Unimodular Completion of Polynomial Matrices

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Unimodular Completion

Let

$$\mathbf{F} = \begin{bmatrix} -2x+1 & -x^2+2x & -x & -x \\ 2x^3 & -x^3+2 & x^3-1 & x^3 \end{bmatrix} \in \mathbb{Z}_5[x]^{2\times 4}$$

Unimodular Completion

Extend so

$$\begin{bmatrix} -2x+1 & -x^2+2x & -x & -x \\ 2x^3 & -x^3+2 & x^3-1 & x^3 \\ & & & \\ & & & \end{bmatrix} \in \mathbb{Z}_5[x]^{4\times 4}$$

is unimodular, i.e. has determinant a constant

Unimodular Completion

Extend so

$$\begin{bmatrix} -2x+1 & -x^2+2x & -x & -x \\ 2x^3 & -x^3+2 & x^3-1 & x^3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathbb{Z}_5[x]^{4\times 4}$$

is unimodular, i.e. has determinant a constant

Program : Fast Polynomial Arithmetic

Fast order bases (roughly $\mathbf{F} \cdot \mathbf{G} \approx 0$) and kernel bases ($\mathbf{F} \cdot \mathbf{N} = 0$):

- W. Z and G. L, Efficient Computation of Order Bases, Proceedings of ISSAC'09 (2009)
- W. Z and G. L, Algorithms for Efficient Order Basis Computation, Journal of Symbolic Computation (2012)
- W. Z, G. L and A. Storjohann, Computing Minimal Nullspace Bases, Proceedings of ISSAC'12 (2012)

Gives fast block elimination : $\mathbf{F} \cdot \mathbf{N} = 0$, $\qquad \mathbf{F} \cdot \mathbf{U} = [\ \mathbf{C},\ \mathbf{0}\ ]$.

Used for fast comp. of Column Bases, Inversion, Determinant, etc

- W. Z and G. L, Computing Column Bases of Polynomial Matrices, Proceedings of ISSAC'13, (2013)
- W. Z, G. L and A. Storjohann, A deterministic algorithm for inverting a polynomial matrix, Submitted (2014)
- W. Z and G. L, This paper, ISSAC 2014 (2014)

Unimodular Completion for $m \times n$ problem.

Given $\mathbf{F} \in \mathbb{K}[x]^{m\times n}$ $\quad (m < n),\quad$ full rank.

Goal: $\quad \mathbf{G} \in \mathbb{K}[x]^{(n-m)\times n}$ $\qquad \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix}$ $\qquad$ is unimodular .

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(1) Existence : only if there is a unimodular $\mathbf{U}$ with $\mathbf{F} \cdot \mathbf{U} = [\ \mathbf{I}_m,\ \mathbf{0}\ ]$

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(1) Existence : only if there is a unimodular $\mathbf{U}$ with $\mathbf{F} \cdot \mathbf{U} = [\ \mathbf{I}_m,\ \mathbf{0}\ ]$

(2) This talk : algorithm with complexity $O^{\sim}(n^{\omega} s)$ ops from $\mathbb{K}$.

- s is average of the m largest column degrees of $\mathbf{F}$
- deterministic

Previous Work

Multivariate Polynomials

- In case of multivariate polynomial matrices solves Serre's conjecture : (Quillen-Suslin theorem)

  Serre's conjecture : Every finitely generated projective module over a polynomial ring is free.

  Reduces to being able to solve

  $$\mathbf{F} \cdot \mathbf{U} \; = \; [\; \mathbf{I_m}, \;\; \mathbf{0} \;] \quad \text{with} \quad \mathbf{U} \quad \text{unimodular}$$

- Logar, A. and Sturmfels, B., Algorithms for the Quillen-Suslin theorem, *Journal of Algebra*, (1992)

  Also
  - H. Park and C. Woodburn J of A (1995)
  - Lombardi H. and Yengui I. JSC (2005)

Multivariate differential polynomial matrices

- Fabianska, A., PhD Thesis (Aachen)
- Fabianska, A. and Quadrat, A. Applications of Quillen-Suslin thm to multidimensional systems theory (2007)
  - Including software : QuillenSuslin
- Cluzeau, T. and Quadrat, A, Isomorphisms and Serre's reduction of linear systems (2013),

Simple Example : Look at 1×2 case

Consider

$$\begin{bmatrix} a(x) & b(x) \\ & \end{bmatrix} \cdot \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Simple Example : Look at 1×2 case

Consider

$$\begin{bmatrix} a(x) & b(x) \end{bmatrix} \cdot \begin{bmatrix} u(x) & \\ v(x) & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a(x) \cdot u(x) + b(x) \cdot v(x) = 1$$

That is: gcd($a(x)$, $b(x)$) $= 1$.

