Invariants of Finite Abelian Groups and their use in Symmetry Reduction of Dynamical Systems

George Labahn

Symbolic Computation Group
Cheriton School of Computer Science
University of Waterloo, Canada.

* Joint with Evelyne Hubert, INRIA Méditerranée.

ICIAM - Beijing - Aug. 13, 2015

Motivation : Invariant Dynamic System

Consider the following dynamical system (with c a parameter) [1]

$$
\begin{aligned}
x_1'(t) &= x_1(t)(\ 1 - c \cdot x_1(t) - x_1(t) \cdot x_2^2(t) - x_1(t) \cdot x_3^2(t)\ ) \\
x_2'(t) &= x_2(t)(\ 1 - c \cdot x_2(t) - x_2(t) \cdot x_1^2(t) - x_2(t) \cdot x_3^2(t)\ ) \\
x_3'(t) &= x_3(t)(\ 1 - c \cdot x_3(t) - x_3(t) \cdot x_1^2(t) - x_3(t) \cdot x_2^2(t)\ )
\end{aligned}
$$

Steady state gives system of polynomial equations

$$
\begin{aligned}
0 &= 1 - c \cdot x_1 - x_1 \ x_2^2 - x_1 \ x_3^2 \\
0 &= 1 - c \cdot x_2 - x_2 \ x_1^2 - x_2 \ x_3^2 \\
0 &= 1 - c \cdot x_3 - x_3 \ x_1^2 - x_3 \ x_2^2
\end{aligned}
$$

[1] Neural network model [SIAM J. Numer. Anal. Noonburg 1989].

Motivation : Invariant Dynamic System

Consider the following dynamical system (with c a parameter) [1]

$$\begin{aligned}
x_1'(t) &= x_1(t)(\ 1 - c \cdot x_1(t) - x_1(t) \cdot x_2^2(t) - x_1(t) \cdot x_3^2(t)\ ) \\
x_2'(t) &= x_2(t)(\ 1 - c \cdot x_2(t) - x_2(t) \cdot x_1^2(t) - x_2(t) \cdot x_3^2(t)\ ) \\
x_3'(t) &= x_3(t)(\ 1 - c \cdot x_3(t) - x_3(t) \cdot x_1^2(t) - x_3(t) \cdot x_2^2(t)\ )
\end{aligned}$$

Steady state gives system of polynomial equations

$$\begin{aligned}
0 &= 1 - c \cdot x_1 - x_1 \ x_2^2 - x_1 \ x_3^2 \\
0 &= 1 - c \cdot x_2 - x_2 \ x_1^2 - x_2 \ x_3^2 \\
0 &= 1 - c \cdot x_3 - x_3 \ x_1^2 - x_3 \ x_2^2
\end{aligned}$$

Solution space of system is invariant under the order 3 permutation

$$(x_1,\ x_2,\ x_3) \ \rightarrow \ (x_2,\ x_3,\ x_1).$$

We wish to work "modulo" this order 3 permutation.

[1] Neural network model [SIAM J. Numer. Anal. Noonburg 1989].

Group Actions and Invariants

Solution space is invariant under the $\mathcal{G} = \mathbb{Z}_3$ linear matrix action

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Goal : Find and rewrite system in terms of invariants

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \left(3x_1^2x_2 + 3x_1x_3^2 + 3x_2^2x_3\right)\alpha^2 + \left(3x_1^2x_3 + 3x_1x_2^2 + 3x_2x_3^2\right)\alpha + \left(x_1^3 + 6x_1x_2x_3 + x_2^3 + x_3^3\right) \\ (x_1x_2 + x_1x_3 + x_2x_3)\alpha^2 + (x_1x_2 + x_1x_3 + x_2x_3)\alpha + \left(x_1^2 + x_2^2 + x_3^2\right) \\ x_1 + x_2 + x_3 \end{bmatrix}.$$

Here α primitive cube root of unity.

Finite Abelian Symmetries

Action : $\mathcal{G} \times \mathbb{K}^n \rightarrow \mathbb{K}^n$, $\mathcal{G}$ finite, abelian matrix group.

