Symbolic Computation of Convolution Integrals of Holonomic Functions

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Joint work with :

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I Introduction

Computer Algebra and Calculus

Calculus was a big success story for computer algebra in early days.

Integration played a major role in success of calculus.

1. Integration usually meant indefinite integration at start
2. Risch algorithm played significant role.
   - Decision procedure was quite intriguing for average researcher
   - Decision procedure was unknown for average user

Indefinite Integration in Maple

1. Fast front end : basically matching patterns for (large set of) common cases
2. Slower back end : Risch
3. Excellent results for elementary functions
4. So-so results for special functions
   - issue for linear differential solver
5. People involved : M. Monagan (front end), G.L., K.O. Geddes (Risch), some students

Issues with Indefinite Integration

Researchers often wanted definite integration
(e.g. integral transforms)

Issues:

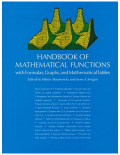
1. fundamental theorem of calculus often did not work
2. sometimes multiple answers depending on parameters of integrand
3. unclear on how to compute such answers (beyond pattern matching)
4. analysis and algebra both involved

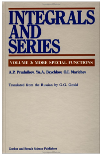
Dynamic Dictionary of Mathematical Functions

Aim of the project

DDMF = Mathematical Handbooks + Computer Algebra + Web

1. Develop and use computer algebra algorithms to generate the formulas;
2. Provide web-like interaction with the document and the computation.

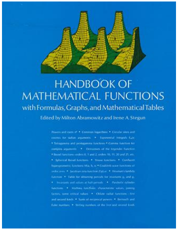




http://ddmf.msr-inria.inria.fr/

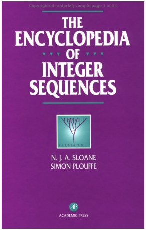
Equations Are a Good Data Structure

- Classical:
  polynomials represent their roots better than radicals.
  Algorithms: Euclidean division and algorithm, Gröbner bases.
- Recent:
  same for linear differential or recurrence equations.
  Algorithms: non-commutative analogues.



About 25% of Sloane's encyclopedia,
60% of Abramowitz & Stegun.

eqn+ini. cond.=data structure



Examples of Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$ [Strehl92]

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$ [GlMo94]

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor!}$$ [Doetsch30]

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k} (q;q)_{n+k}}$$ [Andrews74]

$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j} (q;q)_i (q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k} (q;q)_{n-k}}$$ [Paule85].

More Identities

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n \quad \text{[Abel1826]}$$

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle, \quad \text{[Frobenius1910]}$$

$$\sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{n} \binom{n}{k} B_{m+k}, \quad \text{[Gessel03]}$$

$$\int_0^\infty x^{k-1} \zeta(n, \alpha + \beta x)\, dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha),$$

$$\int_0^\infty x^{\alpha-1} \operatorname{Li}_n(-xy)\, dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)},$$

$$\int_0^\infty x^{s-1} \exp(xy) \Gamma(a, xy)\, dx = \frac{\pi y^{-s}}{\sin((a+s)\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

Computer Algebra Algorithms

Aim

- Prove these identities automatically (fast?);
- Compute the rhs given the lhs;
- Explain why these identities exist.

Examples:

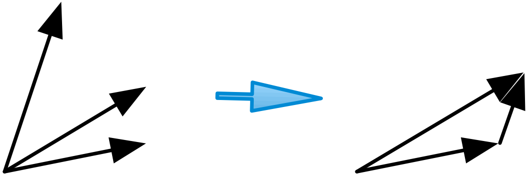
- 1st slide: Zeilberger's algorithm and variants;
- 2nd slide (1st 3): Majewicz, Kauers, Chen & Sun;
- last 3: recent generalization of previous ones (with Chyzak & Kauers).

Ideas

Confinement in finite dimension + Creative telescoping.

