On Simultaneous Row and Column Reduction of Higher-Order Linear Differential Systems

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Motivation

Consider system of differential equations

$$a_n(x)\vec{y}^{\,n}(x) + \cdots + a_1(x)\vec{y}^{\,\prime}(x) + a_0(x)\vec{y}(x) = \vec{f}(x)$$

with $a_i(x) \in \mathbb{K}^{m\times n}[[x]]$ and $\vec{y}(x) \in \mathbb{K}^n[[x]]$. (i.e. working locally about point 0).

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Usually convert to 1^{st} order systems

$$A(t)Y'(t) = B(t)Y(t) + C(t)$$

Various techniques used to build solutions or solution types (e.g. existence of rational function or exponential solutions).

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We instead want to work directly with the higher order system.

View problem as

$$L(D)\vec{y}(x) = \vec{f}(x)$$

with $L(D) \in \mathbb{K}^{m\times n}[[x]][D]$ and transform $L(D)$.

Outline

Matrix Normal Forms

Given : $\mathbf{L}(D) \in \mathbb{K}^{m \times m}[D]$.

Do row and column operations

$$\mathbf{L}(D) \qquad = \text{ easier}$$

Matrix Normal Forms

Given : $\mathbf{L}(D) \in \mathbb{K}^{m \times m}[D]$.

Do row and column operations $\mathbf{U}(D)$ and $\mathbf{V}(D)$ so that

$$\mathbf{U}(D)\mathbf{L}(D)\mathbf{V}(D) = \text{ easier}$$

(easier = $\mathbf{L}^*(D) \in \mathbb{K}^{m \times m}[D]$ in some sort of normal form)

$\mathbf{U}(D), \mathbf{V}(D) \in \mathbb{K}^{m \times m}[D]$ invertible

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$\mathbf{U}(D), \mathbf{V}(D) \in \mathbb{K}^{m\times m}[D]$ invertible

Gives $\mathbf{L}^*(D)\vec{y}^*(x) = \vec{f}^*(x)$ with $\mathbf{L}^*(D) \in \mathbb{K}^{m\times m}[[x]][D]$

Normal Forms : Hermite

Hermite forms of $\mathbf{L}$.

$$\mathbf{U} \cdot \mathbf{L} = \begin{bmatrix} h_{11} & \cdots & \cdots & h_{1k} \\ 0 & h_{22} & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{kk} \end{bmatrix}$$

with h_{ii} larger than h_{ij}.

Problem : Degrees of elements growth

Normal Forms : Smith

Smith form of $\mathbf{L}$.

$$\mathbf{U} \cdot \mathbf{L} \cdot \mathbf{V} = \begin{bmatrix} s_{11} & 0 & & 0 \\ 0 & s_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & s_{kk} \end{bmatrix}$$

with $s_{i,i}$ dividing $s_{i+1,i+1}$.

Problem : Trivial in case of differential domain, i.e. $s_{11} = \cdots = s_{k-1,k-1} = 1$ (Johannes Middeke)

Normal Forms : Popov

Popov form of $\mathbf{L}$.

$$\mathbf{U} \cdot \mathbf{L} = \begin{bmatrix} p_{11} & \cdots & \cdots & p_{1k} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ p_{k1} & \cdots & \cdots & p_{kk} \end{bmatrix}$$

with each row having a highest degree pivot

Problem :

Normal Forms : Popov

Popov form of $\mathbf{L}$.

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No Problem : We like this one.

Example : Algorithms Based on Popov Forms

Popov transforms $L(D)$ so that $L(D)\vec{y}(x) = \vec{f}(x)$ converts naturally to larger first order system.

- Popov to First Order
- Solution method of First Order
- Convert first order solution method back to original higher order system
  (possible since conversion to Popov is reversible)

First Order System : Harris Algorithm (1968)

$$L(D) = a_1(x)D + a_2(x) \qquad \text{with } a_i(x) \in \mathbb{K}^{n\times n}[[x]]$$

Harris algorithm produces matrices $S, T \in \mathbb{K}^{n\times n}[[x]]$ such that

$$S \cdot L(D) \cdot T = \begin{bmatrix} L_1(D) & 0 & \cdots & \cdots & 0 \\ 0 & c_1(x) & 0 & & \\ & & \ddots & & \\ & & & c_k(x) & \\ 0 & \cdots & & & 0 \end{bmatrix}$$

with $L_1(D) = x^p I_q D + b(x)$, $b(x) \in \mathbb{K}^{q\times q}[[x]]$ and $c_i(x) \in \mathbb{K}^{n_i\times n_i}((x))$ invertible for all i.