This talk:

- Determine important constructions for group actions
  - rational invariants, rewrite rules
- Integer linear algebra solves finite abelian symmetry problems
  - Gives complete and elegant description of above
- Given finite abelian action for systems can determine reduction
- Given systems can find finite abelian group action (if possible)

Related work

- K. Gatermann (ISSAC 1990)
  - Using group actions to reduce Gröbner bases comp.
- J-C Faugère and J. Svartz (ISSAC 2013)
  - Using abelian group actions to reduce polynomial systems.
- E. Hubert and G. Labahn (ISSAC 2012, FoCM 2013)
  - scaling symmetries: e.g. $(\mathbb{K}^*)^2 \times \mathbb{K}^5 \to \mathbb{K}^5$

$$(\alpha,\beta) \times (z_1,\, z_2,\, z_3,\, z_4,\, z_5) \to \left(\alpha^6\, z_1,\, \beta^3\, z_2,\, \frac{\beta}{\alpha^4}\, z_3,\, \frac{\alpha}{\beta^4}\, z_4,\, \alpha^3\beta^3\, z_5\right).$$

- E. Hubert and G. Labahn (To appear : Math of Comp) (this talk)

Finite Abelian Group Actions

- Special Form of Finite Abelian Groups
- Diagonalization

Basic Facts : Finite Abelian Linear Groups

$\mathcal{G}$: finite abelian subgroup of $\mathrm{GL}_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

(i) Group is diagonalizable .

Basic Facts : Finite Abelian Linear Groups

$\mathcal{G}$: finite abelian subgroup of $\mathrm{GL}_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

(i) Group is diagonalizable .

- I.e., $\exists$ matrix R such that $R^{-1} \cdot \mathcal{G} \cdot R$ all diagonal matrices

Basic Facts : Finite Abelian Linear Groups

$\mathcal{G}$: finite abelian subgroup of $\mathrm{GL}_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

(i) Group is diagonalizable .

- I.e., $\exists$ matrix R such that $R^{-1} \cdot \mathcal{G} \cdot R$ all diagonal matrices

- Original linear group action :

$$\begin{array}{ccc} \mathcal{G} \times \mathbb{K}^n & \rightarrow & \mathbb{K}^n \\ (\ G,\ x\ ) & \mapsto & G \cdot x \end{array}$$

- New action : (with $\mathcal{D} = R^{-1} \cdot \mathcal{G} \cdot R$ and $\mathbf{x} = R \cdot \mathbf{z}$)

$$\begin{array}{ccc} \mathcal{D} \times \mathbb{K}^n & \rightarrow & \mathbb{K}^n \\ (\ D,\ z\ ) & \mapsto & D \cdot z \end{array}$$

is diagonal action

$$\begin{array}{ccc} \mathcal{D} \times \mathbb{K}^n & \rightarrow & \mathbb{K}^n \\ (\ \mathit{diagonal}(\ d_1,\ \ldots,\ d_n\ ),\ (z_1, \ldots, z_n)\ ) & \mapsto & (d_1 \cdot z_1,\ \ldots,\ d_n \cdot z_n) \end{array}$$

Basic Facts : Finite Abelian Linear Groups

$\mathcal{G}$: finite abelian subgroup of $\mathrm{GL}_n(\mathbb{K})$ (order $p = p_1 \cdots p_s$)

(ii) Group isomorphism : $\mathcal{D} \leftrightarrow \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_s}$

Explicit via exponents :

$$\begin{array}{ccc} \mathbb{Z}_{p_1} \times \ldots \times \mathbb{Z}_{p_s} & \to & \mathcal{D} \\ (m_1, \ldots, m_s) & \mapsto & D_1^{m_1} \cdots D_s^{m_s} \end{array}$$

Running Example (cont)

Polynomial system

$$\begin{aligned} f_1 &= x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3 + 12 \\ f_2 &= x_1x_2 + x_2x_3 + x_1x_3 - 15 \\ f_3 &= x_1x_2x_3 - 13 \end{aligned}$$

(i) $\mathbb{Z}_3$ linear action $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(ii) Diagonalize $\mathbb{Z}_3$ (with α a primitive cube root of unity) via

- Diagonalize via $R = \begin{bmatrix} \alpha & \alpha^2 & 1 \\ \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \end{bmatrix}$

- Change coordinates $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha^2 & 1 \\ \alpha^2 & \alpha & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$.

