Fraction-free Computation of Simultaneous Padé Approximants

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Simultaneous Padé Approximants

Given power series $A_1(z), \ldots, A_m(z)$ and integers $n_0, \ldots, n_m$

Find polynomials $U_1(z), ..., U_m(z)$ and $V(z)$ with :

- $A_1(z) \approx \frac{U_1(z)}{V(z)}, \quad \ldots, \quad A_m(z) \approx \frac{U_m(z)}{V(z)}$.
- Degree bounds for $U_1(z), \ldots, U_m(z), V(z)$

Simultaneous Padé Approximants (more precise)

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- $A_i(z) \approx \frac{U_i(z)}{V(z)}$ means $A_i(z)V(z) - U_i(z) = \mod z^{N+1}$
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$$\deg (U_i) \leq N - n_i \quad \deg (V) \leq N - n_0,$$

where $N = n_0 + \cdots + n_m$.

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where $N = n_0 + \cdots + n_m$.

Important fact : Closely related to Hermite-Padé approximants.

Example : e^z and e^{2z} and e^{3z}

$$
\begin{aligned}
e^z &= \frac{-2\,z^2 + 6\,z - 6}{6\,z^3 - 11\,z^2 + 12\,z - 6} + O(z^4) \\
e^{2z} &= \frac{z^2 - 6}{6\,z^3 - 11\,z^2 + 12\,z - 6} + O(z^4) \\
e^{3z} &= \frac{-2z^2 - 6\,z - 6}{6\,z^3 - 11\,z^2 + 12\,z - 6} + O(z^4)
\end{aligned}
$$

Here $(n_0, n_1, n_2, n_3) = (0, 1, 1, 1)$ so $N = 3$.

Degree bounds : denom $= 3$, numers $= 2,\ 2,\ 2$, order $= 4$.

Where used?

- Transcendence of important numbers : e.g. e, π, etc
- Inversion formulae for structured matrices
- Vector rational reconstruction problem:

$$\mathbb{K}^n(z) \rightarrow \mathbb{K}^n[[z]] \text{ - solve problem - } \mathbb{K}^n[[z]] \rightarrow \mathbb{K}^n(z)$$

(e.g. linear system solving)

- etc.

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People

- 1850-1890: Hermite
  - used these (informally) to prove transcendence of e (1873)
- 1890s: H. Padé
  - student of Hermite
  - first systematic study of rational approximation (1893)
  - Padé table
- 1920-1967: K. Mahler
  - a formal study in general case
  - relationships between groups of simultaneous Padé approximants and Hermite-Padé approximants

How to Compute Simultaneous Padé Approximants

- linear system of equations for coefficients
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Last two methods used to find *all* solutions to approximation problem. Here *all* $\equiv$ *basis as a module*

All three methods work with Fraction-Free arithmetic.

Linear System : $A(z)$ and $B(z)$ and $C(z)$

Can always solve via linear system for coefficients

$$(a_0 + a_1 z + \cdots)(v_0 + v_1 z + v_2 z^2 + v_3 z^3) - (u_0 + u_1 z + u_2 z^2) = O(z^4)$$

$$(b_0 + b_1 z + \cdots)(v_0 + v_1 z + v_2 z^2 + v_3 z^3) - (\hat{u}_0 + \hat{u}_1 z + \hat{u}_2 z^2) = O(z^4)$$

$$(c_0 + c_1 z + \cdots)(v_0 + v_1 z + v_2 z^2 + v_3 z^3) - (u_0^* + u_1^* z + u_2^* z^2) = O(z^4)$$

$$\left[\begin{array}{cccc|ccc|ccc|ccc}
a_0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_1 & a_0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_2 & a_1 & a_0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
b_0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
b_1 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
b_2 & b_1 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
c_1 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
c_2 & c_1 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
c_3 & c_2 & c_1 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
v_0 \\ v_1 \\ v_2 \\ v_3 \\ \hline u_0 \\ u_1 \\ u_2 \\ \hline \hat{u}_0 \\ \hat{u}_1 \\ \hat{u}_2 \\ \hline u_0^* \\ u_1^* \\ u_2^*
\end{array}\right] = 0.$$

Notice that matrix is highly structured.