Second Order System

$$L(D) = a_2(x)D^2 + a_1(x)D + a_2(x) \qquad \text{with } a_i(x) \in \mathbb{K}^{n\times n}[[x]]$$

Theorem: Barkatou, El Bacha,Pflügel (ISSAC 2010)

There exist $S \in \mathbb{K}((x))^{m\times m}[D]$ and $T \in \mathbb{K}((x))^{n\times n}[D]$ invertible such that

$$S \cdot L(D) \cdot T = \begin{bmatrix} a_{11}(x)D^2 + b_{11}(x)D + c_{11}(x) & c_{12}(x) & 0 & 0 \\ c_{21}(x) & b_{22}(x)D + c_{22}(x) & 0 & 0 \\ 0 & 0 & c_{33}(x) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with $a_{11}(x)$, $b_{22}(x)$ and $c_{33}(x)$ invertible matrices.

Second Order System (cont.)

Thus second order system $L(D)\vec{y}(x) = \vec{f}(x)$ is converted to

$$\begin{bmatrix} a_{11}(x)D^2 + b_{11}(x)D + c_{11}(x) & c_{12}(x) & 0 & 0 \\ c_{21}(x) & b_{22}(x)D + c_{22}(x) & 0 & 0 \\ 0 & 0 & c_{33}(x) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \\ \vec{z}_3 \\ \vec{z}_4 \end{bmatrix} \cdot \begin{bmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vec{f}_3 \\ \vec{f}_4 \end{bmatrix}.$$

Resulting system is decoupled into differential, algebraic and arbitary components.

Decoupling?

The decoupled system

$$\begin{bmatrix} a_{11}(x)D^2 + b_{11}(x)D + c_{11}(x) & c_{12}(x) \\ c_{21}(x) & b_{22}(x)D + c_{22}(x) \end{bmatrix} \begin{bmatrix} \vec{z}_1 \\ \vec{z}_2 \end{bmatrix} = 0,$$

can be transformed to the first-order systems of ODEs:

$$\left(\begin{bmatrix} I & & \\ & a_{11}(x) & \\ & & b_{22}(x) \end{bmatrix} D + \begin{bmatrix} 0 & -I & 0 \\ c_{11}(x) & b_{11}(x) & c_{12}(x) \\ c_{21}(x) & 0 & c_{22}(x) \end{bmatrix} \right) \begin{bmatrix} \vec{z}_1 \\ \vec{z}_1\,' \\ \vec{z}_2 \end{bmatrix} = 0.$$

Row Reduction

Definition: $L(D), \tilde{L}(D) \in \mathbb{K}((x))^{m\times n}[D]$ are said to be equivalent if there exist $U(D) \in \mathbb{K}((x))^{m\times m}[D]$ and $V(D) \in \mathbb{K}((x))^{n\times n}[D]$ invertible s.t. $L(\tilde{D}) = U(D) \cdot L(D) \cdot V(D)$.

Row operations on our operator $L(D)$:

1. multiplying a row of $L(D)$ by a nonzero element of $\mathbb{K}((x))$,
2. adding to any row of $L(D)$ another row multiplied by an element of $\mathbb{K}((x))[D]$,
3. interchanging any two rows of $L(D)$.

Row Reduced Form

Want to find $\mathbf{U}(D) \in \mathbb{K}^{m\times m}[[x]][D]$, invertible, such that

$$\mathbf{L}^*(D) = \mathbf{U}(D) \cdot \mathbf{L}(D) = D^{\vec{n}} \ell_{\vec{n}}(x) + \text{ l.o.t. s}$$

where $\ell_{\vec{n}}(x) \in \mathbb{K}^{m\times m}[[x]]$ is nonsingular.

