

Construction of Rational Invariants of Finite Abelian Groups and its Applications

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Motivation : Invariant Dynamic System

Consider the following dynamical system (with c a parameter) ¹

$$x_1'(t) = x_1(t)(1 - c \cdot x_1(t) - x_1(t) \cdot x_2^2(t) - x_1(t) \cdot x_3^2(t))$$

$$x_2'(t) = x_2(t)(1 - c \cdot x_2(t) - x_2(t) \cdot x_1^2(t) - x_2(t) \cdot x_3^2(t))$$

$$x_3'(t) = x_3(t)(1 - c \cdot x_3(t) - x_3(t) \cdot x_1^2(t) - x_3(t) \cdot x_2^2(t))$$

Steady state gives system of polynomial equations

$$0 = 1 - c \cdot x_1 - x_1 x_2^2 - x_1 x_3^2$$

$$0 = 1 - c \cdot x_2 - x_2 x_1^2 - x_2 x_3^2$$

$$0 = 1 - c \cdot x_3 - x_3 x_1^2 - x_3 x_2^2$$

¹ Neural network model [SIAM J. Numer. Anal. Noonburg 1989].

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Solution space of system is **invariant** under the order 3 permutation

$$(x_1, x_2, x_3) \rightarrow (x_2, x_3, x_1).$$

~~We wish to work “modulo” this order 3 permutation.~~

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Easier Example : Invariant Polynomial System

Consider the following system of polynomial equations

$$x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3 + 12 = 0$$

$$x_1x_2 + x_2x_3 + x_1x_3 - 15 = 0$$

$$x_1x_2x_3 - 13 = 0$$

Each equation is invariant under the $\mathcal{G} = \mathbb{Z}_3$ linear matrix action

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

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Goal both examples : Find and rewrite system in terms of **invariants**

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} (3x_1^2x_2 + 3x_1x_3^2 + 3x_2^2x_3)\alpha^2 + (3x_1^2x_3 + 3x_1x_2^2 + 3x_2x_3^2)\alpha + (x_1^3 + 6x_1x_2x_3 + x_2^3 + x_3^3) \\ (x_1x_2 + x_1x_3 + x_2x_3)\alpha^2 + (x_1x_2 + x_1x_3 + x_2x_3)\alpha + (x_1^2 + x_2^2 + x_3^2) \\ x_1 + x_2 + x_3 \end{bmatrix}.$$

Here α primitive cube root of unity.

Finite Abelian Symmetries

Action : $\mathcal{G} \times \mathbb{K}^n \rightarrow \mathbb{K}^n$, $\mathcal{G}$ finite, abelian matrix group.

This talk:

- Determine important constructions for group actions
 - rational invariants, rewrite rules
- Integer linear algebra solves finite abelian symmetry problems
 - Gives complete and elegant description of above
- Given finite abelian action for systems can determine reduction
- Given systems can find finite abelian group action (if possible)

Related work

- K. Gattermann (ISSAC 1990)
 - Using group actions to reduce Gröbner bases comp.
- J-C Faugère and J. Svartz (ISSAC 2013)
 - Using abelian group actions to reduce polynomial systems.
- E. Hubert and G. Labahn (ISSAC 2012, FoCM 2013)
 - scaling symmetries: e.g. $(\mathbb{K}^*)^2 \times \mathbb{K}^5 \rightarrow \mathbb{K}^5$
$$(\alpha, \beta) \times (z_1, z_2, z_3, z_4, z_5) \rightarrow \left(\alpha^6 z_1, \beta^3 z_2, \frac{\beta}{\alpha^4} z_3, \frac{\alpha}{\beta^4} z_4, \alpha^3 \beta^3 z_5 \right).$$
- E. Hubert and G. Labahn ([This talk](#) - Preprint. [HAL-00921905])

Finite Abelian Group Actions

- Special Form of Finite Abelian Groups
- Diagonalization

Basic Facts : Finite Abelian Linear Groups

$\mathcal{G}$: finite abelian subgroup of $\mathrm{GL}_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

(i) Group is diagonalizable .

