SNAP:
A Maple 8 package
for arithmetic with numeric polynomials

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Introduction: Symbolic-Numeric Algorithms for Polynomials

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Data: Univariate polynomials with real floating-point coefficients

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Computations: Division, remainder, GCD, ...

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Goal: Implement provably fast and numerically stable algorithms

Example: numerical coprimeness test

```
> with(SNAP):
> a := 0.1*z^2+1.5*z-0.2:
> b := 0.2*z^3+0.15:
> AreCoprime(a,b,z,0.5);
```

false

```
> AreCoprime(a,b,z,0.1);
```

true

Why look at symbolic-numeric problems?

- Linear systems theory
- Control theory
- Computer aided design
- ...

SNAP functionalities

AreCoprime
DistanceToCommonDivisors
DistanceToSingularPolynomials
EpsilonGCD
EuclideanReduction
IsSingular
QuasiGCD
Quotient
Remainder

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Division
Distance problems
Tests
Approximate GCDs

Algorithmic Structure of SNAP package

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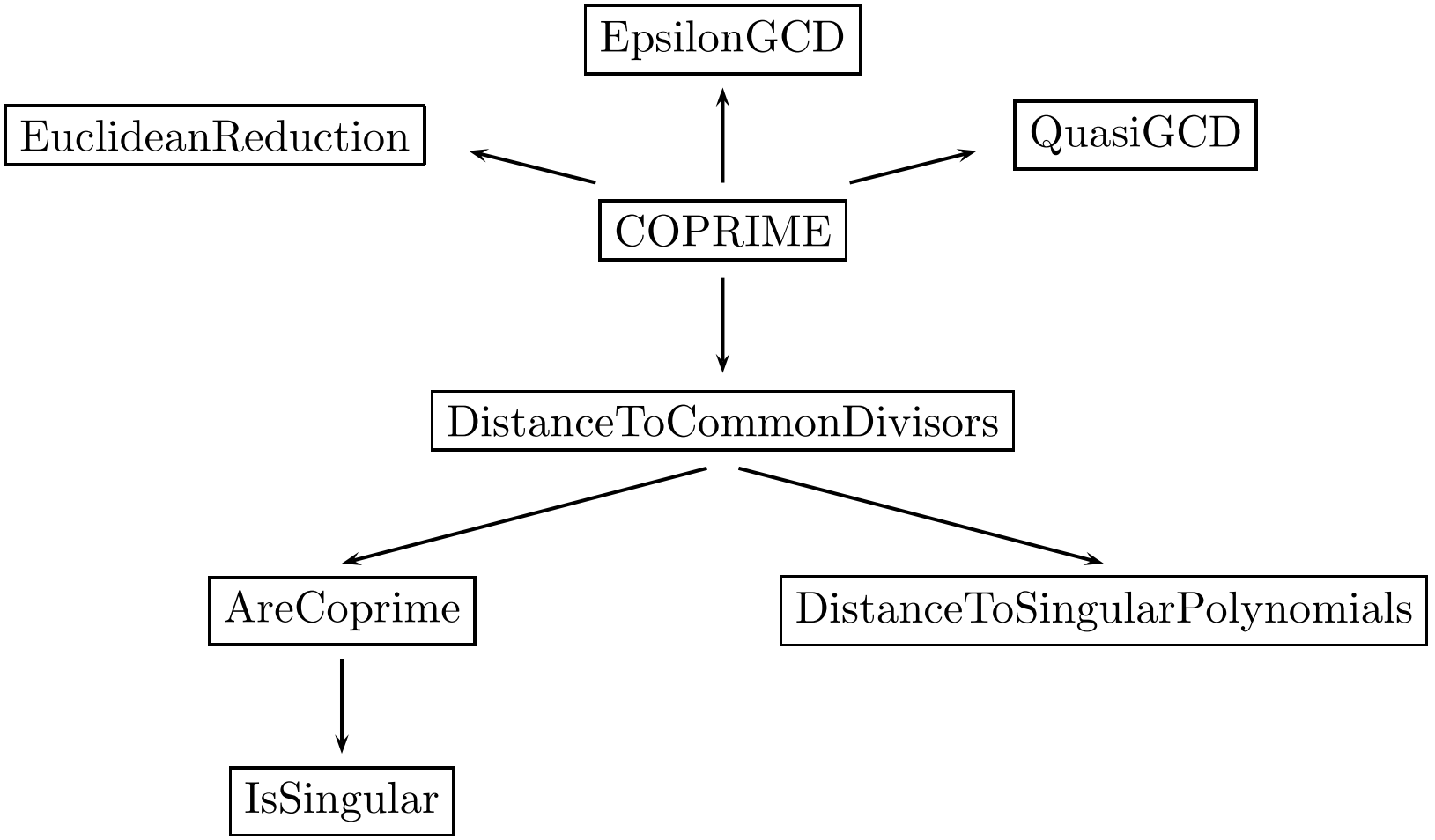
- Central procedure is COPRIME algorithm which solves :

$$\begin{aligned} a(z) \cdot s(z) + b(z) \cdot t(z) &\approx 1 \\ a(z) \cdot u(z) + b(z) \cdot v(z) &\approx z^{m+n-1} \end{aligned}$$

