Order Bases

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Outline

Outline

The Problem

Simplest case : Given $A(z)$ find $U(z)$, $V(z)$ so that

$$a_0 + a_1 z + a_2 z^2 + \cdots = \frac{U(z)}{V(z)} + O(z^N)$$

- Usually some relationship between degrees and order.
- Sometimes more than one power series.
- Sometimes coefficients a_i not scalars.
- Often done as $A(z)V(z) - U(z) = O(z^\sigma)$

Example : e^z

$$
\begin{aligned}
e^z &= \frac{1 + 1/2z}{1 - 1/2z} + O(z^3) \\
&= \frac{1 + 1/2z + 1/12z^2}{1 - 1/2z - 1/12z^2} + O(z^5) \\
&= \frac{1 + 1/2z + 1/12z^2 + 1/120z^3}{1 - 1/2z - 1/12z^2 + 1/120z^3} + O(z^7)
\end{aligned} \tag{1}
$$

Simple Case : Early Example by Hermite

For $f(x)$ a polynomial of degree ℓ :

$$\int_0^1 f(x)e^{-zx}dx = \frac{p(z) - e^{-z}q(z)}{z^{\ell+1}}$$

with

$$\begin{aligned} p(z) &= f(0)z^{\ell} + f'(0)z^{\ell-1} + \cdots + f^{(\ell)}(0), \\ q(z) &= f(1)z^{\ell} + f'(1)z^{\ell-1} + \cdots + f^{(\ell)}(1). \end{aligned}$$

If $f(x) = x^n(x-1)^m$ then

- deg $p(z) \leq m$
- deg $q(z) \leq n$
- $e^z p(z) - q(z) = O(z^{m+n+1})$

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Example : Hermite-Padé Approximation

Given power series $A_0(z), \ldots, A_m(z)$ and integers $n_0, \ldots, n_m$

Find $P_0(z), ..., P_m(z)$ with deg $P_i(z) \leq n_i$ and

- $$A_0(z)P_0(z) + \cdots + A_m(z)P_m(z) \approx 0$$

  that is:

- $$A_0(z)P_0(z) + \cdots + A_m(z)P_m(z) = r_0 z^{N+1} + r_1 z^{N+2} + \cdots$$

  with $N = n_0 + \cdots + n_m + m$.

Examples

Given power series $y(z)$ find $P_0(z), P_1(z), P_2(z)$ such that

- $P_0(z)y(z) + P_1(z)y'(z) + P_2(z)y^{(2)}(z) \approx 0$
- $P_0(z) \;+\; P_1(z)y(z) \;+\; P_2(z)y^2(z) \approx 0$

  (generalized rational reconstruction)

Simultaneous Padé Approximants

Given power series $A_1(z), \ldots, A_m(z)$ and integers $n_0, \ldots, n_m$

find $U_1(z), ..., U_m(z)$ and $V(z)$ with some degree bounds :

$$A_1(z) \approx \frac{U_1(z)}{V(z)}, \quad A_2(z) \approx \frac{U_2(z)}{V(z)}, \quad \ldots, \quad A_m(z) \approx \frac{U_1(z)}{V(z)}.$$

Example : e^z and e^{2z} and e^{3z}

$$
\begin{aligned}
e^z &= \frac{2\,z^2 - 6\,z + 6}{-6\,z^3 + 11\,z^2 - 12\,z + 6} + O(z^4) \\
e^{2z} &= \frac{-z^2 + 6}{-6\,z^3 + 11\,z^2 - 12\,z + 6} + O(z^4) \\
e^{3z} &= \frac{2z^2 + 6\,z + 6}{-6\,z^3 + 11\,z^2 - 12\,z + 6} + O(z^4)
\end{aligned}
$$

Another early example from Hermite

For $f(x)$ a polynomial of degree ℓ :

$$\int_0^k f(x)e^{-zx}dx = \frac{p(z) - e^{-kz}q_k(z)}{z^{\ell+1}}$$

with

$$\begin{aligned} p(z) &= f(0)z^{\ell} + f'(0)z^{\ell-1} + \cdots + f^{(\ell)}(0), \\ q_k(z) &= f(k)z^{\ell} + f'(k)z^{\ell-1} + \cdots + f^{(\ell)}(k). \end{aligned}$$