Simultaneous-Padé as Vector Hermite-Padé

$$\begin{aligned} A(z)U_0(z) - U_1(z) &= O(z^4) \\ B(z)U_0(z) - U_2(z) &= O(z^4) \\ C(z)U_0(z) - U_3(z) &= O(z^4) \end{aligned}$$

same as

$$\begin{bmatrix} A(z) \\ B(z) \\ C(z) \end{bmatrix} U_0(z) + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} U_1(z) + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} U_2(z) + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} U_3(z) = 0(z^4)$$

Computation

- Hermite-Padé (general):
  - Beckermann-Labahn (SIMAX - 1994)
  - P. Giorgi, C-P. Jeannerod, G. Villard (ISSAC 2003)
    - fast polynomial matrix arithmetic
  - Beckermann-Labahn (SIMAX - 2000)
    - fraction-free arithmetic
  - Zhou-Labahn (Next talk)
    - fast arithmetic
- Simultaneous-Padé
  - Olesky and Storjohann (2007)

Cost

If input has size $O(\kappa)$ then :

- Fraction-Free Gaussian Elimination (FFGE) :
  - Bit complexity of operations: $O(\kappa^2 m^6 N^5)$
- B-L [SIMAX 2000] :
  - Bit complexity of operations: $O(\kappa^2 m^5 N^4)$
  - Size of objects : $O(\kappa m N)$
- Today
  - Bit complexity of operations: $O(\kappa^2 m^2 N^4)$
  - Size of objects : $O(\kappa m N)$

Order Basis

Input : Vector of power series $\mathbf{A}(z) \in \mathbb{K}^{1\times m}[z]$ and integer σ

Order Basis : Matrix polynomial $\mathbf{M}(z) \in \mathbb{K}m \times m[z]$

$$\mathbf{A}(z) \cdot \mathbf{M}(z) = z^{\sigma}\mathbf{R}(z)$$

- All solutions $\mathbf{V}(z) \in \mathbb{K}^{m\times 1}[z]$ of $\mathbf{A}(z) \cdot \mathbf{V}(z) = O(z^{\sigma})$ is a combination of columns of $\mathbf{M}(z)$.

$$\mathbf{V}(z) = \alpha_1(z)\mathbf{M}^{(1)}(z) + \cdots + \alpha_m(z)\mathbf{M}^{(m)}(z)$$

- degree constraints become degree constraints on $\alpha_i(z)$.

BL [SIMAX 2000]

For given vector of power series and given vector of degree bounds $\vec{n}$ it computes an order basis for problem

$$A_0(z)U_0(z) + \cdots + A_m(z)U_m(z) = O(z^{N+1})$$

with degree bounds

$$\deg\,(U_i) \leq n_i - 1.$$

- Order bases for Hermite-Padé approximant problem
- Fraction-free method (FFFG)
- Works for vector and matrix power series

Simultaneous-Padé

Solve as

$$\begin{bmatrix} A_1(z) \\ A_2(z) \\ \vdots \\ A_m(z) \end{bmatrix} U_0(z) + \begin{bmatrix} -A_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} U_1(z) + \begin{bmatrix} 0 \\ -A_0(z) \\ \vdots \\ 0 \end{bmatrix} U_2(z) + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -A_0(z) \end{bmatrix} U_m(z) = O(z^{N+1}).$$

Simultaneous-Padé becomes vector Hermite-Padé problem.

Use FFFG to solve in exact arithmetic.

Duality

Right and Left Matrix Padé duality:

$$A(z)V(z) - U(z) = O(z^K) \text{ and } \hat{V}(z)A(z) - \hat{U}(z) = O(z^K)$$

Hermite-Padé and Simultaneous-Padé duality:

$$A_0(z)U_0(z) + [A_1(z), \cdots, A_m(z)] \begin{bmatrix} U_1(z) \\ \vdots \\ U_m(z) \end{bmatrix} = O(z^{N+1})$$

$$V(z)[A_1(z), \cdots, A_m(z)] - [U_1(z), \cdots, U_m(z)]\, A_0(z) = O(z^{(N+1, \cdots, N+1)})$$

Useful for inversion formulae (L [LAA 1992])

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Hermite-Padé and Simultaneous-Padé duality:

$$A_0(z)U_0(z) + [A_1(z), \cdots, A_m(z)] \begin{bmatrix} U_1(z) \\ \vdots \\ U_m(z) \end{bmatrix} = O(z^{N+1})$$

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Duality : Much Better

Hermite-Padé and Simultaneous-Padé order bases are duals to each other (B-L [JCAM 1997]) .

$$\mathbf{A}(z)\mathbf{M}(z) = \begin{bmatrix} A_0(z) & A_1(z) & \cdots & A_m(z) \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} U_{00}(z) & \cdots & U_{0,m}(z) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ U_{m0}(z) & \cdots & U_{m,m}(z) \end{bmatrix} = O(z^{(N+1,0,\cdots,0)})$$

$$\mathbf{A}^*(z)\mathbf{M}^*(z) \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -A_1(z) & A_0(z) & & \\ \vdots & & \ddots & \\ -A_m(z) & & & A_0(z) \end{bmatrix} \begin{bmatrix} V_{00}(z) & \cdots & V_{0,m}(z) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ V_{m0}(z) & \cdots & V_{m,m}(z) \end{bmatrix} = O(z^{(0,N+1,\cdots,N+1)})$$

where $\mathbf{P}^*(z) = \operatorname{adj}(\mathbf{P}(z))^T = \operatorname{cof}(\mathbf{P}(z))$.