- $\mathbf{L}^*(D)$ is said to be in row-reduced form
- $\vec{n}$ is the row order of $\mathbf{L}^*(D)$
- $\ell_{\vec{n}}(x)$ is the leading row coefficient matrix

Reducing Row Degree

If $L(D)$ not row-reduced implies : there exists nonzero vector of the form $v = (v_1, \ldots, v_m) \in \mathbb{K}[[x]]^{1\times m}$ such that $v\,\ell_{\vec{n}} = 0$.

$$U_{11}(D) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ v_1\,D^{n_j-n_1} & \cdots & v_{j-1}\,D^{n_j-n_{j-1}} & v_j & v_{j+1}\,D^{n_j-n_{j+1}} & \cdots & v_m\,D^{n_j-n_m} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}.$$

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Need $n_j \geq n_i$ whenever $v_i \neq 0$. Tells us how to choose j !

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Need $n_j \geq n_i$ whenever $v_i \neq 0$. Tells us how to choose j !

Note: $U_{11}(D)L(D)$ has lower row degree. Continue until row reduced.

B-L (1997), Abramov-Bronstein (2001), B-C-L (2006)

Simultaneous Row and Column Reduction

Row reduction followed by column reduction does not imply simultaneous Row and Column reduction.

We have two algorithms for computing simultaneous row-column reduced forms

- Row then column then row then $\cdots$
- A better use row, column combinations

Both depend on : Leading row and column matrix becomes

$$\begin{bmatrix} \ell c(L_{11}) & 0 \\ * & * \end{bmatrix} \text{ and } \begin{bmatrix} \ell c(L_{11}) & * \\ 0 & * \end{bmatrix},$$

Then iterate.

Row then Column then Row $\cdots$

Consider the matrix differential operator given by

$$L = \begin{bmatrix} D^3 + x & 2D^2 & x^2 + x \\ D^2 & x\,D^2 & 2x^2 + 1 \\ D & x\,D & 1 \end{bmatrix},$$

Row reduction gives:

$$L^{(1)} = U_1\,L = \begin{bmatrix} D^3 + x & 2D^2 & x^2 + x \\ 0 & -D & -D + 2x^2 + 1 \\ D & x\,D & 1 \end{bmatrix}.$$

Not column reduced.

Row then Column then Row $\cdots$

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Column reduction then gives;

$$L^{(2)} = L^{(1)} V_1 = \begin{bmatrix} 2x & 2D^2 & x^2 + x \\ D^2 & -D & -D + 2x^2 + 1 \\ -xD^2 + 2D & x\,D & 1 \end{bmatrix}.$$

Not row reduced.

Row then Column then Row $\cdots$

Consider the matrix differential operator given by

$$L = \begin{bmatrix} D^3 + x & 2D^2 & x^2 + x \\ D^2 & x\,D^2 & 2x^2 + 1 \\ D & x\,D & 1 \end{bmatrix},$$

Row reduction then gives;

$$L^{(3)} = U_2 L^{(2)} = \begin{bmatrix} 2x & 2D^2 & x^2 + x \\ 2D & 0 & -xD + 2x^3 + x + 1 \\ -xD^2 + 2D & xD & 1 \end{bmatrix}.$$

Now row and column reduced.

Form of Simultaneous Row-Column Form

Theorem

Let $L(D) \in \mathbb{K}[[x]]^{m\times n}[D]$ be simultaneous row-column reduced.
Can permute the rows and columns of $L(D)$ so it has block form :

$$\left[\begin{array}{ccc|c} L_{11} & \cdots & L_{1k} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ L_{k1} & \cdots & L_{kk} & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right]$$

L_{ii} *are block square matrices satisfying*

(a) L_{ii} *row & column reduced, same row and column degree,*

(b) $deg(L_{ii}) > deg(L_{i+1,i+1})$ *for all* i,

(c) $deg(L_{ij}) \leq deg(L_{ii})$ *for* $j < i$, $deg(L_{ij}) < deg(L_{ii})$ *for* $j > i$,

(d) $deg(L_{ij}) \leq deg(L_{jj})$ *for* $i < j$, $deg(L_{ij}) < deg(L_{jj})$ *for* $i > j$.

