3.36pt

# Order Bases:
# Fraction-Free Computation


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# Thursday Outline

3.36pt

1. What about yesterday?
2. Recursive Computation
3. Determinant Representations
4. Closest Normal Points
5. Fraction-Free Computation

# Preamble

In lecture 4 we show how we approach the order problem

$$a_1(z)p_1(z) + \cdots + a_m(z)p_m(z) = O(z^\sigma)$$

but in an exact arithmetic environment where coefficient growth can be an issue. As an example, we have that the power series $a_i(z) \in \mathbb{K}[[z]]$ where $\mathbb{K}$ is an integral domain such as the integers or else a multivariate domain $\mathbb{F}[u_1, \ldots, u_k]$ of parameters over a field $\mathbb{F}$. In such cases we wish to ensure we do not do arithmetic over the quotient field as we wish to avoid all coefficient gcd computations.

In this case we return to making use of linear algebra with structured matrices. The main tool is the construction of Mahler Systems which are order bases which are in a Popov normal form. We show how such order bases arise naturally when solving with nonsingular structures along a specific path of computation which depends on our specification of our degree bounds. We include hints as to what needs to be done when we encounter singular submatrices. The resulting algorithm is efficient and easy to implement as it turns out to be a simple small variation of the sigma-basis algorithm from yesterday.

In fact this lecture only got to page 20, that is, we stopped just before trying to address the singular case. We still include the slides which would have been used had I used more time for this algorithm description..

This work was done jointly with Bernhard Beckermann.

# Fraction-Free Guassian Elimination (FFGE)

$$A = \begin{bmatrix} a & b & c & \cdots & \cdots \\ d & e & f & \cdots & \cdots \\ g & h & i & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix} \approx \begin{bmatrix} a & b & c & \cdots & \cdots \\ 0 & \tilde{e} & \tilde{f} & \cdots & \cdots \\ 0 & \tilde{h} & \tilde{i} & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

- Only using cross multiplication results in exponential growth of coefficients: $O(2^n \cdot N^2)$ in $n \times n$ case ( $N$ bound for size of entries).

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  Gives linear growth of coefficient size. Cost $\mathcal{O}(n^5 \cdot N^2)$.
- FFGE computes the Cramer solution of a linear problem.

# Example : Euclidean Gcd

$a(z), b(z) \in \mathbb{Z}[z]$ given by (Knuth)

$$\begin{aligned} a(z) &= z^8 + z^6 - 3z^4 - 3z^3 + 8z^2 + 2z - 5 \\ b(z) &= 3z^6 + 5z^4 - 4z^2 - 9z + 21 \end{aligned}$$

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Euclidean Algorithm over $\mathbb{Q}[z]$ (monic)

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Euclidean Algorithm over $\mathbb{Q}[z]$ (monic)

$$r_1(z) = z^4 - \frac{1}{5} z^2 + \frac{3}{5}$$

$$r_2(z) = z^2 + \frac{25}{13} z - \frac{49}{13}$$

$$r_3(z) = z - \frac{6150}{4663}$$

$$r_4(z) = 1$$

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Euclidean-like Algorithm over $\mathbb{Z}[z]$ (Cross multiplication)

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Euclidean-like Algorithm over $\mathbb{Z}[z]$ (Cross multiplication)

$$r_1(z) = 15\,z^4 - 3\,z^2 + 9$$

$$r_2(z) = 15795\,z^2 + 30375\,z - 59535$$

$$r_3(z) = 1254542875143750\,z - 1654608338437500$$

$$r_4(z) = 12593338795500743100931141992187500$$

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Fraction-free Algorithm over $\mathbb{Z}[z]$ (Removing contents)

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Fraction-free Algorithm over $\mathbb{Z}[z]$ (Removing contents)

$$r_1(z) = 5\,z^4 - z^2 + 3$$

$$r_2(z) = 13\,z^2 + 25\,z - 49$$

$$r_3(z) = 4663\,z - 6150$$

$$r_4(z) = 1$$

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Fraction-free Algorithm over $\mathbb{Z}[z]$ (Subresultants)

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Fraction-free Algorithm over $\mathbb{Z}[z]$ (Subresultants)

$$\begin{aligned} r_1(z) &= 15\,z^4 - 3\,z^2 + 9 \\ r_2(z) &= 65\,z^2 + 125\,z - 245 \\ r_3(z) &= 9326\,z - 12300 \\ r_4(z) &= 260708 \end{aligned}$$

# What I hope to describe today

Given $\mathbf{A}(z) = [a_1(z), \ldots, a_m(z)]$ a vector of power series from $\mathbb{K}[[z]]$, $\vec{n} = (n_1, \ldots, n_m)$ a degree bound, $\mathbb{K}$ integral domain.