If $f(x) = x^{n-1}(x-1)^n \cdots (x-m)^n$ and $N = mn + n - 1$ then

- $\deg p(z) \leq N - (n-1)$
- $\deg q_k(z) \leq N - n$
- $e^{kz}p(z) - q_k(z) = O(z^{N+1})$

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Charles Hermite




Most General Setting

Given matrix $\mathbf{A}(z)$, orders, $\vec{\sigma} = (\sigma_1, \ldots, \sigma_m)$ find all solutions of

$$\mathbf{A}(z) \cdot \mathbf{Q}(z) = z^{\vec{\sigma}} \mathbf{R}(z).$$

Often also have degree constraints $\vec{n} = (n_1, \ldots, n_m)$ and want

$$\deg Q_i(z) \leq n_i.$$

$\mathbf{R}(z)$ called residual.

Example

Let

$$\mathbf{A}(z) = \begin{bmatrix} \frac{1}{2} + z^2 - z^4 & 1 + \sin(z^2)^4 & \frac{1}{\sqrt{1+z^2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\vec{\sigma} = (z^8, 1, 1)$. Then a basis for all solutions given by

$$\mathbf{M}(z) = \begin{bmatrix} z^4 + \frac{11\,z^2}{2} & -\frac{10\,z^2}{19} + \frac{2}{19} & \frac{9\,z^2}{19} - \frac{11}{76} \\ -\frac{59\,z^2}{4} & z^2 - \frac{33}{19} & -\frac{5\,z^2}{4} + \frac{59}{152} \\ 12\,z^2 & \frac{32}{19} & z^2 - \frac{6}{19} \end{bmatrix}$$

with $\det \mathbf{M}(z) = z^8$. In this case the first 4 terms of the order residual $\mathbf{R}$ of $\mathbf{M}$ are given by

$$\mathbf{R}(z) = \begin{bmatrix} -\frac{19}{4} - \frac{367}{32}\,z^2 - \frac{189}{64}\,z^4 + O\left(z^6\right) & -\frac{97}{76} + \frac{89}{152}\,z^2 + O\left(z^4\right) & -\frac{13}{1216} - \frac{1093}{1216}\,z^2 + O\left(z^4\right) \\ -\frac{59\,z^2}{4} & -\frac{33}{19} + z^2 & \frac{59}{152} - \frac{5\,z^2}{4} \\ 12\,z^2 & \frac{32}{19} & -\frac{6}{19} + z^2 \end{bmatrix}.$$

Order Bases

Gives all solutions of all rational approximation problems.

Rational approximation problems appear in:

- Transcendence of *e* and other famous numbers
- Inversion formulae for structured matrices
- Linear diophantine equations (and hence to GCDs)
- Guessing recurrence formulae (e.g. Gfun)
- Reconstruction of power series to polynomial problems (e.g. DFactor)
- Matrix normal forms (Popov, etc)
- Fast polynomial matrix arithmetic
- . . .

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Outline

People

- 1730-1870: Lambert, Lagrange, Hankel, Frobenius, ...
  - connection to continued fractions, etc
- 1850-1890: Hermite
  - used these (informally) to prove transcendence of e (1873)
- 1890s: Padé
  - student of Hermite
  - first systematic study of rational approximation (1893)
  - Padé table

Outline

Transcendence of e

Suppose have integers $a_0, a_1, \ldots, a_m$:

$$a_m e^m + \cdots + a_1 e + a_0 = 0$$

Find integers $d_0, d_1, \ldots, d_m$ such that

$$d_0 e^k - d_k = \eta_k \quad \text{i.e. } e^k \approx \frac{d_k}{d_0}$$

with η_k small. Gives

$$\begin{aligned} a_m d_m + \cdots + a_1 d_1 + a_0 d_0 &= -a_m \eta_m - \cdots - a_1 \eta_1 \\ \textit{nonzero integer} &= \textit{small} \end{aligned}$$

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Transcendence of e

Recall : $p_k(z) - q_k(z)e^{-kz} = z^{\ell+1} \int_0^k f(x)e^{-zx}dx$

with $f(x)$ of degree ℓ and

$$\begin{aligned} p_k(z) &= f(0)z^{\ell} + f'(0)z^{\ell-1} + \cdots + f^{(\ell)}(0), \\ q_k(z) &= f(k)z^{\ell} + f'(k)z^{\ell-1} + \cdots + f^{(\ell)}(k). \end{aligned}$$

