invariants with variables a, b

For a given ODE to we then compute its invariant $\hat{I}(x)$, match to set of existing common invariants and build solution.

Heuristic Special Function Solver

E.g. match

$$x^2 I(x) = a^2\ b^2\ x^{2b} + (1/4 - b^2\ v^2)$$

for general invariant form of Bessels equation or match

$$x^2 I(x) = \frac{A}{ax^b - 1} + \frac{B}{ax^b + 1} + \frac{C}{(ax^b - 1)^2} + \frac{D}{(ax^b + 1)^2}$$

for general invariant form of Legendre equation.

Example (Kamke 2.220)

Solve:

$$y''(x) - \frac{f'(x)}{f(x)} \cdot y'(x) + \Big(\quad g'(x)^2 + \frac{g'(x)^2}{g(x)^2} \cdot (\frac{1}{4} - v^2) + \frac{3}{4} \cdot \frac{f'(x)^2}{f(x)^2}$$

$$- \frac{f''(x)}{2f(x)} - \frac{3}{4}\frac{g''(x)^2}{g'(x)^2} + \frac{g'''(x)}{2g'(x)}\Big) \cdot y(x) = 0$$

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Solve: Invariant form of original ode

$$y''(x) - \Big(\frac{3g''(x)^2 \cdot g(x)}{g'(x)} - 2g'''(x) \cdot g(x) - (1 - 4v^2) \cdot \frac{g'(x)^3}{g(x)} - 4 \cdot g'(x)^3 \cdot g(x)\Big) \cdot y(x) = 0$$

Example (Kamke 2.220)

Solve: With substitution of $z = g(x)$

$$w''(z) - x''(z) \cdot w'(z) + (\frac{3x''(z)}{4x'(z)} - \frac{1}{2}x'''(z) \quad +(1 + (\frac{1}{4} - v^2)) \cdot x'(z)) \cdot w(z) = 0.$$

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Solve: New invariant form

$$w''(z) + \frac{4z^2 - 4v^2 + 1}{4z^2} \cdot w(z) = 0.$$

Example (Kamke 2.220)

Solution for our equation is then

$$w(z) = c_1\sqrt{z} \cdot x'(z) \cdot J_v(z) + c_2\sqrt{z} \cdot x'(z) \cdot Y_v(z).$$

Example (Kamke 2.220)

Solution for our equation is then

$$w(z) = c_1\sqrt{z} \cdot x'(z) \cdot J_v(z) + c_2\sqrt{z} \cdot x'(z) \cdot Y_v(z).$$

Inverting substitution gives solution to initial invariant equation. Solution of invariant equation gives final solution as

$$y(x) = c_1\sqrt{\frac{g(x) \cdot f(x)}{g'(x)}} \cdot J_v(g(x)) + c_2\sqrt{\frac{g(x) \cdot f(x)}{g'(x)}} \cdot Y_v(g(x)).$$

Additional Improvements (1997-present)

- a new *exponential* solver (M. van Hoeij 1997)
- handling functions in coefficients (G. Labahn 1998)
  - known (e.g. $sin(x)$) or unknown (e.g. $f(x)$)
- using differential factorization (G. Labahn 1999)
- LCLM differential factorizations for orders 3 and 4 (MVH 2000)

- finding symmetric products for orders 3 and 4 (MVH 2000)
- recognizing MeijerG ODE for higher order (G. Labahn 2001)
  - basis in terms of Meijer G functions (with ax^b arguments)
  - Meijer G functions converted to special functions
- improved recognition of MeijerG ODE (E. Cheb-Terrab 2001)
  - extend to include $\frac{ax^b+c}{dx^b+e}$ arguments
- new Kovacic algorithm (MVH 2001)
- blended in with newer version of dsolve (E. Cheb-Terrab 2001)

How Good?

- Handles 90% of Kamke's linear ODES. Kamke has approximately
  - 500 second order examples,
  - 82 third order examples,
  - 44 fourth order examples,
  - 11 fifth order examples.
- Handles 95% of Kamke that one can expect Maple to do.
- Implies dsolve/linear can do nearly all *easy* problems. Rest?

Linear ODEs having doubly-periodic coefficients

- Introduction
- Decision procedure : I
- Picard's Theorem and consequences
- Second order ODEs
- Decision procedure : II

Introduction

Problem: Solve

$$a_n(x)y^{(n)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0$$

where $a_i(x)$ are all *doubly-periodic* with periods T and T'.

i.e. $a_i(x+T) = a_i(x+T') = a_i(x)$ for all x.

E.g. Weierstrass $\wp$ function and Jacobi $sn, cn, \ldots$ functions.