Simple Example : Look at 1×2 case

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$$\begin{bmatrix} a(x) & b(x) \end{bmatrix} \cdot \begin{bmatrix} u(x) & w(x) \\ v(x) & z(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a(x) \cdot u(x) + b(x) \cdot v(x) = 1$$

That is: gcd($a(x),\ b(x)$) $=$ 1.

$$a(x) \cdot w(x) + b(x) \cdot z(x) = 0$$

That is: lcm($a(x),\ b(x)$) $=$ $a(x) \cdot b(x)$.

Simple Example : Look at 1×2 case

Consider

$$\begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} \cdot \begin{bmatrix} u(x) & w(x) \\ v(x) & z(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a(x) \cdot u(x) + b(x) \cdot v(x) = 1$$

That is: gcd($a(x),\ b(x)$) $=\ 1$.

$$a(x) \cdot w(x) + b(x) \cdot z(x) = 0$$

That is: lcm($a(x),\ b(x)$) $=\ a(x) \cdot b(x)$. So gcd($u(x),\ v(x)$) $=\ 1$

$$c(x) \cdot u(x) + d(x) \cdot v(x) = 1$$

General Case : Tools Used : I

$$\mathbf{P} = \begin{bmatrix} 4x+3x^2 & 4+x+3x^2 & x^3 & 0 \\ 1 & 2x^3 & 0 & x^4 \\ 4x & 4+x & 0 & 0 \\ x & 1 & 0 & 0 \end{bmatrix}$$

- Column degree : $\text{cdeg}(\mathbf{P})$ Example : $\text{cdeg}(\mathbf{P}) = (2,3,3,4)$.
- Shifted column degree : $\text{cdeg}_{\vec{u}}(\mathbf{P}) = \text{cdeg}(x^{\vec{u}} \cdot \mathbf{P})$. Example : $\text{cdeg}_{(-2,-3,-1,-1)}(\mathbf{P}) = (0,0,1,1)$.
- Leading coefficient matrix

$$\text{lcoeff}\,(\mathbf{P}) = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Shifted leading coefficient matrix

$$\text{lcoeff}_{(-2,-3,-1,-1)}(\mathbf{P}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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- $\mathbf{P}$ is $(-2,-3,-1,-1)$-column reduced of shifted degree $(0,0,1,1)$

Tools Used : II

- Kernel bases for $\mathbf{A}$. Basis for module :

$$\{\mathbf{p} \in \mathbb{K}[x]^n \mid \mathbf{A} \cdot \mathbf{p} = 0\}$$

- Kernel basis represented by matrix (put into columns) $\mathbf{M}$:

Tools Used : II

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$$\{\mathbf{p} \in \mathbb{K}[x]^n \mid \mathbf{A} \cdot \mathbf{p} = 0\}$$

- Kernel basis represented by matrix (put into columns) $\mathbf{M}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2+2x & 3+4x \\ 0 & x^2 & 1+3x^4 & 1+4x+3x^4 \end{bmatrix} \in \mathbb{Z}_5[x]^{2\times 4}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & x^2 \\ 2x+2 & 3x^4+1 \\ 4x+3 & 3x^4+4x+1 \end{bmatrix}$$

- Minimal kernel basis if matrix $\mathbf{M}$ is column reduced
- Shifted minimal kernel basis if matrix $\mathbf{M}$ is shifted column reduced
- Example : $\mathbf{M}$ is $(2,3,1,1)$-minimal kernel basis for $\mathbf{A}$.