II Confinement in Finite Dimension

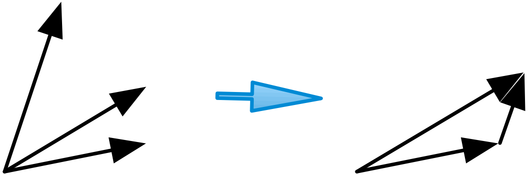
Confinement Provokes Identities



Obvious

$k+1$ vectors in dimension $k \to$ an identity.

Confinement Provokes Identities



Obvious

$k+1$ vectors in dimension $k \rightarrow$ an identity.

Idea: confine a function and all its derivatives.

First Algorithmic Proof: $\sin^2 + \cos^2 = 1$

```
> series(sin(x)^2+cos(x)^2-1,x,4);
```

$$O(x^4)$$

Why is this a proof?

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Why is this a proof?

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
5. Cauchy's theorem concludes.

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What about $\sin' = \cos$?

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Second algorithmic proof (same idea): $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

```
> for n to 5 do
   fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n od;
```

Third Proof: Contiguity of Hypergeometric Series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \underbrace{\frac{(a)_n(b)_n}{(c)_n n!}}_{u_{a,n}} z^n, \qquad (x)_n := x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \rightarrow \; z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \rightarrow S_a F := F(a+1,b;c;z) = \frac{z}{a}F' + F.$$

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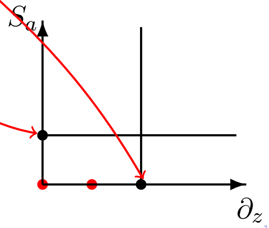

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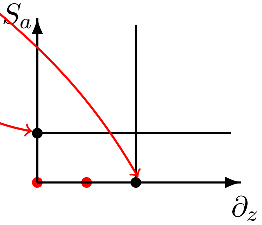
$$\frac{u_{a+1,n}}{u_{a,n}}=\frac{n}{a}+1\to S_aF:=F(a+1,b;c;z)=\frac{z}{a}F'+F.$$

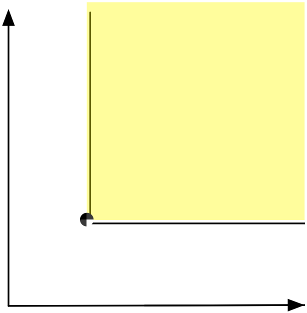


Gauss 1812: contiguity relation.
dim=2 $\Rightarrow S_a^2F, S_aF, F$ linearly dependent:
(Coordinates in $\mathbb{Q}(a,b,c,z)$.)

$$(a+1)(z-1)S_a^2F+((b-a-1)z+2-c+2a)S_aF+(c-a-1)F=0.$$

Gröbner Basis: Euclidean Division in Several Variables

1. Monomial ordering: order on $\mathbb{N}^k$, compatible with $+$, 0 minimal.



Gröbner Basis: Euclidean Division in Several Variables

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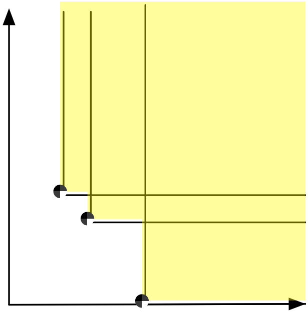
2. Gröbner basis of a (left) ideal $\mathcal{I}$: corners of stairs.



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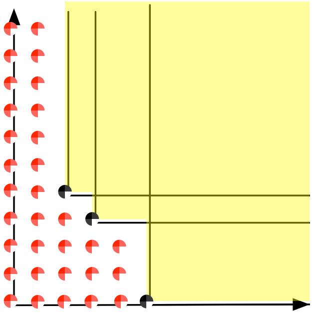

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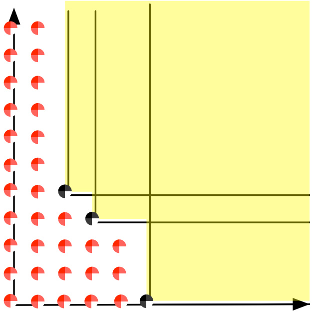
4. Reduction of P: Rewrite $P \bmod \mathcal{I}$ on this basis.