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- Original linear group action :

$$\begin{aligned} \mathcal{G} \times \mathbb{K}^n &\rightarrow \mathbb{K}^n \\ (G, x) &\mapsto G \cdot x \end{aligned}$$

- New action : (with $\mathcal{D} = R^{-1} \cdot \mathcal{G} \cdot R$ and $x = R \cdot z$)

$$\begin{aligned} \mathcal{D} \times \mathbb{K}^n &\rightarrow \mathbb{K}^n \\ (D, z) &\mapsto D \cdot z \end{aligned}$$

is diagonal action

$$\mathcal{D} \times \mathbb{K}^n \rightarrow \mathbb{K}^n$$

$$(\text{diagonal}(d_1, \dots, d_n), (z_1, \dots, z_n)) \mapsto (d_1 \cdot z_1, \dots, d_n \cdot z_n)$$

Basic Facts : Finite Abelian Linear Groups

$\mathcal{G}$: finite abelian subgroup of $\mathrm{GL}_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

(ii) Group isomorphism : $\mathcal{D} \leftrightarrow \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}$

Explicit via exponents :

$$\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s} \rightarrow \mathcal{D}$$

$$(m_1, \dots, m_s) \mapsto D_1^{m_1} \cdots D_s^{m_s}$$

Running Example (cont)

Polynomial system

$$f_1 = x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3 + 12$$

$$f_2 = x_1x_2 + x_2x_3 + x_1x_3 - 15$$

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(i) $\mathbb{Z}_3$ linear action $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$

(ii) Diagonalize $\mathbb{Z}_3$ (with α a primitive cube root of unity) via

- Diagonalize via $R = \begin{bmatrix} \alpha & \alpha^2 & 1 \\ \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \end{bmatrix}$

- Change coordinates $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha^2 & 1 \\ \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$

Running Example (cont)

(iii) This converts original system

$$f_1 = x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3 + 12$$

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to new system

$$f_1 = 3z_1z_2 + 3z_3 - 3z_3^2 + 12,$$

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(iv) Key idea : $\mathbb{Z}_3$ action is now

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \rightarrow \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \alpha \cdot z_1 \\ \alpha^2 \cdot z_2 \\ z_3 \end{bmatrix}$$

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Integer Linear Algebra

- Exponent and Order matrices
- Matrix notation and its properties
- Hermite Normal Form

Exponent and Order Matrices

Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_7 \times \mathbb{Z}_5$:

$$(z_1, z_2, z_3, z_4, z_5) \rightarrow (\alpha^6 z_1, \beta^3 z_2, \frac{\beta}{\alpha^4} z_3, \frac{\alpha}{\beta^4} z_4, \alpha^3 \beta^3 z_5).$$

Exponent and Order matrices:

$$A := \begin{bmatrix} 6 & 0 & -4 & 1 & 3 \\ 0 & 3 & 1 & -4 & 3 \end{bmatrix} \quad P := \begin{bmatrix} 7 & \\ & 5 \end{bmatrix}$$

Exponent matrix notation:

$$(\alpha, \beta)^A = (\alpha^6, \beta^3, \alpha^{-4} \beta^1, \alpha^1 \beta^{-4}, \alpha^3 \beta^3)$$

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Write diagonal action as : $(\lambda, \mathbf{z}) \rightarrow \lambda^A \star \mathbf{z}$

Star operator $\star$ is pointwise multiplication

Hermite Normal Form

Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_6 \times \mathbb{Z}_3$:

$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix}$$

$$[A, -P]$$

exponent
matrix

Hermite Normal Form

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$$\begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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exponent
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$\rightarrow$

$$[H_i, 0]$$

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$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix} \left[\begin{array}{cc|ccc} 1 & 1 & 3 & 2 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 \end{array} \right] = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[A, -P]$$

exponent
matrix

$$\begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}$$

unimodular
multiplier

$$[H_i \ 0]$$

Hermite
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(Unimodular means $W = V^{-1} \in \mathbb{Z}^{5 \times 5}$)