- Can use for efficient numerical coprimeness test
- Solved via a numerically stable Euclidean-like reduction algorithm
- Can use to often find certified approximate GCDs


EpsilonGCD
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Efficient numerical coprimeness test

Timings (in seconds) for polynomials of degrees m, n

$m = n + 1$	LinearAlgebra*	SNAP
50	0.33	0.34
100	1.30	1.34
250	8.6	8.7
500	43	33
10^3	307	126
$2 \cdot 10^3$	3360	515

Pentium III 800 MHz with 512MB of RAM, under Linux

* = solve a Sylvester linear system with `LinearAlgebra[LinearS olve]`.

DistanceToCommonDivisors($a(z), b(z), \epsilon$)

Given $a(z) = \sum_{i=0}^{m} a_i z^i$ and $b(z) = \sum_{i=0}^{n} b_i z^i$, the set

$$P = \{(p, q) \, : \, \deg p \leq m, \, \deg q \leq n, \, \deg \gcd(p, q) > 0\}$$

and a suitable norm $\| \cdot \|$, estimate the distance

$$\text{dist} = \inf_{P} \|(a, b) - (p, q)\| \quad ?$$

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Can solve this using linear algebra.

Norms

View $c \in \mathbb{C}[z]$, $c(z) = c_0 + .. + c_n z^n$ as the vector $\vec{c} = (c_0, .., c_n)^T$.

- Norm for $\mathbb{C}[z]$: $\quad ||c|| = ||\vec{c}||_1 = \sum_j |c_j|$
- Norm for $\mathbb{C}[z]^{r \times s}$: $\quad ||(c_{j,k})|| = ||(||c_{j,k}||)||_1 = \max_k \sum_j ||c_{j,k}||$.

Linear algebra formulation

- Sylvester matrix for $a(z)$ and $b(z)$:

$$S = \left[\begin{array}{cc|ccc} a_0 & & b_0 & & \\ a_1 & a_0 & b_1 & b_0 & \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & & b_2 & b_1 \\ & a_3 & & & b_2 \end{array}\right]$$

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- Well known: $\det S \neq 0 \iff \gcd(a, b) = 1$

$$\iff \text{can solve } a(z)s(z) + b(z)t(z) = 1.$$

Linear algebra formulation (cont'd)

Also well known: classical lower bound $\frac{1}{||S^{-1}||_1} \leq \text{dist}$.

Proof: $||S||_1 = ||(a,b)||$ where $||B||_1 = \max_{x \neq 0} \frac{||Bx||_1}{||x||}$

$$\implies \quad \text{dist} = \inf\{||S - S(p,q)||_1 : S(p,q) \text{ singular}\}$$

$$\geq \min\{||S - B||_1 : \ B \text{ singular}\} = \frac{1}{||S^{-1}||_1}$$

□

Note: $||S||_1$ can be computed via SVD.

A better lower bound [Beckermann & Labahn - 1998]

Sylvester matrix for $a(z)$ and $b(z)$ and its inverse:

$$S = \left[\begin{array}{cc|ccc} a_0 & & b_0 & & \\ a_1 & a_0 & b_1 & b_0 & \\ a_2 & a_1 & b_2 & b_1 & b_0 \\ a_3 & a_2 & & b_2 & b_1 \\ & a_3 & & & b_2 \end{array}\right] \qquad S^{-1} = \left[\begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{array}\right]$$

$$\frac{1}{\|S^{-1}\|_1} \leq \frac{1}{\kappa} \leq \text{dist} \quad \text{defined by } \kappa = \max(\textstyle\sum_i |*_i|, \sum_i |*_i|).$$

EuclideanReduction($a(z), b(z), \epsilon$**)**

EuclideanReduction($a(z), b(z), \epsilon$)

- Numerically stable euclidean reduction.
  - algorithm "jumps" remainders resulting from ill-conditioned subproblems
  - finds such ill-conditioned subsystems via estimating condition numbers
  - fast : quadratic rather than cubic
  - used to determine if two numeric polynomials are coprime
  - can be used for additional numerical GCD computations