I want to describe the following process:

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- Each $\mathbf{M}_{(\sigma_k, \vec{v}_k)}(z)$ has a special degree structure
- Structured linear algebra our guide

# Furthermore

- $\mathbf{M}_{(\sigma_k, \vec{v}_k)}(z)$ to $\mathbf{M}_{(\sigma_{k+1}, \vec{v}_{k+1})}(z)$ uses fraction-free arithmetic

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- Elements of $\mathbf{M}_{(\sigma_k,\vec{v}_k)}(z)$ are all determinant polynomials
- Coefficients of $M_{(\sigma_k,\vec{v}_k)}(z)$ grow linearly
- FF algorithm essentially same as Sigma Basis algorithm with an additional normalization step at each iteration

## Theorem (Properties of the algorithm FFFG)

*For each* $k$ *let* $\sigma_k$ *be the order at step* $k$*, a degree bound* $\vec{n}_k$ *and* $\vec{\nu}_{\sigma_k}$ *the closest normal point to* $\vec{n}_k$*. Then*

**(a)** *For all* $k, \sigma_k$*, the vector space of solutions of the linear system of type* $(\sigma_k, \vec{n}_k)$ *is spanned by the vectors associated with*

$$\mathbf{M}^{(j)}_{(\sigma,\nu_k)}(z),\ z\cdot\mathbf{M}^{(j)}_{(\sigma,\nu_k)}(z), \cdots,\ z^{\vec{n}^{(j)}_k - \vec{\nu}^{(j)}_\sigma - 1}\mathbf{M}^{(j)}_{(\sigma,\nu_k)}(z) \quad j = 1, ..., m.$$

**(b)** *Any* $\mathbf{P}(z)$ *of type* $(\sigma, \vec{n}_k)$ *satisfies*

$$\mathbf{P}(z) = \alpha_1(z)\mathbf{M}^{(1)}_{(\sigma,\nu_k)}(z) + \cdots + \alpha_m(z)\mathbf{M}^{(m)}_{(\sigma,\nu_k)}(z)$$

*with* $\deg \alpha_j(z) < \vec{n}^{(j)}_k - \vec{\nu}^{(j)}_\sigma$.

**(c)** *For all* $k, \sigma \geq 0$ *we have* $\operatorname{rank}\ \underline{K}(\vec{n}_k, \sigma) = |\min(\vec{\nu}_\sigma, \vec{n}_k)|$.

# Recursive Computation

Closest normal points

Independent linear functionals

# Matrix Reduction view of Algorithm

Reduce (when everything goes well) according to submatrices :

$$\left[\begin{array}{ccc|ccc|ccc}
a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\
a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6
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a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
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Points:

(1,0,0)

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a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
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\end{array}\right]$$

Points:

(1,0,0) , (1,1,0)

# Matrix Reduction view of Algorithm

Reduce (when everything goes well) according to submatrices :

$$\left[\begin{array}{ccc|ccc|ccc}
{\color{red}a_0} & 0 & 0 & {\color{red}b_0} & 0 & 0 & {\color{red}c_0} & 0 & 0 \\
{\color{red}a_1} & a_0 & 0 & {\color{red}b_1} & b_0 & 0 & {\color{red}c_1} & c_0 & 0 \\
{\color{red}a_2} & a_1 & a_0 & {\color{red}b_2} & b_1 & b_0 & {\color{red}c_2} & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
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\end{array}\right]$$

Points:

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a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
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Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1)

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a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
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Points:

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a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6
\end{array}\right]
$$

Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1) , (2,2,1) , (2,2,2) , (3,2,2) , (3,3,2)

# Matrix Reduction view of Algorithm

Reduce (when everything goes well) according to submatrices :

$$\left[\begin{array}{ccc|ccc|ccc}
a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\
a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6
\end{array}\right]$$

Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1) , (2,2,1) , (2,2,2) , (3,2,2) , (3,3,2) , (3,3,3).