Running Example (cont)

(iii) This converts original system

$$
\begin{aligned}
f_1 &= x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3 + 12 \\
f_2 &= x_1x_2 + x_2x_3 + x_1x_3 - 15 \\
f_3 &= x_1x_2x_3 - 13
\end{aligned}
$$

to new system

$$
\begin{aligned}
f_1 &= 3z_1z_2 + 3z_3 - 3z_3^2 + 12, \\
f_2 &= -3z_1z_2 + 3z_3^2 - 15 \\
f_3 &= z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 13
\end{aligned}
$$

(iv) Key idea : $\mathbb{Z}_3$ action is now

$$
\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \rightarrow \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \alpha \cdot z_1 \\ \alpha^2 \cdot z_2 \\ z_3 \end{bmatrix}
$$

Running Example (cont)

(iii) This converts original system

$$\begin{aligned} f_1 &= x_1 + x_2 + x_3 - x_1x_2 - x_2x_3 - x_1x_3 + 12 \\ f_2 &= x_1x_2 + x_2x_3 + x_1x_3 - 15 \\ f_3 &= x_1x_2x_3 - 13 \end{aligned}$$

to new system

$$\begin{aligned} f_1 &= 3z_1z_2 + 3z_3 - 3z_3^2 + 12, \\ f_2 &= -3z_1z_2 + 3z_3^2 - 15 \\ f_3 &= z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 13 \end{aligned}$$

(iv) Key idea : $\mathbb{Z}_3$ action is now

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \rightarrow \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \alpha \cdot z_1 \\ \alpha^2 \cdot z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \alpha^1 \cdot z_1 \\ \alpha^2 \cdot z_2 \\ \alpha^0 \cdot z_3 \end{bmatrix}.$$

Integer Linear Algebra

- Exponent and Order matrices
- Matrix notation and its properties
- Hermite Normal Form

Exponent and Order Matrices

Diagonal action : $(\alpha,\beta) \in \mathbb{Z}_7 \times \mathbb{Z}_5$:

$$(z_1,\, z_2,\, z_3,\, z_4,\, z_5) \;\rightarrow\; (\alpha^6\, z_1,\, \beta^3\, z_2,\, \frac{\beta}{\alpha^4}\, z_3,\, \frac{\alpha}{\beta^4}\, z_4,\, \alpha^3\beta^3\, z_5\,).$$

Exponent and Order matrices:

$$A := \begin{bmatrix} 6 & 0 & -4 & 1 & 3 \\ 0 & 3 & 1 & -4 & 3 \end{bmatrix} \qquad P := \begin{bmatrix} 7 & \\ & 5 \end{bmatrix}$$

Exponent matrix notation:

$$(\,\alpha,\;\beta\,)^A = \left(\,\alpha^6\,,\;\beta^3\,,\;\alpha^{-4}\beta^1\,,\;\alpha^1\beta^{-4}\,,\;\alpha^3\beta^3\,\right)$$

Exponent and Order Matrices

Diagonal action : $(\alpha,\beta) \in \mathbb{Z}_7 \times \mathbb{Z}_5$:

$$(z_1,\, z_2,\, z_3,\, z_4,\, z_5) \;\to\; (\alpha^6\, z_1,\, \beta^3\, z_2,\, \frac{\beta}{\alpha^4}\, z_3,\, \frac{\alpha}{\beta^4}\, z_4,\, \alpha^3\beta^3\, z_5\,).$$

Exponent and Order matrices:

$$A := \begin{bmatrix} 6 & 0 & -4 & 1 & 3 \\ 0 & 3 & 1 & -4 & 3 \end{bmatrix} \qquad P := \begin{bmatrix} 7 & \\ & 5 \end{bmatrix}$$

Exponent matrix notation:

$$(\,\alpha,\,\beta\,)^A = \left(\,\alpha^6\,,\,\beta^3\,,\,\alpha^{-4}\beta^1\,,\,\alpha^1\beta^{-4}\,,\,\alpha^3\beta^3\,\right)$$

Write diagonal action as : $(\,\lambda,\,\mathbf{z}\,) \to \lambda^A \star \mathbf{z}$

Star operator $\star$ is pointwise multiplication

Hermite Normal Form

Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_6 \times \mathbb{Z}_3$:

$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix}$$

$$[\ A,\ -P\ ]$$

exponent
matrix

Hermite Normal Form

Diagonal action : $(\alpha,\beta) \in \mathbb{Z}_6 \times \mathbb{Z}_3$:

$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$[\ A,\ -P\ ] \quad \rightarrow \quad [\ H_i \quad 0\ ]$$

exponent matrix

Hermite normal form

Hermite Normal Form

Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_6 \times \mathbb{Z}_3$:

$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix} \left[\begin{array}{cc|ccc} 1 & 1 & 3 & 2 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 \end{array}\right] = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$[\; A, \; -P \;] \qquad \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix} \qquad [\; H_i \quad 0 \;]$$

exponent matrix — unimodular multiplier — Hermite normal form

(Unimodular means $W = V^{-1} \in \mathbb{Z}^{5\times 5}$)

Hermite Normal Form

Diagonal action : $(\alpha, \beta) \in \mathbb{Z}_6 \times \mathbb{Z}_3$:

$$\begin{bmatrix} 4 & -1 & -3 & -6 & 0 \\ -1 & 4 & -3 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 & 1 \\ 1 & 0 & & 2 & 1 \\ 0 & 0 & & & 1 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \end{bmatrix}$$

$[\ A,\ -P\ ]$	$\begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}$	$[\ H_i \quad 0\ ]$
exponent matrix	unimodular multiplier	Hermite normal form

Note : V not unique but can be normalized. Implies V_n is special

Finite Abelian Group Actions

- Rational invariants
- Rewrite rules

Rational Invariants $\mathbb{K}(z)^A$

Definition: $F(z)$ is *invariant* under $\mathbf{z} \mapsto \lambda^A \star \mathbf{z}$ if $F(\lambda^A \star \mathbf{z}) = F(\mathbf{z})$

Rational Invariants $\mathbb{K}(z)^A$

Definition: $F(z)$ is *invariant* under $\mathbf{z} \mapsto \lambda^A \star \mathbf{z}$ if $F(\lambda^A \star \mathbf{z}) = F(\mathbf{z})$

Let

$$[\ A,\ -P\ ] \cdot \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix} = [\ H_i \quad 0\ ]$$

Invariant Laurent monomials:

$$\mathbf{z}^v = z_1^{v_1} \cdots z_n^{v_n}, \quad v \in \mathbb{Z}^n$$

$$(\lambda^A \star \mathbf{z})^v = \lambda^{Av}\mathbf{z}^v = \mathbf{z}^v \quad \Leftrightarrow \quad A \cdot v = 0 \mod P$$

Rational Invariants $\mathbb{K}(z)^A$

Definition: $F(z)$ is *invariant* under $\mathbf{z} \mapsto \lambda^A \star \mathbf{z}$ if $F(\lambda^A \star \mathbf{z}) = F(\mathbf{z})$

Let

$$[\ A,\ -P\ ] \cdot \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix} = [\ H_i \quad 0\ ]$$

Invariant Laurent monomials:

$$\mathbf{z}^v = z_1^{v_1} \cdots z_n^{v_n}, \quad v \in \mathbb{Z}^n$$

$$(\lambda^A \star \mathbf{z})^v = \lambda^{Av} \mathbf{z}^v = \mathbf{z}^v \quad \Leftrightarrow \quad A \cdot v = 0 \mod P$$

Lemma: Rational Invariants: $F(z) \in \mathbb{K}(z)^A$:

$$F(z) = \frac{\sum_{v \in \ colspan_{\mathbb{Z}} V_n} a_v \, z^v}{\sum_{v \in \ colspan_{\mathbb{Z}} V_n} b_v \, z^v}$$

Rational Invariants and Rewrite Rules

Theorem: $A \in \mathbb{Z}^{s \times n}$, $\quad [A\ ,\ \ -P] \cdot V = [H, 0]$,

$$V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}, \qquad W = V^{-1} = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix}$$

(a) $y = [z_1, \ldots, z_n]^{V_n}$ form generating set of rational invariants.

(b) V normalized : components of $y = [z_1, \ldots, z_n]^{V_n}$ are polynomials.