Certified approximate GCDs (Rupprecht 's example)

$$a(z) = (z^4 - 1.000001)(z^3 - 3.000001z + 0.99999999)$$

$$b(z) = (z^4 - 0.9999999)(z^4 - 3.0000003z - 2.9999999)$$

`SNAP[EpsilonGCD](a,b,z)` with 10 digits yields the ϵ-GCD

$$z^4 + 5.198691325\,10^{-8} z^3 + 5.804634175\,10^{-7} z^2 + 9.873862873\,10^{-7} z - 1.000000584$$

with $\epsilon = 0.0006481143605.$

Satisfactory answers from Euclidean reduction

$$a(z) = z^6 - 12.4z^5 + 62.53z^4 - 163.542z^3 + 232.9776z^2 - 170.69184z + 50.18112$$

$$b(z) = z^5 - 17.6z^4 + 118.26z^3 - 372.992z^2 + 538.3333z - 274.09272$$

`SNAP[EuclideanReduction](a,b,z,tau=1e-8)` with 10 digits yields the reduced pair

$$0.2500000000z^2 - 0.8750003765z + 0.6600005057,$$
$$-0.90726410^{-7}z + 0.133529610^{-6}.$$

Euclidean algorithm

- Typical polynomial remainder sequence:

$$\begin{bmatrix} a & b \end{bmatrix} \xrightarrow{\times U^{(1)}} \begin{bmatrix} a^{(1)} & b^{(1)} \end{bmatrix} \xrightarrow{\times U^{(2)}} \begin{bmatrix} a^{(2)} & b^{(2)} \end{bmatrix} \xrightarrow{\times U^{(3)}} \begin{bmatrix} a^{(3)} & b^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{array}{cccccccc} 3 & 2 & & 2 & 1 & & 1 & 0 & & 0 & -\infty \end{array}$$

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The first column of $U^{(1)}U^{(2)}U^{(3)}$ is $\begin{bmatrix} s(z) \\ t(z) \end{bmatrix}$ s.t. $as + bt = 1$.

- Fast but unstable!

Linear algebra interpretation

$$a(z) = 1 + \epsilon z + (1+\epsilon)\, z^2 + z^3 + z^4, \qquad b(z) = -2 + 3\, z - z^2 + z^3$$

$$S = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ \epsilon & 1 & 0 & 3 & -2 & 0 & 0 \\ 1+\epsilon & \epsilon & 1 & -1 & 3 & -2 & 0 \\ 1 & 1+\epsilon & \epsilon & 1 & -1 & 3 & -2 \\ 1 & 1 & 1+\epsilon & 0 & 1 & -1 & 3 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Euclid's algorithm solves linear subsystems.

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Euclid's algorithm solves linear subsystems.

Problem: what to do with ill-conditioned subsystems?

COPRIME algorithm [Beckermann & Labahn - 1998]

Look-ahead process: if $|U^{(2)}(0)|$ too small, jump over $\begin{bmatrix} a^{(2)} & b^{(2)} \end{bmatrix}$:

$$\begin{bmatrix} a & b \end{bmatrix} \overset{\times U^{(1)}}{\longrightarrow} \begin{bmatrix} a^{(1)} & b^{(1)} \end{bmatrix} \overset{\times U^{(2)}}{\longrightarrow} \begin{bmatrix} a^{(2)} & b^{(2)} \end{bmatrix} \overset{\times U^{(3)}}{\longrightarrow} \begin{bmatrix} a^{(3)} & b^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

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- Unstability related to the ill-conditioning of some $U^{(i)}$ at $z = 0$.
- Effective way of deducing $\begin{bmatrix} a^{(3)} & b^{(3)} \end{bmatrix}$ from $\begin{bmatrix} a^{(1)} & b^{(1)} \end{bmatrix}$.
- Skipping all the small $|U^{(i)}(0)|$ yields a small residual error.
- Provably stable (weak stability).

What about speed?

- At most $\max(m, n)$ jumps where $m = \deg a$ and $n = \deg b$.
- One jump of size s costs $O(s^3)$ flops.

$\hookrightarrow$ In most cases, $s \leq 3 \ll m+n$ and we have a $O((m+n)^2)$ algo.

$\hookrightarrow$ Worst-case: one large jump (i.e. $s \sim m+n$) costs $O((m+n)^3)$.

Summary

- COPRIME algorithm:
  - Fast in most cases
  - Proven weak (or forward) stability
  - Distance estimate + coprimeness test

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- COPRIME algorithm:
  - Fast in most cases
  - Proven weak (or forward) stability
  - Distance estimate + coprimeness test
- Not only a numerical coprimeness test:
  - ϵ-GCD
  - quasi-GCD

  can be guaranteed from the last accepted basis $[a^{(k)}\ b^{(k)}]$.

Concluding remarks

- User's guide available at www.ens-lyon.fr/∼cpjeanne

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- Extensions:
  - Other kinds of approximate GCDs (≥ 6 definitions)
  - Complex polynomials, multivariate polynomials, systems of polynomials
  - Approximate factorization
  - Symbolic-numerics for matrix polynomials