Closed form solutions of linear odes having elliptic function coefficients

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Introduction

- Consider the nth order homogeneous linear ODE

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  where the $A_i(x)$ are elliptic functions.

- e.g., *Lamé's equation* (Kamke 2.26):

$$y''(x) - [n(n+1)\wp(x) + B]y(x) = 0$$

  where n is a positive integer, B any constant, and $\wp(x)$ the Weierstrass $\wp$ function.

Examples from Kamke

1.49: $y' + a\wp'(x)y^3 + 6a\wp(x)y^2 + (2a{+}1)\,\dfrac{\wp''(x)}{\wp'(x)}y + 2\,(a{+}1) = 0$

2.27: $y'' + \big(a\ \mathsf{sn}^2 x + b\big)\,y = 0$

2.28: $y'' = \left(\dfrac{1}{30}\wp^{(4)}(x) + \dfrac{7}{3}\wp''(x) + a\wp(x) + b\right) y$

2.72: $y'' + a\wp'(x)y' + [\alpha + \beta\wp(x) - 4na\wp^2(x)]y = 0$

2.73: $y'' + \dfrac{\wp^3 - \wp\wp' - \wp''}{\wp' + \wp^2}y' + \dfrac{(\wp')^2 - \wp\wp' - \wp\wp''}{\wp' + \wp^2}y = 0$

2.74: $y'' + k^2\dfrac{\mathsf{sn}x\ \mathsf{cn}x}{\mathsf{dn}x}y' + n^2 y\,\mathsf{dn}^2 x = 0$

and also 2.439–2.441, 3.9–3.14, 3.28, 4.10

Felix Klein on the younger generation

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Elliptic Functions

- $f(x)$ is *doubly-periodic of the first kind* if there exist two periods T, T' such that
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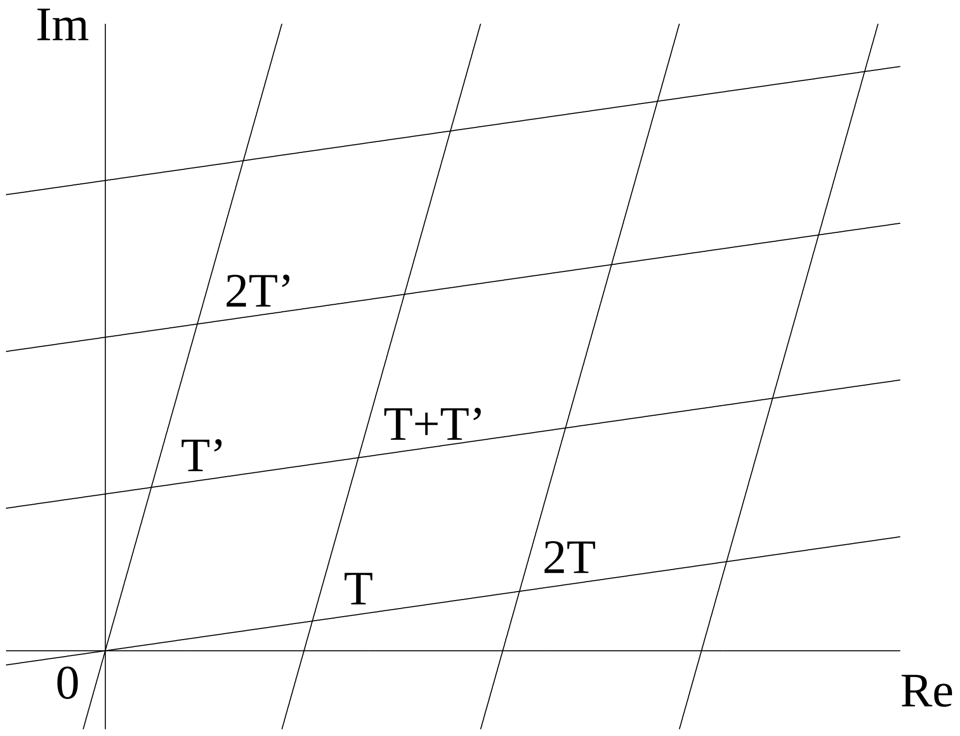

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  $\Rightarrow \wp'', \wp''', \ldots$ expressible in terms of $\wp$ and $\wp'$:

  $$\wp'' = 6\wp^2 - \tfrac{1}{2}g_2, \quad \wp''' = 12\wp\wp', \quad \text{etc.}$$

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- Any elliptic function can always be expressed as

$$R_1(\wp) + R_2(\wp)\wp' \quad \in \quad \mathbb{K}(\wp, \wp'),$$

where $R_1(\wp)$, $R_2(\wp)$ are rational functions of $\wp$

Picard's Theorem (c. 1879)

- Consider the nth order homogeneous linear ODE

$$A_n(x)y^{(n)}(x) + \cdots + A_1(x)y'(x) + A_0(x)y(x) = 0 \qquad (1)$$

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- If the general solution of (1) is uniform (i.e., path-independent), then (1) possesses at least one solution that is doubly-periodic of the second kind.

Doubly-Periodic of the Second Kind

- $F(x)$ is *doubly-periodic of the second kind* if there exist two periods T, T', and two constants s, s' such that

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- hence $\frac{F'(x)}{F(x)}$ is doubly-periodic of first kind:

$$\frac{F'(x+T)}{F(x+T)} = \frac{sF'(x)}{sF(x)} = \frac{F'(x)}{F(x)} \qquad \Longrightarrow \quad \frac{F'(x)}{F(x)} \in \mathbb{K}(\wp, \wp')$$

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- $\Rightarrow F(x) = e^{\int^x f(u)du}$, for some $f \in \mathbb{K}(\wp, \wp')$.