Examples from Kamke

2.26: $y'' - (n(n+1)\wp(x) + B)y = 0$

2.27: $y'' + \big(n(n+1)\ sn^2(x) + b\big)\, y = 0$

2.28: $y'' = \left(\frac{1}{30}\wp^{(4)}(x) + \frac{7}{3}\wp''(x) + a\wp(x) + b\right) y$

2.72: $y'' + a\wp'(x)y' + [\alpha + \beta\wp(x) - 4na\wp^2(x)]y = 0$

2.73: $y'' + \frac{\wp^3(x) - \wp(x)\wp'(x) - \wp''(x)}{\wp'(x) + \wp^2(x)} y' + \frac{\left(\wp'(x)\right)^2 - \wp(x)\wp'(x) - \wp(x)\wp''(x)}{\wp'(x) + \wp^2(x)} y = 0$

2.74: $y'' + k^2 \frac{sn(x)\ cn(x)}{dn(x)} y' + n^2 y\, dn^2(x) = 0$

and also 2.439–2.441, 3.9–3.14, 3.28, 4.10

Representations of Doubly-periodic Functions

- Represented as rational functions of $\wp(x), \wp'(x)$

$$\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3;$$

- Implies that all doubly-periodic functions written as $R_1(\wp) + R_2(\wp)\wp'$, where $R_1(\wp)$, $R_2(\wp)$ are rational functions of $\wp$.

- Similar statement can be made for sn, cn, dn etc.

First Decision Procedure

- Look for solution of form $R_1(\wp) + R_2(\wp)\wp'$
- Substitute into ODE; simplify to form $X(\wp)+Y(\wp)\wp' = 0$, where $X(\wp)$, $Y(\wp)$ are linear in R_1, R_2 and their derivatives w.r.t. $\wp$.
- Set $X = 0$, $Y = 0$ to obtain a system of 2 coupled ODEs for R_1, R_2:

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} R_1(\wp) \\ R_2(\wp) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where the M_{ij} are polynomials in $D_\wp$, with coefficients rational in $\wp$.

First Decision Procedure (cont.)

- Uncouple using Ore algebra techniques, yielding two linear ODEs.
- Find basis for rational solutions $R_1(\wp)$ and $R_2(\wp)$.
- Match R_1 and R_2 solutions to obtain $R_1(\wp) + R_2(\wp)\wp'$ for original ODE.
- Basically same as Singer's algorithm for solving in domain $\mathsf{K}(x, \sqrt{w(x)})[D]$

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- Gives a decision procedure for doubly-periodic solutions (good).

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- Match R_1 and R_2 solutions to obtain $R_1(\wp) + R_2(\wp)\wp'$ for original ODE.
- Basically same as Singer's algorithm for solving in domain $\mathsf{K}(x, \sqrt{w(x)})[D]$
- Gives a decision procedure for doubly-periodic solutions (good).
- Does not work well in practice (bad).

Linear ODEs having doubly-periodic coefficients

(Picard's Theorem : 1877) If the general solution of a linear ODE having doubly-periodic coefficients is single-valued, then there exists a basis of solutions which are doubly-periodic of the second kind.

- $F(x)$ is doubly-periodic of the second kind if

$$F(x+T) = sF(x),\ F(x+T') = s'F(x).$$

  for two constants s, s'.

- Expressed in closed form using the Weierstrass $\wp$, ζ, and σ functions: $\zeta(x) = -\int \wp(x)dx$ and $\sigma(x) = \exp(\int \zeta(x)dx)$.

How to find second kind of solution? Case of Second Order.

- Let y_1, y_2 be any solutions of $y''(x) + A(x)y(x) = 0$.
- Then y_1^2, $y_1 y_2$, y_2^2 are solutions of

$$Y'''(x) + 4A(x)Y'(x) + 2A'(x)Y(x) = 0. \tag{1}$$

- From a solution Y of (1), can recover solutions y_1, y_2 of original ode:

$$y_1(x) = \exp \int \frac{Y' - C}{2Y} dx, \qquad y_2(x) = \exp \int \frac{Y' + C}{2Y} dx,$$

where

$$C^2 = (Y')^2 - 2YY'' - 4AY^2 \text{ with } C \neq 0.$$

Why should it be easier to find Y?

- Let y_1, y_2 be solutions of $y''(x) + A(x)y(x) = 0$, doubly-periodic of the second kind. Then

$$y_1(x+T) = s_1 y_1(x), \qquad y_2(x+T) = s_2 y_2(x),$$
$$y_1(x+T') = s'_1 y_1(x), \qquad y_2(x+T') = s'_2 y_2(x).$$

- Because ODE lacks a term in y' we get: $s_1 s_2 = 1, \qquad s'_1 s'_2 = 1.$

$\Rightarrow y_1(x+T)y_2(x+T) = s_1 y_1(x) s_2 y_2(x) = y_1(x)y_2(x).$

- Similarly for T': $\Rightarrow Y(x) = y_1(x)y_2(x)$ is doubly-periodic.