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- Cost : $O^{\sim}(n^{\omega} s)$ Z, L, Storjohann (ISSAC 2012)

Tools Used : III

- Order bases for $\mathbf{A}$ of order $\vec{\sigma}$. Basis for module :

$$\left\{\mathbf{p} \in \mathbb{K}[x]^n \mid \mathbf{A} \cdot \mathbf{p} = O(x^{\vec{\sigma}})\right\}$$

- Order basis represented by matrix (put into columns)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2+2x & 3+4x \\ 0 & x^2 & 1+3x^4 & 1+4x+3x^4 \end{bmatrix} \in \mathbb{Z}_5[x]^{2\times 4}$$

$$\mathbf{P} = \begin{bmatrix} 4x+3x^2 & 4+x+3x^2 & x^3 & 0 \\ 1 & 2x^3 & 0 & x^4 \\ 4x & 4+x & 0 & 0 \\ x & 1 & 0 & 0 \end{bmatrix}$$

- Reduced order basis if matrix $\mathbf{P}$ is column reduced
- Shifted reduced order basis if matrix $\mathbf{P}$ is shifted column reduced
- Example : $\mathbf{P}$ is $(-2,-3,-1,-1)$-reduced order basis for $\mathbf{A}$ of cdeg $(0,0,1,1)$.

Tools Used : III

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- Example : $\mathbf{P}$ is $(-2,-3,-1,-1)$-reduced order basis for $\mathbf{A}$ of cdeg $(0,0,1,1)$.
- Cost : $O^{\sim}(n^{\omega}s)$ Z, L (ISSAC 2009, JSC 2012)

Tools Used : IV

Reversing Polynomials : $p(x^{-1})x^k$:

$$
\begin{aligned}
p(x) &= p_0 + p_1 x + \cdots + p_k x^k \\
p^*(x) &= p_k + \cdots + p_1 x^{k-1} + p_0 x^k
\end{aligned}
$$

Tools Used : IV

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Tools Used : IV

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Reversing Matrix Polynomials (e.g. $\cdot x^{-\vec{u}} \cdot \mathbf{P}(x^{-1}) \cdot x^{\vec{v}}$):

$$\mathbf{P}(x) = \begin{bmatrix} 4x + 3x^2 & 4 + x + 3x^2 & x^3 & 0 \\ 1 & 2x^3 & 0 & x^4 \\ 4x & 4 + x & 0 & 0 \\ x & 1 & 0 & 0 \end{bmatrix}$$

Tools Used : IV

Reversing Polynomials : $p(x^{-1})x^k$:

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Reversing Matrix Polynomials (e.g. $\cdot x^{-\vec{u}} \cdot \mathbf{P}(x^{-1}) \cdot x^{\vec{v}}$):

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\mathbf{P}(x) = \begin{bmatrix} 4x+3x^2 & 4+x+3x^2 & x^3 & 0 \\ 1 & 2x^3 & 0 & x^4 \\ 4x & 4+x & 0 & 0 \\ x & 1 & 0 & 0 \end{bmatrix}
$$

$$
\mathbf{P}^c(x) = x^{(2,3,1,1)} \cdot \mathbf{P}(x^{-1}) \cdot x^{(0,0,1,1)} = \begin{bmatrix} 4x+3 & 4x^2+x+3 & 1 & 0 \\ x^3 & 2 & 0 & 1 \\ 4 & 4x+1 & 0 & 0 \\ 1 & x & 0 & 0 \end{bmatrix}.
$$

Note : cdeg$_{(-2,-3,-1,-1)}(\mathbf{P}(x)) = (0,0,1,1)$.

Why is reversing convenient?

$\mathbf{P}$ $\vec{u}$-order basis with $\mathrm{cdeg}_{\vec{u}} = \vec{v}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2+2x & 3+4x \\ 0 & x^2 & 1+3x^4 & 1+4x+3x^4 \end{bmatrix} \in \mathbb{Z}_5[x]^{2\times 4}$$

$$\mathbf{P} = \begin{bmatrix} 4x+3x^2 & 4+x+3x^2 & x^3 & 0 \\ 1 & 2x^3 & 0 & x^4 \\ 4x & 4+x & 0 & 0 \\ x & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}(x)\cdot\mathbf{P}(x) = \begin{bmatrix} 0 & 0 & x^3 & 0 \\ 0 & 0 & 0 & x^6 \end{bmatrix}$$

Why is reversing convenient?