$\rightarrow$ An access to (finite dimensional) vector spaces

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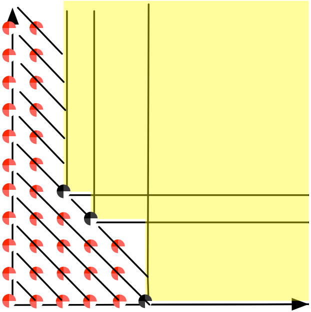

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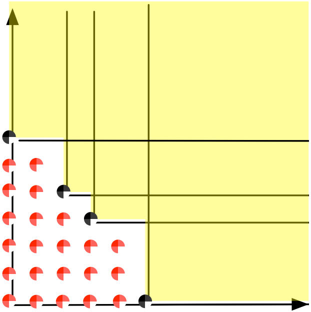
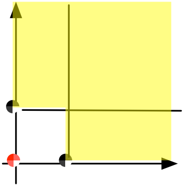
6. D-finiteness: $\dim = 0$.



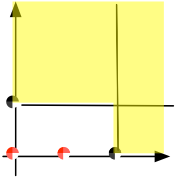
$\rightarrow$ An access to (finite dimensional) vector spaces

Examples

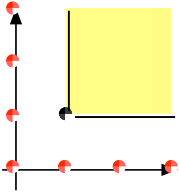
Binomial coeffs $\binom{n}{k}$ wrt S_n, S_k
Hypergeometric sequences



Bessel $J_\nu(x)$ wrt S_ν, ∂_x
Orthogonal pols wrt S_n, ∂_x



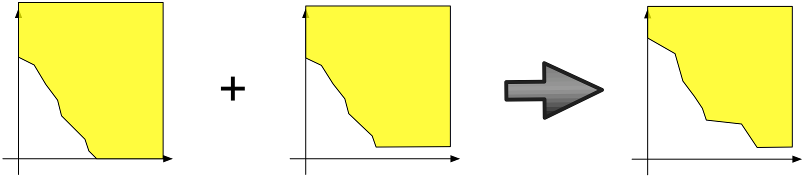
Stirling nbs wrt S_n, S_k



Abel type wrt S_m, S_r, S_k, S_s
$\mathrm{hgm}(m,k)(k+r)^k(m-k+s)^{m-k}\frac{r}{k+r}$

$\dim = 2$ in space of dim 4.

Closure Properties



Proposition

$$\dim \operatorname{ann}(f+g) \leq \max(\dim \operatorname{ann} f, \dim \operatorname{ann} g),$$
$$\dim \operatorname{ann}(fg) \leq \dim \operatorname{ann} f + \dim \operatorname{ann} g,$$
$$\dim \operatorname{ann} \partial f \leq \dim \operatorname{ann} f.$$

Algorithms by linear algebra.

Fourth Algorithmic Proof: Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$): recurrence of order 2;
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!, k \in \mathbb{N}$ generated over $\mathbb{Q}(x,n)$ by

   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

   $\rightarrow$ recurrence of order at most 4;
3. Translate into differential equation.

III Creative Telescoping

Summation by Creative Telescoping

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

Creative Telescoping (Zeilberger 90)

$$F_n = \sum_k u_{n,k} =?$$

IF one knows $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)) \cdot u_{n,k} = 0,$$

then the sum "telescopes", leading to $A(n, S_n) \cdot F_n = 0.$

Creative Telescoping (Zeilberger 90)

$$I(x) = \int_\Omega u(x,y)\,dy =?$$

IF one knows $A(x, \partial_x)$ and $B(x, y, \partial_x, \partial_y)$ such that

$$(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)) \cdot u(x,y) = 0,$$

then the integral “telescopes”, leading to $A(x, \partial_x) \cdot I(x) = 0.$

Creative Telescoping (Zeilberger 90)

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$$(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)) \cdot u(x,y) = 0,$$

then the integral "telescopes", leading to $A(x, \partial_x) \cdot I(x) = 0.$

Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

Richard P. Feynman 1985

Creative telescoping="differentiation" under integral+"integration" by parts

Diff. under $\int$ + Integration by Parts $\rightarrow$ Algorithm?