Second Decision Procedure

Solve: $L(D) = A_n(x)y^{(n)}(x) + \cdots + A_0(x)y(x) = 0$ with $A_i(x)$ doubly-periodic.

- Change variables from x to $\wp$. Changes problem to:

  solve $L(D) = a_n(x)y^{(n)}(x) + \cdots + a_0(x)y(x) = 0$ with $a_i(x) \in K(x, \sqrt{w(x)})$.

- Solve by using local information from singularities

- First ensure that $a_n(x) \in K[x]$ and find singularities of $a_n(x)$ (over algebraic curve defined by $\sqrt{w(x)}$)

- For each singularity:
  - determine a local parameterization,
  - find indicial equation
  - find the smallest negative integer or half-integer root (depending on type of local parameterization)

- Use these to construct universal denominator

- Use singularity at ∞ to determine degrees of numerator

- Use method of undetermined coefficients to find numerator(s) $a(x) + b(x)\sqrt{w(x)}$. Invert change of variables for $\wp$ and $\wp'$.

- Note: Many computational challenges for efficiency!

Other Possibilities

For case of higher order equations can try:

- Use symmetric powers for higher order case
  - Follow methods used in rational function case for exp solutions
  - Computational problems
- Use LCLM differential factorization (similar to Kronecker-Trager algorithm)
  - Works with higher order ODES via: LCLM(L, conj(L)).
  - Computationally difficult to do LCLM differential factorization

Further Work

- Try direct methods for finding doubly-periodic solutions of the second kind based on analyzing local singularities

- Try to find doubly-periodic solutions faster via substitution techniques (in the case where there are parameters)
  - What about unlucky evaluations?

Example (if time permits)

$$L(D)(y(x)) = \Big(32\,x^7 + 1224\,x^6 - 832\,x^8 + 580\,x^9 + 288\,x^2 - 1200\,x^3 + 1784\,x^4 - 1732\,x^5 - 160\,x^{10} + 16\,x^{11}$$
$$+ \Big(4930\,x^5 + 672\,x^8 + 8548\,x^3 + 12\,x^{10} - 3888\,x^2 - 1284\,x^7 - 146\,x^9 - 9476\,x^4 - 88\,x^6 + 720\,x\Big)$$
$$\sqrt{x\,(x-2)\,(x-3)}\Big)\,y''(x) + \Big(-16\,x^{11} + 864\,x - 3952\,x^6 - 836\,x^9 + 8424\,x^3 - 7472\,x^4 + 312\,x^7 + 1392\,x^8 - 4320\,x^2$$
$$+5892\,x^5 + 192\,x^{10} + \Big(2520 - 10843\,x^5 - 43560\,x^3 + 164\,x^9 - 14220\,x + 33852\,x^2 - 12\,x^{10}$$
$$+31474\,x^4 - 899\,x^8 - 715\,x^6 + 2239\,x^7\Big)\,\sqrt{x\,(x-2)\,(x-3)}\Big)\,y'(x)$$
$$+ \Big(-864\,x + 4032\,x^2 - 7224\,x^3 + 5688\,x^4 + 2728\,x^6 - 4160\,x^5 - 344\,x^7 - 32\,x^{10} + 256\,x^9 - 560\,x^8$$
$$+ \Big(-2520 - 18\,x^9 - 29964\,x^2 - 955\,x^7 + 227\,x^8 + 803\,x^6 + 5913\,x^5 + 35012\,x^3 + 13500\,x$$
$$-21998\,x^4\Big)\,\sqrt{x\,(x-2)\,(x-3)}\Big)\,y(x) = 0 \qquad (2)$$

Example (cont.)

- Singularities of $\hat{a}_2(x) = A_2(x)^2 - B_2(x)^2 w(x)$ where $w(x) = x(x-2)(x-3)$ which introduce negative integer roots are 0 and 1.

- At $x = 1$, $\sqrt{\omega(x)} = \sqrt{2} \neq 0 \Rightarrow \sqrt{\omega(x)}$ has a Taylor's series expansion

$$\sqrt{\omega(x)} = \sqrt{2} - \frac{\sqrt{2}}{4}(x-1) - \frac{17}{32}\sqrt{2}(x-1)^2 + \frac{15}{128}\sqrt{2}(x-1)^3 - \ldots \tag{3}$$

- Hence $A(x) + B(x)\sqrt{\omega(x)}$ has a series expansion at $x = 1$:

$$A(x) + B(x)\sqrt{\omega(x)} = (x-1)^N + c_1(x-1)^{N+1} + c_2(x-1)^{N+2} + \ldots \tag{4}$$

- Indicial equation here is $N(N+2)$.