Sim. Padé approx when $f(x) = x^{n-1}(x-1)^n \cdots (x-m)^n$

- $\eta_k = e^k p_k(1) - q_k(1) = e^k F(0) - F(k)$.
- η_k can be made small (via controlling n)

Transcendence of e

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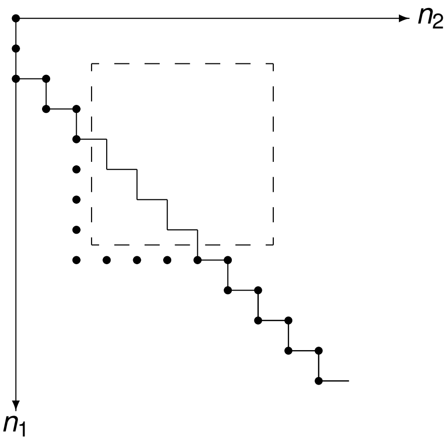
Henri Padé

Staircase path of computation in a Padé table:



Outline

Associated Linear System

$$A(z)V_n(z) - U_m(z) = z^{m+n+1}W(z)$$

$$(a_0+a_1z+\cdots)(v_0+\cdots+v_nz^n)-(u_0+\cdots+u_mz^m) = z^{m+n+1}w_0+z^{m+n+2}w_1+\cdots$$

Same as

$$\begin{bmatrix} a_{m-n+1} & \cdots & \cdots & a_{n-1} & a_n \\ a_{m-n+2} & \cdots & \cdots & a_n & a_{n+1} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n-1} & \cdots & \cdots & & a_{m+n-2} \\ a_n & \cdots & \cdots & a_{m+n-2} & a_{m+n-1} \end{bmatrix} \cdot \begin{bmatrix} v_n \\ v_{n-1} \\ \vdots \\ \vdots \\ v_2 \\ v_1 \end{bmatrix} = -v_0 \begin{bmatrix} a_{m+1} \\ a_{m+2} \\ \vdots \\ \vdots \\ a_{m+n-1} \\ a_{m+n} \end{bmatrix}$$

Similarly for a_i square matrices.
Similarly have structured linear system for other approx problems.
Nice when coefficient matrix is nonsingular.

Example : Hermite-Padé

$$a(x) \cdot p(x) + b(x) \cdot q(x) + c(x) \cdot r(x) = O(x^8)$$

with deg $p(x) \leq 2$, deg $q(x) \leq 3$, deg $r(x) \leq 1$

$$\left[\begin{array}{ccc|cccc|cc} a_0 & 0 & 0 & b_0 & 0 & 0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & 0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & 0 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & b_0 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & b_1 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & b_2 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & b_3 & c_6 & c_5 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & b_4 & c_7 & c_6 \end{array}\right] \cdot \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ \hline q_0 \\ q_1 \\ q_2 \\ q_3 \\ \hline r_0 \\ r_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

Note

- All Padé approximants known in scalar case
  - Padé table in scalar case has a type of block structure
- Padé approximants related to diophantine equations
  - There are algorithms corresponding to Euclidean algorithm
  - Fast way to compute Padé approximants in scalar case
- Nothing known about structure of matrix Padé case or Hermite-Padé case by 1990
- Use in inversion formulas for Hankel and Toeplitz matrices known in scalar and block cases.

Example : Inverses of Block Hankel Matrices

Inverse of block Hankel matrix, $H_{m,n}$, given by

$$\begin{bmatrix} v_{n-1} & \cdots & v_0 \\ \vdots & & \\ v_0 & & \end{bmatrix} \begin{bmatrix} q_{n-1}^* & \cdots & q_0^* \\ & & \vdots \\ & \ddots & \\ & & q_{n-1}^* \end{bmatrix} - \begin{bmatrix} q_{n-2} & \cdots q_0 & 0 \\ \vdots & & \\ q_0 & & \\ 0 & & \end{bmatrix} \begin{bmatrix} v_n^* & \cdots & v_1^* \\ & \ddots & \vdots \\ & & v_n^* \end{bmatrix}$$

where

$$H_{m,n}[v_n, \cdots, v_0]^T = -[a_{m+1}, \cdots, a_{m+n+1}]^T, \quad H_{m,n}n[q_{n-1}, \cdots, q_0]^T = [0, \cdots, 0, 1]^T,$$

$$[v_n^*, \cdots, v_0^*]H_{m,n}n = -[a_{m+1}, \cdots, a_{m+n+1}], \quad [q_{n-1}^*, \cdots, q_0^*]H_{m,n}n = [0, \cdots, 0, 1].$$