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$$\begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}$$

unimodular
multiplier

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Note : V not unique but can be normalized. Implies V_n is special

Finite Abelian Group Actions

- Rational invariants
- Rewrite rules

Rational Invariants $\mathbb{K}(z)^A$

Definition: $F(z)$ is *invariant* under $z \mapsto \lambda^A \star z$ if $F(\lambda^A \star z) = F(z)$

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$$[\textcolor{red}{A}, -\textcolor{green}{P}] \cdot \begin{bmatrix} \textcolor{red}{V}_i & \textcolor{blue}{V}_n \\ P_i & P_n \end{bmatrix} = [\textcolor{red}{H}_i & 0]$$

Invariant Laurent monomials:

$$\mathbf{z}^v = z_1^{v_1} \cdots z_n^{v_n}, \quad v \in \mathbb{Z}^n$$

$$(\lambda^A \star \mathbf{z})^v = \lambda^{Av} \mathbf{z}^v = \mathbf{z}^v \Leftrightarrow A \cdot v = 0 \pmod{P}$$

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Let

$$[A, -P] \cdot \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix} = [H_i \quad 0]$$

Invariant Laurent monomials:

$$z^v = z_1^{v_1} \cdots z_n^{v_n}, \quad v \in \mathbb{Z}^n$$

$$(\lambda^A \star z)^v = \lambda^{Av} z^v = z^v \Leftrightarrow A \cdot v = 0 \pmod{P}$$

Lemma: Rational Invariants: $F(z) \in \mathbb{K}(z)^A$:

$$F(z) = \frac{\sum_{v \in \text{colspan}_{\mathbb{Z}} V_n} a_v z^v}{\sum_{v \in \text{colspan}_{\mathbb{Z}} V_n} b_v z^v}$$

Rational Invariants and Rewrite Rules

Theorem: $A \in \mathbb{Z}^{s \times n}$, $[A, -P] \cdot V = [H, 0]$,

$$V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}, \quad W = V^{-1} = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix}$$

- (a) $y = [z_1, \dots, z_n]^{V_n}$ form generating set of rational invariants.
- (b) V normalized : components of $y = [z_1, \dots, z_n]^{V_n}$ are polynomials.
- (c) Rewrite rule : $F \in \mathbb{K}(z)^A \implies F(z) = F(y^{(W_d - P_d P_u^{-1} W_u)})$

Why?

Rational Invariants and Rewrite Rules

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Why? $v = V_n(W_d - P_d P_u^{-1} W_u)v$. any term z^v with $v \in \text{colspan}_{\mathbb{Z}} V_n$:

$$\begin{aligned} z^v &= z^{V_n(W_d - P_d P_u^{-1} W_u)v} \\ &= (z^{V_n})^{(W_d - P_d P_u^{-1} W_u)v} \\ &= (y^{(W_d - P_d P_u^{-1} W_u)})^v \end{aligned}$$

Then use Lemma.

Example

Polynomials in $\mathbb{K}[z_1, z_2, z_3]$:

$$\begin{aligned} f_1 &= 3z_1z_2 + 3z_3 - 3z_3^2 + 12, \\ f_2 &= -3z_1z_2 + 3z_3^2 - 15, \\ f_3 &= z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 13 \end{aligned}$$

Exponent matrix and order matrix. $A = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$ $P = \begin{bmatrix} 3 \end{bmatrix}$

Unimodular matrices :

$$V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix} = \left[\begin{array}{c|ccc} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \end{array} \right]$$

$$W = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ \hline 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

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- Invariants: $y_1 = z_1^3, \quad y_2 = z_1^1 z_2^1, \quad y_3 = z_3^1$

- Rewrite rule: $(z_1, z_2, z_3) \rightarrow (y_1^{1/3}, \frac{y_2^1}{y_1^{1/3}}, y_3^1).$

- Laurent Polynomials in y_1, y_2, y_3 :

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- 2 solns for new system : $(8, -4, 1), (-8, -4, 1)$. 6 solns for original
: $(2, -2, 1), (-2, 2, 1), (2\eta, -2\eta^2, 1), (-2\eta, -2\eta^2, 1), (2\eta^2, -2\eta, 1), (-2\eta^2, -2\eta, 1)$

- works well because V_n is triangular.