$\mathbf{P}^c = x^{-\vec{u}}\mathbf{P}(x^{-1})x^{\vec{v}}$ gives unimodular matrix

$$\mathbf{A}^c = \begin{bmatrix} 1 & 0 & 2x+2 & 3x+4 \\ 0 & 1 & x^4+3 & x^4+4x^3+3 \end{bmatrix} \in \mathbb{Z}_5[x]^{2\times 4}$$

$$\mathbf{P}^c = \begin{bmatrix} 4x+3 & 4x^2+x+3 & 1 & 0 \\ x^3 & 2 & 0 & 1 \\ 4 & 4x+1 & 0 & 0 \\ 1 & x & 0 & 0 \end{bmatrix}.$$

$$\mathbf{A}^c(x) \cdot \mathbf{P}^c(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Procedure : Step 1

Reversing (carefully)

$$\begin{bmatrix} \mathbf{F}(x) \\ \mathbf{G}(x) \end{bmatrix} \cdot \begin{bmatrix} ?? & \mathbf{M}(x) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{F}(x^{-1}) \\ \mathbf{G}(x^{-1}) \end{bmatrix} \cdot x^{\vec{s}} \cdot x^{-\vec{s}} \cdot \begin{bmatrix} ?? & \mathbf{M}(x^{-1}) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{F}^c(x) \\ \mathbf{G}^c(x) \end{bmatrix} \cdot \begin{bmatrix} ?? & x^{-\vec{s}} \cdot \mathbf{M}(x^{-1}) \cdot x^{\vec{b}+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & x^{\vec{b}+1} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{F}^c(x) \\ \mathbf{G}^c(x) \end{bmatrix} \cdot \begin{bmatrix} ?? & \mathbf{M}^c(x) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & x^{\vec{b}+1} \end{bmatrix}$$

The Procedure : Step 2

From $$\begin{bmatrix} \mathbf{F}^c(x) \\ \mathbf{G}^c(x) \end{bmatrix} \cdot \begin{bmatrix} ?? & \mathbf{M}^c(x) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & x^{\vec{b}+1} \end{bmatrix}$$ we get:

(a) $$\mathbf{F}^c(x) \cdot \mathbf{M}^c(x) = 0$$

- Use a $\vec{s}$-minimal kernel basis for $\mathbf{M}^c(x)$.

The Procedure : Step 2

From $\begin{bmatrix} \mathbf{F}^c(x) \\ \mathbf{G}^c(x) \end{bmatrix} \cdot \begin{bmatrix} ?? & \mathbf{M}^c(x) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & x^{\vec{b}+1} \end{bmatrix}$ we get:

(a) $$\mathbf{F}^c(x) \cdot \mathbf{M}^c(x) = 0$$

(b) $$\begin{bmatrix} \mathbf{N}(x) \\ \mathbf{G}^c(x) \end{bmatrix} \cdot \mathbf{M}^c(x) = \begin{bmatrix} 0 \\ x^{\vec{b}+1} \end{bmatrix}$$

- Use a $\vec{s}$-minimal kernel basis for $\mathbf{M}^c(x)$.
- Use order basis on the left to get $\mathbf{G}^c(x)$.
- Turns out : $\begin{bmatrix} \mathbf{F}(x) \\ \mathbf{G}(x) \end{bmatrix}$ will be unimodular .

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- Use a $\vec{s}$-minimal kernel basis for $\mathbf{M}^c(x)$.
- Use order basis on the left to get $\mathbf{G}^c(x)$.
- Turns out : $\begin{bmatrix} \mathbf{F}(x) \\ \mathbf{G}(x) \end{bmatrix}$ will be unimodular .
- Efficiency : Need to do order bases incrementally

Steps + Complexity

Overall cost : $O^{\sim}(n^{\omega}s)$

(1) kernel bases computation $\mathbf{M}$

- Cost : $O^{\sim}(n^{\omega}s)$
- Partition : $\mathbf{M} = [\mathbf{M}_1,\ \mathbf{M}_2, \ldots, \mathbf{M}_{\lceil \log(n-m) \rceil}]$

  (increasing $\vec{s}$-column degrees)

(2) order bases computation $\mathbf{N}_{i-1} \cdot \mathbf{M}_i$:

- Incremental orders
- Cost : $O^{\sim}(n^{\omega}s)$

(3) polynomial matrix multiplication

- Multiply $\mathbf{N}_{i-1} \cdot \mathbf{M}_i$
- Cost : $O^{\sim}(n^{\omega}s)$

Future work

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- Implement in appropriate fast environment

  Added bonus : likely allows for moving soft O to big O

- Fast unimodular completion for :
  - differential and Ore matrix polynomials
  - multivariate matrix polynomials
  - multivariate differential and Ore matrix polynomials