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where $Q_{n-1}(z)$, $V_n(z)$, $Q^*_{n-1}(z)$, $V^*_n(z)$

$$\begin{aligned} A(z)Q_{n-1}(z) - P_{m-1}(z) &= z^{m+n-1}R(z) \text{ with } R(0) = I \\ A(z)V_n(z) - U_m(z) &= z^{m+n+1}W(z) \text{ with } V_n(0) = I \\ Q^*_{n-1}(z)A(z) - P^*_{m-1}(z) &= z^{m+n-1}R^*(z) \text{ with } R^*(0) = I \\ V^*_n(z)A(z) - U^*_m(z) &= z^{m+n+1}W^*(z) \text{ with } V^*_n(0) = I \end{aligned}$$

Inversion Formulae : Fast, Stable, Numeric Algorithms

- Inversion formula determine condition number when properly scaled
- Allows one to build on stable subproblems
- Gives fast, provably numerically stable algorithms

Outline

People

- K. Mahler(1925-1969), J. Coates(1965) , J. Della Dora (1980), . . .
  - (strong conditions assumed)
- B. Beckermann, myself, A. Bultheel, M. van Barel
- B. Salvy and P. Zimmermann (gfun), (and H. Derksen) ; M. Rubey
- M. van Hoeij (use in differential factorization)
- G. Villard (matrix normal forms)
- P. Giorgi, C-P. Jeannerod, G. Villard (fast polynomial matrix arithmetic)
- H. Cheng; E. Kaltofen and students (W. Turner,G. Yuhasz); E. Schost . . .

Bernhard Beckermann


北京友谊宾馆
BEIJING FRIENDSHIP HOTEL
The 2005 International Symposium on Symbolic and Algebraic Computation
July 24-27, 2005, Beijing

Order Bases

Idea : look at order condition independently of degree bounds,

$$R_\sigma = \{\mathbf{Q}(z) \in F^{(m)}[z] \mid \mathbf{A}(z) \cdot \mathbf{Q}(z) = O(z^\sigma)\}$$

Find basis of R_σ as a *module* over $F[z]$.

Basis always has m elements - write as columns of an $m \times m$ matrix polynomial $\mathbf{M}(z)$.

Order Bases

Vector case ($s \times 1$). Convert to scalar series.

$$R_s = \{\mathbf{Q}(z) \in F^{(m)}[z] \mid \mathbf{A}(z) \cdot \mathbf{Q}(z^s) = O(z^\sigma)\}$$

Correct solutions handled via "s-trick.

$$A_1(z)Q_1(z^s) + \cdots + A_m(z)Q_m(z^s) = O(z^\sigma)$$

Conveniently converts vector problem to scalar problem.

Order Bases

Degree bounds? Given $\vec{n} = (n_1, \ldots, n_m)$:

Then

$$\mathbf{Q}(z) \quad = \quad \alpha_1(z)\mathbf{M}_1(z) + \cdots + \alpha_m(z)\mathbf{M}_m(z)$$

with

$$\deg \alpha_i(z) \leq \text{ defect } \mathbf{Q}(z) - \text{ defect } \mathbf{M}_i(z)$$

Here defect is a measure of the difference between degrees and bounds n_i.

Implies $\mathbf{M}(z)$ describes all solutions of $\mathbf{A}(z)\mathbf{Q}(z) = O(z^\sigma)$

Outline

Computation Complexity

Hermite-Padé approx. with degree bounds $(n_1, \ldots n_m)$

Set $N = n_1 + \cdots + n_m$. Then :

- Linear algebra : $O(N^3)$
- Sigma basis : $O(N \log^2 N)$ i.e. - $O(N^{1+\epsilon})$ in scalar case (BL - 1994)
- MBasis : $O(m^\omega N^{1+\epsilon})$ - in case of matrix input (GJV - 2003)
- Generating set : $O(m^\omega (N/m)^{1+\epsilon})$ (Stor. 2006)

Sigma Basis Algorithm [SIMAX - BL]

1. Start with Order basis $= \mathbf{I}$ and order $= 0$.
2. Of all the columns that need to have order increased:
   - pick one with minimal defect.
   - use to eliminate other columns needing order increase.
3. Multiply pivot column by z. Continue. Quadratic complexity.