Computation Process

Input : $\mathbf{A}(z)$, $\vec{n}$.

Process for Hermite-Padé : Computes Order bases of type

$$\vec{v}^{(0)}, \vec{v}^{(1)}, \ldots \quad \ldots, \vec{v}^{(N+1)}$$

$$\mathbf{I} = \mathbf{M}(\vec{v}^{(0)}, z) \quad \rightarrow \quad \mathbf{M}(\vec{v}^{(1)}, z) \rightarrow \quad \cdots \quad \rightarrow \mathbf{M}(\vec{v}^{(N+1)}, z)$$

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$$\mathbf{I} = \mathbf{M}^{*}(\vec{v}^{(0)}, z) \quad \rightarrow \quad \mathbf{M}^{*}(\vec{v}^{(1)}, z) \rightarrow \quad \cdots \quad \rightarrow \mathbf{M}^{*}(\vec{v}^{(N+1)}, z)$$

Computation Process

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Process for Hermite-Padé : Computes Order bases of type

$$\vec{v}^{(0)}, \vec{v}^{(1)}, \ldots \quad \ldots, \vec{v}^{(N+1)}$$

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$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

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Recursion (Hermite-Padé)

$$\mathbf{M}(\vec{v}^{(i+1)}, z) \quad = \quad \mathbf{M}(\vec{v}^{(i)}, z)\mathbf{A}(z)\mathbf{B}(z)$$

$$\begin{bmatrix} m_{11}^{(i+1)}(z) & \cdots m_{1,m}^{(i+1)}(z) \\ \vdots & \vdots \\ m_{m1}^{(i+1)}(z) & \cdots m_{m,m}^{(i+1)}(z) \end{bmatrix} = \begin{bmatrix} m_{11}^{(i)}(z) & \cdots m_{1,m}^{(i)}(z) \\ \vdots & \vdots \\ m_{m1}^{(i)}(z) & \cdots m_{m,m}^{(i)}(z) \end{bmatrix} \begin{bmatrix} * & & \\ * & * & \\ & \ddots & \\ * & & * \end{bmatrix} \begin{bmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Recursion (Simultaneous-Padé)

Take adjoints and then transposes to get

$$\mathbf{M}^*(\vec{v}^{(i+1)}, z) \quad = \quad \mathbf{M}^*(\vec{v}^{(i)}, z)\mathbf{A}^*(z)\mathbf{B}^*(z)$$

$$\begin{bmatrix} \hat{m}_{11}^{(i+1)}(z) & \cdots \hat{m}_{1,m}^{(i+1)}(z) \\ \vdots & \vdots \\ \hat{m}_{m1}^{(i+1)}(z) & \cdots \hat{m}_{m,m}^{(i+1)}(z) \end{bmatrix} = \begin{bmatrix} \hat{m}_{11}^{(i)}(z) & \cdots \hat{m}_{1,m}^{(i)}(z) \\ \vdots & \vdots \\ \hat{m}_{m1}^{(i)}(z) & \cdots \hat{m}_{m,m}^{(i)}(z) \end{bmatrix} \begin{bmatrix} * & & * \\ 0 & * & * \\ \vdots & * & \\ 0 & * & * \end{bmatrix} \begin{bmatrix} 1 & & & \\ & z & & \\ & & \ddots & \\ & & & z \end{bmatrix}$$

The Algorithm

At each iteration for Hermite-Padé

- Increase order of each row using ‘special’ pivot column π
- Increase order of ‘special’ pivot column π
- Normalize order basis to get special shifted *Popov* form

At each iteration for Simultaneous-Padé

- Increase order of ‘special’ row using fraction-free Gaussian elimination on first term of residual
- Increase order of ‘special’ pivot row
- Normalize order basis to get special shifted *Popov* form

Future Research

- Fraction-Free $\rightarrow$ modular methods
- Duality for alternative order basis algorithm for Hermite-Padé and vector Hermite-Padé approximation problem
- Use alternative order basis algorithm for noncommutative case of Ore matrix polynomials
- Use above algorithm to create faster algorithms for matrix polynomial and matrix Ore normal forms (Popov, Hermite)