# Construction: $(n_1, \ldots, n_m) \to (n_1, \ldots, n_m) + \vec{e}_j$

Our algorithm assumes that we have a specific path of computation. We start by assuming that each coefficient matrix for our linear system of equations encountered is nonsingular.

This leads naturally to the solving of $m$ linear systems at each step which in turn results in an $m \times m$ matrix polynomial $\mathbf{M}$ having certain important properties.

For instructional purposes it is better to illustrate the process with specific values of $m$ and degree bounds $n_i$.

# Construction: $(3, 2, 2) \rightarrow (3, 3, 2)$

Reduce according to submatrices :

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Points:

(3,2,2)

# Construction: $(3,2,2) \rightarrow (3,3,2)$

Reduce according to submatrices :

$$\left[\begin{array}{ccc|ccc|ccc}
a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\
a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6
\end{array}\right]$$

Points:

(3,2,2) , (3,3,2)

# Construction: $(3, 2, 2) \to (3, 3, 2)$

- Assume we have an 'order basis' at location $(3, 2, 2)$
- Then show how to compute 'order basis' at location $(3, 3, 2)$.
- We assume that everything is 'good' at both locations
- We make sure arithmetic is fraction-free and efficient
- Let Linear Algebra tell us what to do at each step
- 'order basis' in this context called Mahler System

# $(3,2,2) \rightarrow (3,3,2)$

Solve:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right]$$

# $(3,2,2) \to (3,3,2)$

Solve:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right]$$

with $d = \det S_{(3,2,2)}$. This gives

$$M = \left[\begin{array}{lll} p(z) & & \\ q(z) & & \\ r(z) & & \end{array}\right] \text{ with degrees } \left[\begin{array}{lll} 3 & & \\ 1 & & \\ 1 & & \end{array}\right]$$

## $(3,2,2) \to (3,3,2)$

Solve:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} s_0 \\ s_1 \\ s_2 \\ t_0 \\ t_1 \\ u_0 \\ u_1 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array}\right]$$

with $d = \det S_{(3,2,2)}$. This gives

$$M = \left[\begin{array}{l} p(z) \\ q(z) \\ r(z) \end{array}\right] \text{ with degrees } \left[\begin{array}{l} 3 \\ 1 \\ 1 \end{array}\right]$$

# $(3,2,2) \to (3,3,2)$

Solve:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} s_0 \\ s_1 \\ s_2 \\ t_0 \\ t_1 \\ u_0 \\ u_1 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array}\right]$$

with $d = \det S_{(3,2,2)}$. This gives

$$M = \left[\begin{array}{ccc} p(z) & s(z) & \\ q(z) & t(z) & \\ r(z) & u(z) & \end{array}\right] \text{ with degrees } \left[\begin{array}{ccc} 3 & 2 & \\ 1 & 2 & \\ 1 & 1 & \end{array}\right]$$

# $(3,2,2) \to (3,3,2)$

Solve:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} g_0 \\ g_1 \\ g_2 \\ h_0 \\ h_1 \\ v_0 \\ v_1 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\right]$$

with $d = \det S_{(3,2,2)}$. This gives

$$M = \left[\begin{array}{ccc} p(z) & s(z) & \\ q(z) & t(z) & \\ r(z) & u(z) & \end{array}\right] \text{ with degrees } \left[\begin{array}{ccc} 3 & 2 & \\ 1 & 2 & \\ 1 & 1 & \end{array}\right]$$

# $(3,2,2) \to (3,3,2)$

Solve:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} g_0 \\ g_1 \\ g_2 \\ h_0 \\ h_1 \\ v_0 \\ v_1 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\right]$$

with $d = \det S_{(3,2,2)}$. This gives

$$M = \left[\begin{array}{ccc} p(z) & s(z) & g(z) \\ q(z) & t(z) & h(z) \\ r(z) & u(z) & v(z) \end{array}\right] \text{ with degrees } \left[\begin{array}{ccc} 3 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

# $(3,2,2) \rightarrow (3,3,2)$

Note: Leading coefficient matrix of $M$ is diagonal with coeffcients $d$.