Ex.: $\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2} J_0(z), \quad (\underbrace{zJ_0'' + J_0' + zJ_0}_{A(z,\partial_z)\cdot J_0} = 0,$

$J_0(0) = 1).$

Diff. under $\int$ + Integration by Parts → Algorithm?

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$$I(z) = \int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt, \quad I'(z) = \int_0^1 -t\frac{\sin zt}{\sqrt{1-t^2}}\,dt,$$

$$I''(z) = \int_0^1 -t^2\frac{\cos zt}{\sqrt{1-t^2}}\,dt = -I(z) + \int_0^1 \sqrt{1-t^2}\cos zt\,dt,$$

$$I''(z) + I(z) = \underbrace{\left[\sqrt{1-t^2}\frac{\sin zt}{z}\right]_0^1}_{0} + \int_0^1 \frac{t}{\sqrt{1-t^2}}\frac{\sin zt}{z}\,dt = -\frac{I'(z)}{z}.$$

Diff. under $\int$ + Integration by Parts → Algorithm?

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$$\operatorname{ann}\frac{\cos zt}{\sqrt{1-t^2}} \ni \underbrace{A(z,\partial_z)}_{\text{no } t,\,\partial_t} - \partial_t \underbrace{\frac{t^2-1}{t}\partial_z}_{\text{anything}}$$

Diff. under $\int$ + Integration by Parts → Algorithm?

Ex.: $\displaystyle\int_0^1 \frac{\cos zt}{\sqrt{1-t^2}}\,dt = \frac{\pi}{2}J_0(z)$, $\big(\underbrace{zJ_0'' + J_0' + zJ_0}_{A(z,\partial_z)\cdot J_0} = 0,$

$J_0(0) = 1\big)$.

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Creative Telescoping

Input: generators of (a subideal of) $\operatorname{ann} f$;
Output: A, B such that $A - \partial_t B \in \operatorname{ann} f$, A free of t, ∂_t.
Algorithm: sometimes. (Why would they exist?)

Telescoping of $\mathcal{I}$ wrt t:

$$T_t(\mathcal{I}) := (\mathcal{I} + \partial_t\mathbb{Q}(z,t)\langle\partial_z,\partial_t\rangle) \cap \mathbb{Q}(z)\langle\partial_z\rangle.$$

Example: Pascal's Triangle Again

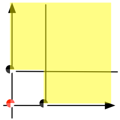
$$(S_nS_k - S_k - 1) \cdot \binom{n}{k} = 0 = (\underbrace{S_n - 2}_{\text{no } k,\, S_k} + (S_k - 1)(S_n - 1)) \cdot \binom{n}{k}.$$

Sum over $k \Rightarrow (S_n - 2)\sum_k \binom{n}{k} = 0$.

Example: Pascal's Triangle Again

$$(S_nS_k - S_k - 1)\cdot\binom{n}{k} = 0 = (S_n - 2 + (S_k-1)(S_n-1))\cdot\binom{n}{k}.$$

Reduce all monomials of degree $\leq s = 2$:



$$1 \to 1, \quad S_n \to \frac{n+1}{n+1-k}1, \quad S_k \to \frac{n-k}{k+1}1$$

$$S_n^2 \to \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}1, \quad S_k^2 \to \frac{(n-k-1)(n-k)}{(k+2)(k+1)}1,$$

$$S_nS_k \to \frac{n+1}{k+1}1.$$

Common denominator: $D_2 = (k+1)(k+2)(n+1-k)(n+2-k)$.