(c) Rewrite rule : $F \in \mathbb{K}(z)^A \implies F(z) = F(y^{(W_d - P_d P_u^{-1} W_u)})$

Why?

Rational Invariants and Rewrite Rules

Theorem: $A \in \mathbb{Z}^{s \times n}$, $\quad [A\ ,\ -P] \cdot V = [H, 0]$,

$$V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}, \qquad W = V^{-1} = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix}$$

(a) $y = [z_1, \ldots, z_n]^{V_n}$ form generating set of rational invariants.

(b) V normalized : components of $y = [z_1, \ldots, z_n]^{V_n}$ are polynomials.

(c) Rewrite rule : $F \in \mathbb{K}(z)^A \implies F(z) = F(y^{(W_d - P_d P_u^{-1} W_u)})$

Why? $\quad v = V_n(W_d - P_d P_u^{-1} W_u)v$. any term z^v with $v \in colspan_{\mathbb{Z}} V_n$:

$$\begin{aligned} z^v &= z^{V_n(W_d - P_d P_u^{-1} W_u)v} \\ &= (z^{V_n})^{(W_d - P_d P_u^{-1} W_u)v} \\ &= (y^{(W_d - P_d P_u^{-1} W_u)})^v \end{aligned}$$

Then use Lemma.

Example

Polynomials in $\mathbb{K}[z_1, z_2, z_3]$:

$$\begin{aligned} f_1 &= 3z_1z_2 + 3z_3 - 3z_3^2 + 12, \\ f_2 &= -3z_1z_2 + 3z_3^2 - 15, \\ f_3 &= z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 13 \end{aligned}$$

Exponent matrix and order matrix. $A = [1\ 2\ 0]$ $\quad P = [3]$

Unimodular matrices :

$$V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix} = \left[\begin{array}{c|ccc} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]$$

$$W = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ \hline 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example

- Polynomials in $\mathbb{K}[z_1, z_2, z_3]$:

$$
\begin{aligned}
f_1 &= 3z_1z_2 + 3z_3 - 3z_3^2 + 12, \\
f_2 &= -3z_1z_2 + 3z_3^2 - 15, \\
f_3 &= z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 13
\end{aligned}
$$

- Invariants: $y_1 = z_1^3, \quad y_2 = z_1^1 z_2^1, \quad y_3 = z_3^1$

- Rewrite rule: $(z_1,\ z_2,\ z_3) \rightarrow (\ y_1^{1/3}, \frac{y_2^1}{y_1^{1/3}},\ \ y_3^1).$

- Laurent Polynomials in y_1, y_2, y_3:

$$
\begin{aligned}
f_1 &= 3y_2 + 3y_3 - 3y_3^2 + 12, \\
f_2 &= -3y_2 + 3{y_3}^2 - 15, \\
f_3 &= y_1 + \frac{y_2^3}{y_1} + {y_3}^3 - 3y_2y_3 - 13
\end{aligned}
$$

Example

- Polynomials in $\mathbb{K}[z_1, z_2, z_3]$:

$$\begin{aligned} f_1 &= 3z_1z_2 + 3z_3 - 3z_3^2 + 12, \\ f_2 &= -3z_1z_2 + 3z_3^2 - 15, \\ f_3 &= z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 - 13 \end{aligned}$$

- Invariants: $y_1 = z_1^3, \quad y_2 = z_1^1 z_2^1, \quad y_3 = z_3^1$

- Rewrite rule: $(z_1,\ z_2,\ z_3) \to (\ y_1^{1/3}, \frac{y_2^1}{y_1^{1/3}},\ \ y_3^1).$

- Laurent Polynomials in y_1, y_2, y_3:

$$\begin{aligned} f_1 &= 3y_2 + 3y_3 - 3y_3^2 + 12, \\ f_2 &= -3y_2 + 3{y_3}^2 - 15, \\ f_3 &= y_1 + \frac{y_2^3}{y_1} + {y_3}^3 - 3y_2y_3 - 13 \end{aligned}$$

- 2 solns for new system : $(8, -4, 1), \ (-8, -4, 1)$. 6 solns for original : $(2, -2, 1),\ (-2, 2, 1),\ (2\eta, -2\eta^2, 1),\ (-2\eta, -2\eta^2, 1),\ (2\eta^2, -2\eta, 1),\ (-2\eta^2, -2\eta, 1)$
  - works well because V_n is triangular.