Double order everytime : obtain superfast version.

Alternatively (GJV)

To get $(\sigma + 1)$-basis from a σ-basis do:

1. we compute the terms in z^{σ} in the residue $\mathbf{R}(z)$. This give us a matrix Δ
2. we compute a row echelon form of Δ
3. we apply some transformations according to the row echelon form. These transformations are of two types :
   - Either $\mathbf{M}_i$ is replaced by a linear combination of some $\mathbf{M}_j$
   - Or all the polynomials in $\mathbf{M}_i$ are multiplied by z

Example : calculation of a 1-basis

$$\mathbf{A}(z) = \begin{bmatrix} z+1 & z^2-1 & 5 \\ 2z-1 & 1 & z \end{bmatrix} \text{ and } \mathbf{M}(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then: $\Delta = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{row echelon form}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So we replace the 2^{nd} column by the 2^{nd} column minus the 1^{st} column, and we shift column 1 and 3 because they where used as pivots to obtain the 1-basis : $\mathbf{M}(z) = \begin{bmatrix} z & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{bmatrix}$

Outline

Fraction-free Computation

Fraction-free computation for given $\vec{n}$.

- set up linear system (structured Krylov matrix) for each order
- find so-called Cramer's solutions
- eliminate known divisors by a type of Sylvester's identity
- Order basis called Mahler system

Also version for Ore case. Also modular versions.

Goal: Try to find Cramer solutions

- e.g. Hermite-Padé problem

$$a(x) \cdot p(x) + b(x) \cdot q(x) + c(x) \cdot r(x) = O(x^6)$$

with deg $p(x) \leq 2$, deg $q(x) \leq 1$, deg $r(x) \leq 1$

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \cdot \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ \hline q_0 \\ q_1 \\ \hline r_0 \\ r_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ d \end{array}\right]$$

where d is determinant of coefficient matrix.

- Solution has determinant representation in nonsingular case:

$$\text{e.g. } p(z) = \det \left[\begin{array}{ccc|ccc|c} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ 1 & z & z^2 & 0 & 0 & 0 & 0 \end{array} \right]$$

- Unique in nonsingular case.
- Recursively build Cramer solutions from Cramer solutions of smaller problems along offdiagonal of associated table.

- Matrix $\mathbf{M}(z)$ of determinantal polynomials with degrees

$$\begin{bmatrix} n_1 & n_1-1 & \cdots & n_1-1 \\ n_2-1 & n_2 & \cdots & n_2-1 \\ \vdots & & \ddots & \vdots \\ n_m-1 & \cdots & \cdots & n_m \end{bmatrix}$$

  and lcoeff of diagonal $=$ determinant of coeff matrix.
- Unique in nonsingular case
- Basic building block of recursions.
- Method 1: via modified Schur complements
  - nonsingular location to nonsingular location in table
  - similar to look ahead
- Method 2: via determinental identities
  - works in singular cases by computing at closest nonsingular locations (look around)

Outline

1. Introduction
   - The Problem
2. History
   - Background : People, Examples, etc
   - Example: Transcendence of e
3. Matrix Rational Approximation
   - Linear Systems
4. Order Bases
   - Background
   - Computation
   - Fraction-Free Computation
   - Matrix Normal Forms
   - Current Activities

Shifted Popov Normal Forms

Shifted Popov form problem : gives

$$\mathbf{A}(z) \cdot \mathbf{U}(z) = \mathbf{P}(z)$$

Embed normal form problem inside part of a Mahler system for

$$[\mathbf{A}(z), -\mathbf{I}] \begin{bmatrix} \mathbf{V}(z) & \mathbf{U}(z) \\ \mathbf{Q}(z) & \mathbf{P}(z) \end{bmatrix} = [O(z^{\vec{n}}), 0].$$

Fraction-free computation of normal forms.

Outline

Currently we are working on:

- Fast algorithms in differential case
- Better fraction-free algorithms
- Use with differential-algebraic problems
  - Popov forms
  - Invariants and algorithms
- Alternate bases
- Multivariate Order Bases