Example (cont.)

- Hence, at this point,

$$A(x) = \frac{\cdots}{(x-1)^2 \cdots}, \qquad B(x) = \frac{\cdots}{(x-1)^2 \cdots} \tag{5}$$

- Note: In fact, $\sqrt{\omega(1)} = \pm\sqrt{2}$, so a second series expansion $\sqrt{\omega(x)} = -\sqrt{2} + \ldots$ should also be used to find an exponent for $(x-1)$. The most negative exponent is then used in (5).

- Note also that one of $A(x)$, $B(x)$ could have a greater (more positive) exponent than -2 in $(x-1)$. The indicial equation gives information on the exponent of the entire solution, not (directly) on $A(x)$ and $B(x)$ separately.

Example (cont.)

At singularity $x = 0$:

- Again, $A(x)$, $B(x)$ are rational in x, so have Laurent expansions at $x = 0$.

- *But* at $x = 0$, $\sqrt{\omega(x)} = 0$. Hence $\sqrt{\omega(x)}$ does not have a Taylor expansion at this point. In fact,
$$\sqrt{\omega(x)} = \sqrt{6}\sqrt{x} - \ldots$$

- x is not a local parameter for $\sqrt{\omega(x)}$ at $x = 0$, i.e., does not have multiplicity 1 at this point.

- change independent variable from x to a local parameter, e.g., let $x = t^2$.

Example (cont.)

- Then $\sqrt{\omega}(t) = \sqrt{6}t - \frac{5}{12}\sqrt{6}t^3 + \dots$ and $A(t)$, $B(t)$ have Laurent series in t.
- $A(x) + B(x)\sqrt{\omega(x)}$ has a series expansion at $t = 0$:

$$A(t) + B(t)\sqrt{\omega(t)} = t^N + c_1 t^{N+1} + c_2 t^{N+2} + \dots$$

- Changing from x to t^2 in (2) yields indicial equation $N(N+5) = 0$
- Hence $N = -5$ and, since $t = x^{1/2}$,

$$A(x) + B(x)\sqrt{\omega(x)} = x^{-5/2} + \dots$$

Example (cont.)

- Since $A(x)$ is rational in x, its exponent in x cannot be fractional. Thus the lowest it can be is -2. Hence

$$A = \frac{\cdots}{(x-1)^2 x^2}$$

- $B(x)$ occurs in the combination $B(x)\sqrt{\omega(x)}$. Since $\sqrt{\omega(x)}$ has exponent $\frac{1}{2}$ in x, the exponent of $B(x)$ could be as low as -3. Hence

$$B = \frac{\cdots}{(x-1)^2 x^3}$$

Example (cont.)

At $x = \infty$

- Bound degree of numerator by investigating behaviour at $x = \infty$ via $x = 1/t$;
- But: find that t is not a local parameter at 0. Instead, use $x = 1/t^2$. Indicial equation is $2N - 6 = 0$
- Hence $N = 3$. As $t = x^{-1/2}$,

$$A(x) + B(x)\sqrt{\omega(x)} = x^{-3/2} + c_1 x^{-2} + \ldots, \quad \text{as } x \to \infty$$

Example (cont.)

- Notice that the powers of x are *decreasing*. For $A(x)$,

$$\begin{aligned} degree(num(A(x))) - degree(denom(A(x))) &\leq -\frac{3}{2} \\ degree(num(A(x))) &\leq -\frac{3}{2} + 4 = \frac{5}{2} \\ \text{Hence} \;\; degree(num(A(x))) &\leq 2. \end{aligned}$$

- For $B(x)\sqrt{\omega(x)}$, note that, asymptotically, $\sqrt{\omega(x)}$ behaves as $x^{3/2}$ as $x \to \infty$. So

$$\begin{aligned} degree(num(B(x))) + \frac{3}{2} - degree(denom(B(x))) &\leq -\frac{3}{2} \\ degree(num(B(x))) &\leq -\frac{3}{2} - \frac{3}{2} + 5 = 2 \end{aligned}$$

Example (cont.)

- Thus

$$A(x) = \frac{a_2x^2 + a_1x + a_0}{(x-1)^2x^2}, \qquad B(x) = \frac{b_2x^2 + b_1x + b_0}{(x-1)^2x^3}$$

- Substitution into (2) leads to a linear system of equations for the a_i, b_i. When solved, a solution of (2) is found to be (after simplifying)

$$\frac{4\,x^2+1}{(x-1)^2\,x^2} + \frac{(3\,x-2)\,\sqrt{x\,(x-2)\,(x-3)}}{(x-1)\,x^3}$$