Example : Invariant Dynamic System

Recall first system invariant under $\mathbb{Z}_3$. Changing $\mathbf{x} = R \cdot \mathbf{z}$ gives

$$
\begin{aligned}
z_1'(t) &= \frac{z_1}{3}\left(1 - 2cz_3 + z_1^3 - 2z_2^3 - 2z_3^3 + 3\frac{z_2^2 z_3^2}{z_1} - \frac{z_2^2}{z_1} \right) \\
z_2'(t) &= \frac{z_2}{3}\left(1 - 2cz_3 - 2z_1^3 + z_2^3 - 2z_3^3 + 3\frac{z_1^2 z_3^2}{z_2} - \frac{z_1^2}{z_2} \right) \\
z_3'(t) &= \frac{z_3}{3}\left(1 - cz_3 + 4z_1^3 + 4z_2^3 + 4z_3^3 - 6\frac{z_1^2 z_2^2}{z_3} - \frac{z_1 z_2}{z_3} \right)
\end{aligned}
$$

In terms of invariants these equations become:

$$
\begin{aligned}
y_1'(t) &= y_1\left(1 - 2cy_3 + y_1 - 2\frac{y_2^3}{y_1} - 2y_3^3 + 3\frac{y_2^2 y_3^2}{y_1} - \frac{y_2^2}{y_1} \right) \\
y_2'(t) &= \frac{y_2}{3}\left(2 - 4cy_3 - y_1 - \frac{y_2^3}{y_1} - 4y_3^3 + 3\frac{y_1 y_3^2}{y_2} + 3\frac{y_2^2 y_3^2}{y_1} - \frac{y_2^2}{y_1} - \frac{y_1}{y_2} \right) \\
y_3'(t) &= \frac{y_3}{3}\left(1 - cy_3 + 4y_1 + 4\frac{y_2^3}{y_1} + 4y_3^3 - 6\frac{y_2^2}{y_3} - \frac{y_2}{y_3} \right)
\end{aligned}
$$

Dynamic Systems

Write $z'(t) = z(t) \star F(z(t))$ with $F(\ \lambda^A \star z\ ) = F(z)$

Determine finite group diagonalization info for invariance of $F(z)$

$$A \in \mathbb{Z}^{r\times n}, \quad P \in \mathbb{Z}^{r\times r}, \quad V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}, \quad W = V^{-1} = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix}$$

Theorem

If $y(t) = z(t)^{V_n}$ is the set of invariants then the reduced dynamic system is given by

$$y'(t) = y \star F(y^{W_d}) \cdot V_n.$$

Then $z(t) = y(t)^{(W_d - P_d {P_u}^{-1} W_u)}$ *solves*

$$z'(t) = z \star F(z).$$

Why?

Dynamic Systems

Write $z'(t) = z(t) \star F(z(t))$ with $F(\lambda^A \star z) = F(z)$

Determine finite group diagonalization info for invariance of $F(z)$

$$A \in \mathbb{Z}^{r\times n}, \quad P \in \mathbb{Z}^{r\times r}, \quad V = \begin{bmatrix} V_i & V_n \\ P_i & P_n \end{bmatrix}, \quad W = V^{-1} = \begin{bmatrix} W_u & P_u \\ W_d & P_d \end{bmatrix}$$

Theorem

If $y(t) = z(t)^{V_n}$ is the set of invariants then the reduced dynamic system is given by

$$y'(t) = y \star F(y^{W_d}) \cdot V_n.$$

Then $z(t) = y(t)^{(W_d - P_d P_u{}^{-1} W_u)}$ *solves*

$$z'(t) = z \star F(z).$$

Why? Use

$$\frac{d}{dt}(z^{V_n}) = z^{V_n} \star \left(z^{-1} \star \frac{dz}{dt}\right) \cdot V_n$$

Future Research Directions

(i) Extend to parameterized dynamic systems

(ii) Extend from Finite Abelian to Finite Solvable Group actions

- e.g. Neural network example invariant under D_3.

(iii) Combine scaling symmetries with finite diagonal actions

- makes use of Smith Normal Form