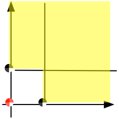
$D_2, D_2S_n, D_2S_k, D_2S_n^2, D_2S_k^2, D_2S_nS_k$ confined in

$$\mathrm{Vect}_{\mathbb{Q}(n)}(1, k1, k^21, k^31, k^41).$$

Example: Pascal's Triangle Again

$$(S_n S_k - S_k - 1) \cdot \binom{n}{k} = 0 = (S_n - 2 + (S_k - 1)(S_n - 1)) \cdot \binom{n}{k}.$$

Reduce all monomials of degree $\leq s = 2$:



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$D_2, D_2 S_n, D_2 S_k, D_2 S_n^2, D_2 S_k^2, D_2 S_n S_k$ confined in

$$\mathrm{Vect}_{\mathbb{Q}(n)}(1, k1, k^2 1, k^3 1, k^4 1).$$

This has to happen for some degree: $\deg D_s = O(s)$.

Polynomial Growth

Definition: Polynomial Growth p

There exists a sequence of polynomials P_s, s.t. for all $(a_1, \ldots, a_k)$ with $a_1 + \cdots + a_k \leq s$, $P_s \partial_1^{a_1} \cdots \partial_k^{a_k}$ reduces to a combination of elements below the stairs with polynomial coefficients of degree $O(s^p)$.

Theorem: ChyzakKauersSalvy2009

$$\dim T_t(\mathcal{I}) \leq \max(\dim \mathcal{I} + p - 1, 0).$$

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Proof. Same as above. Set $q := \dim \mathcal{I} + p$.

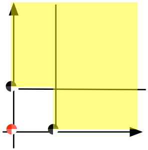
- In degree s, dim $O(s^q)$ below stairs.
- Number of monomials in $\partial_t, \partial_{i_1}, \ldots, \partial_{i_q}$: $O(s^{q+1})$;

$\Rightarrow$ any q variables linearly dependent $\Rightarrow \dim \leq q - 1$.

This proof gives an algorithm. Also, bounds available.

Examples (all with $p = 1$)

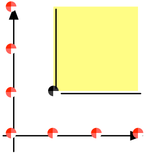
- Proper hypergeometric [Wilf & Zeilberger 1992]:



$$Q(n,k)\xi^k \frac{\prod_{i=1}^{u} (a_i n + b_i k + c_i)!}{\prod_{i=1}^{v} (u_i n + v_i k + w_i)!},$$

Q polynomial, $\xi \in \mathbb{C}$, a_i, b_i, u_i, v_i integers.

- Differential D-finite (definite integration);
- Stirling: ok for $n \geq 3$, e.g., Frobenius:



$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle .$$

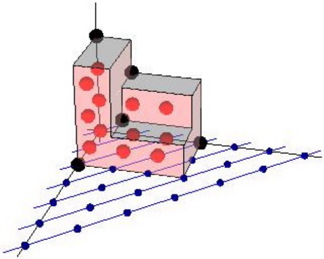
- Abel type: $\dim = 2 \rightarrow$ ok for $n \geq 4$, e.g., Abel:

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n.$$

IV Conclusion

Conclusion

- Summary:
  - Linear differential/recurrence equations as a data structure;
  - Confinement in vector spaces + creative telescoping → identities.
- Also:
  - Fast algorithms: Zeilberger 1990 (hypergeom); Chyzak 2000 (D-finite) Us 2009 (non-D-finite).
  - Bounds → identities;
  - Fast algorithms for special classes;
  - Efficient numerical evaluation.



- Open questions:
  - Replace polynomial growth by something intrinsic;
  - Exploit symmetries;
  - Structured Padé-Hermite approximants;
  - Understand non-minimality.