Classical Solution of Lamé's Equation

$$y''(x) - [n(n+1)\wp(x) + B]y(x) = 0$$

- One solution is

$$y(x) = \frac{\sigma(x-a_1)\sigma(x-a_2)\cdots\sigma(x-a_n)}{\sigma^n(x)} e^{x\sum_{i=1}^n \zeta(a_i)},$$

where $\sigma(x)$ and $\zeta(x)$ are the Weierstrass sigma and zeta functions, respectively, and a_1, a_2,. . . ,a_n are constants satisfying the n equations:

$$\frac{\wp'(a_1)+\wp'(a_2)}{\wp(a_1)-\wp(a_2)} + \frac{\wp'(a_1)+\wp'(a_3)}{\wp(a_1)-\wp(a_3)} + \cdots + \frac{\wp'(a_1)+\wp'(a_n)}{\wp(a_1)-\wp(a_n)} = 0$$

$$\frac{\wp'(a_2)+\wp'(a_1)}{\wp(a_2)-\wp(a_1)} + \frac{\wp'(a_2)+\wp'(a_3)}{\wp(a_2)-\wp(a_3)} + \cdots + \frac{\wp'(a_2)+\wp'(a_3)}{\wp(a_2)-\wp(a_n)} = 0$$

$$\vdots$$

$$\frac{\wp'(a_{n-1})+\wp'(a_1)}{\wp(a_{n-1})-\wp(a_1)} + \frac{\wp'(a_{n-1})+\wp'(a_2)}{\wp(a_{n-1})-\wp(a_2)} + \cdots + \frac{\wp'(a_{n-1})+\wp'(a_n)}{\wp(a_{n-1})-\wp(a_n)} = 0$$

$$(2n-1)\sum_{i=1}^{n} \wp(a_i) = B.$$

- A second solution is $y(-x)$.

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- Picard’s Theorem implies these exist for many odes of interest
- e.g., Lamé’s Equation, for $n = 1$:

$$D_x - \frac{\wp'(u) - \sqrt{4B^3 - g_2 B - g_3}}{2(\wp(u) - B)}, \quad D_x - \frac{\wp'(u) + \sqrt{4B^3 - g_2 B - g_3}}{2(\wp(u) - B)}$$

i.e., solutions

$$y_1(x), \quad y_2(x) = \exp\left(\frac{1}{2}\int^x \frac{\wp'(u) \pm \sqrt{4B^3 - g_2 B - g_3}}{\wp(u) - B} du\right)$$

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- we give an efficient algorithm for 2nd order case

Finding elliptic function solutions

- Find solutions of the form $R_1(\wp) + R_2(\wp)\wp'$ for

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- ...but solutions will not often be of this form

Product of Solutions

- suppose $A_{n-1}(x) = 0$ in

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- then $Y(x) := y_1(x) \cdots y_n(x)$ is doubly-periodic of *first* kind

$$\Longrightarrow \; Y(x) \in \mathbb{K}(\wp, \wp')$$

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- Then solutions y_1, y_2 of (2) are:

$$y_1(x) = \exp \int^x \frac{Y' - C}{2Y}\, du, \qquad y_2(x) = \exp \int^x \frac{Y' + C}{2Y}\, du$$

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- Solutions to Lamé's equation (4):

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- Let y_1, y_2 be two independent hyperexponential solutions of

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- divide by $y_1y_2 = Y$ and rearrange, giving:

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- $\Rightarrow$ **symmetric power method still works for this case**

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  $\Rightarrow$ **symmetric power method fails**

One Approach

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- let $\hat{L} =$ symmetric product $(L, \overline{L})$. 4th order ODE, in $\mathbb{K}(z)[D_z]$
- then $(D_z - r_1 - \overline{r}_1) \in \mathbb{K}(z)[D_z]$ is right hand factor of $\hat{L}$

Recovering Factors of L

- if $D_z - c(z)$ is a right hand factor of $\hat{L}$, set

$$v(z) = \frac{c(z)}{2}$$

$$t(z) = v(z)^2\omega(z) + v'(z)\omega(z) + \frac{1}{2}v(z)\omega'(z)$$

$$u(z) = \frac{1}{2b(z)}(a'(z) + 4a(z)v(z) - 4t(z)v(z) - t'(z))$$

- if $a(z) = u(z)^2 + t(z)$, then we have the factorization

$$L = (D_x + s(x))\,(D_x - s(x))$$

where $s(x) = u\,(\wp(x)) + v\,(\wp(x))\,\wp'(x)$

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- $\Rightarrow v(z) = \frac{1}{z}$, $u(z) = 1$
- $\Rightarrow$ right hand factor of (6) is $D - \left(1 + \frac{\wp'}{\wp}\right)$, i.e., a solution is

$$y_1(x) = e^{\int^x (1 + \frac{\wp'(u)}{\wp(u)}) du} = e^x \wp$$

Conclusion

- complete algorithm for factoring 2nd order odes in $\overline{\mathbb{K}}(\wp, \wp')[D_x]$
- two hyperexponential solutions (symmetric power): in Maple
- one hyperexponential solution: soon in Maple

Future Work

- higher order odes - e.g., Beke-like algorithm
- for second order, Kovacic-like algorithm