Example : Operator of Degree 8

$$\begin{bmatrix}
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8
\end{bmatrix}$$

Example; Row Reduced Form

$$\begin{bmatrix}
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{bmatrix}$$

Example; Row Reduced Form

$$
\left[\begin{array}{ccccccccccccc}
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\hline
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
\hline
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
\hline
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\hline
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right]
$$

Example

$$
\left[\begin{array}{ccccccccccccc}
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
\hline
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right]
$$

Example

$$\left[\begin{array}{cccc|ccccccccc}
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
\hline
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right]$$

Form of Leading Coefficient

$$
\left[\begin{array}{cccc|ccccccccc}
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & *
\end{array}\right]
$$

Example

$$\left[\begin{array}{cccc|ccccccccc}
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
\hline
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
\hline
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
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2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right]$$

Form of Leading Coefficient

$$
\left[\begin{array}{cccc|cc|ccccccc}
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
* & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & *
\end{array}\right]
$$

Example

$$
\left[\begin{array}{cccc|cc|ccccccc}
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
\hline
6 & 6 & 6 & 6 & 6 & 6 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
\hline
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right]
$$

Example

$$\left[\begin{array}{cccc|cc|c|cc|cccc}
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ \hline
6 & 6 & 6 & 6 & 6 & 6 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ \hline
5 & 5 & 5 & 5 & 5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 \\ \hline
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \\ \hline
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right]$$

Example : Block Form

$$\begin{bmatrix} 8 & 7 & 7 & 7 & 7 \\ 6 & 6 & 5 & 5 & 5 \\ 5 & 5 & 5 & 4 & 4 \\ 3 & 3 & 3 & 3 & 2 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 7 & 7 & 7 & 6 \\ 6 & 6 & 5 & 5 & 4 \\ 5 & 5 & 5 & 4 & 3 \\ 3 & 3 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 7 & 7 & 7 & 5 \\ 6 & 6 & 5 & 5 & 3 \\ 5 & 5 & 5 & 4 & 2 \\ 3 & 3 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 7 & 7 & 7 & 5 \\ 6 & 6 & 5 & 5 & 3 \\ 5 & 5 & 5 & 2 & 1 \\ 3 & 3 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 7 & 7 & 7 & 5 \\ 6 & 6 & 4 & 3 & 2 \\ 5 & 5 & 5 & 2 & 1 \\ 3 & 3 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} \color{green}{8} & 7 & 7 & 7 & 5 \\ 6 & \color{green}{6} & 4 & \color{red}{2} & \color{red}{1} \\ 5 & 5 & \color{green}{5} & 2 & 1 \\ 3 & 3 & 3 & \color{green}{3} & 1 \\ 2 & 2 & 2 & 2 & \color{green}{2} \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 6 & 6 & 5 & 4 \\ 6 & 6 & 4 & 2 & 1 \\ 5 & 5 & 5 & 2 & 1 \\ 3 & 3 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 6 & 5 & 3 & 2 \\ 6 & 6 & 4 & 2 & 1 \\ 5 & 5 & 5 & 2 & 1 \\ 3 & 3 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 5 & 4 & 2 & 1 \\ 6 & 6 & 4 & 2 & 1 \\ 5 & 5 & 5 & 2 & 1 \\ 3 & 3 & 3 & 3 & 1 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 5 & 4 & 2 & 1 \\ 5 & 6 & 4 & 2 & 1 \\ 4 & 4 & 5 & 2 & 1 \\ 2 & 2 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Example : Block Form

$$\begin{bmatrix} 8 & 5 & 4 & 2 & 1 \\ 5 & 6 & 4 & 2 & 1 \\ 4 & 4 & 5 & 2 & 1 \\ 2 & 2 & 2 & 3 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Future Directions

- Handle coefficient growth in computation
  - Do all arithmetic in $\mathbb{K}[[x]][D]$ (not $\mathbb{K}((x))[D]$)
  - Control growth via fraction-free algorithm
  - Method : structured linear systems of equations
  - Difficulties : everything mentioned in Mark's talk
- Direct methods for higher order eqns via normal forms
  - Translate : convert (H) $\rightarrow$ solve (F) $\rightarrow$ convert (H)