Example : Invariant Dynamic System

Recall first system invariant under $\mathbb{Z}_3$. Changing $\mathbf{x} = R \cdot \mathbf{z}$ gives

$$\begin{aligned} z_1'(t) &= \frac{z_1}{3} \left(1 - 2cz_3 + z_1^3 - 2z_2^3 - 2z_3^3 + 3\frac{z_2^2 z_3^2}{z_1} - \frac{z_2^2}{z_1} \right) \\ z_2'(t) &= \frac{z_2}{3} \left(1 - 2cz_3 - 2z_1^3 + z_2^3 - 2z_3^3 + 3\frac{z_1^2 z_3^2}{z_2} - \frac{z_1^2}{z_2} \right) \\ z_3'(t) &= \frac{z_3}{3} \left(1 - cz_3 + 4z_1^3 + 4z_2^3 + 4z_3^3 - 6\frac{z_1^2 z_2^2}{z_3} - \frac{z_1 z_2}{z_3} \right) \end{aligned}$$

In terms of invariants these equations become:

$$\begin{aligned} y_1'(t) &= y_1 \left(1 - 2cy_3 + y_1 - 2\frac{y_2^3}{y_1} - 2y_3^3 + 3\frac{y_2^2 y_3^2}{y_1} - \frac{y_2^2}{y_1} \right) \\ y_2'(t) &= \frac{y_2}{3} \left(2 - 4cy_3 - y_1 - \frac{y_2^3}{y_1} - 4y_3^3 + 3\frac{y_1 y_3^2}{y_2} + 3\frac{y_2^2 y_3^2}{y_1} - \frac{y_2^2}{y_1} - \frac{y_1}{y_2} \right) \\ y_3'(t) &= \frac{y_3}{3} \left(1 - cy_3 + 4y_1 + 4\frac{y_2^3}{y_1} + 4y_3^3 - 6\frac{y_2^2}{y_3} - \frac{y_2}{y_3} \right) \end{aligned}$$

Dynamic Systems

Write $z'(t) = z(t) \star F(z(t))$ with $F(\lambda^A \star z) = F(z)$

Determine finite group diagonalization info for invariance of $F(z)$

$$A \in \mathbb{Z}^{r \times n}, \quad P \in \mathbb{Z}^{r \times r}, \quad V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}, \quad W = V^{-1} = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix}$$

Theorem

If $y(t) = z(t)^{V_n}$ is the set of invariants then the reduced dynamic system is given by

$$y'(t) = y \star F(y^{W_d}) \cdot V_n.$$

Then $z(t) = y(t)^{(W_d - P_d P_u^{-1} W_u)}$ solves

$$z'(t) = z \star F(z).$$

Why?

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$$A \in \mathbb{Z}^{r \times n}, \quad P \in \mathbb{Z}^{r \times r}, \quad V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}, \quad W = V^{-1} = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix}$$

Theorem

If $y(t) = z(t)^{V_n}$ is the set of invariants then the reduced dynamic system is given by

$$y'(t) = y \star F(y^{W_d}) \cdot V_n.$$

Then $z(t) = y(t)^{(W_d - P_d P_u^{-1} W_u)}$ solves

$$z'(t) = z \star F(z).$$

Why? Use

$$\frac{d}{dt}(z^{V_n}) = z^{V_n} \star (z^{-1} \star \frac{dz}{dt}) \cdot V_n$$

Future Research Directions

- (i) Extend to parameterized and dynamic systems
- (ii) Extend from Finite Abelian to Finite Solvable Group actions
 - e.g. Neural network example invariant under D_3 .
- (iii) Combine scaling symmetries with finite diagonal actions
 - makes use of Smith Normal Form