Note each column has order 7

$$M = \begin{bmatrix} p(z) & s(z) & g(z) \\ q(z) & t(z) & h(z) \\ r(z) & u(z) & v(z) \end{bmatrix} \text{ with degrees } \begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

# $(3,2,2) \rightarrow (3,3,2)$

- Goal is the to go from $M_{(3,2,2)}$ to $M_{(3,3,2)}$
- Degree structures are

$$\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

- Need to worry about degree structures
- Need to increase order of each column.
- Need to normalize correctly
- Need to use fraction-free arithmetic

# $(3,2,2) \to (3,3,2)$

- Goal is the to go from $M_{(3,2,2)}$ to $M_{(3,3,2)}$
- Degree structures are

$$\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \to \quad \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

- Need to worry about degree structures
- Need to increase order of each column.
- Need to normalize correctly
- Need to use fraction-free arithmetic

First two needs are easy. Last two are not easy.

# $(3,2,2) \rightarrow (3,3,2)$

- Goal is the to go from $M_{(3,2,2)}$ to $M_{(3,3,2)}$
- Degree structures are

$$\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

- Need to worry about degree structures
- Need to increase order of each column.
- Need to normalize correctly
- Need to use fraction-free arithmetic

First two needs are easy. Last two are not easy.

Require initial 'residual' and leading coeffs of other terms.

# Determinant Representations

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right]$$

# Determinant Representations

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right]$$

$$\text{Set } p(z) = \det \left[\begin{array}{cccc|cc|cc} a_0 & 0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & 0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & c_6 & c_5 \\ 1 & z & z^2 & z^3 & 0 & 0 & 0 & 0 \end{array}\right]$$

# Determinant Representations

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right]$$

$$\text{Set } q(z) = \det \left[\begin{array}{cccc|cc|cc} a_0 & 0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & 0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & c_6 & c_5 \\ 0 & 0 & 0 & 0 & 1 & z & 0 & 0 \end{array}\right]$$

# Determinant Representations

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right]$$

Set $r(z) = \det \left[\begin{array}{cccc|cc|cc} a_0 & 0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & 0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & c_6 & c_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \end{array}\right]$

# Determinant Representations

$$\det \left[\begin{array}{cccc|cc|cc} a_0 & 0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & 0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & c_6 & c_5 \\ a(z) & a(z)z & a(z)z^2 & a(z)z^3 & b(z) & b(z)z & c(z) & c(z)z \end{array}\right] = O(z^7).$$

Notice that $a(z)p(z) + b(z)q(z) + c(z)r(z) = \det S_{(4,2,2)} z^7 + O(z^8)$

# Determinant Representations

$$\det \left[\begin{array}{ccc|ccc|cc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 \\ a(z) & a(z)z & a(z)z^2 & b(z) & b(z)z & b(z)z^2 & c(z) & c(z)z \end{array}\right] = O(z^7).$$

Notice that $a(z)s(z) + b(z)t(z) + c(z)u(z) = \det S_{(3,3,2)} z^7 + O(z^8)$

# Determinant Representations

$$\det \left[\begin{array}{ccc|cc|ccc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 & c_4 \\ a(z) & a(z)z & a(z)z^2 & b(z) & b(z)z & c(z) & c(z)z & c(z)z^2 \end{array}\right] = O(z^7).$$

Notice that $a(z)g(z) + b(z)h(z) + c(z)v(z) = \det S_{(3,2,3)} z^7 + O(z^8)$

# Uses of Determinant Representations

- Similar determinant representations for all the entries in $M$
- First column first term of residual is $\det S_{(4,2,2)}$
- Second column first term of residual is $\det S_{(3,3,2)}$
- Third column first term of residual is $\det S_{(3,2,3)}$

# Uses of Determinant Representations

- Similar determinant representations for all the entries in $M$
- First column first term of residual is $\det S_{(4,2,2)}$
- Second column first term of residual is $\det S_{(3,3,2)}$
- Third column first term of residual is $\det S_{(3,2,3)}$
- Leading coefficents of $M$ are

$$\begin{bmatrix} \det S_{(3,2,2)} & \det S_{(2,3,2)} & \det S_{(2,2,3)} \\ \det S_{(4,1,2)} & \det S_{(3,2,2)} & \det S_{(3,1,3)} \\ \det S_{(4,2,1)} & \det S_{(3,3,1)} & \det S_{(3,2,2)} \end{bmatrix}$$

(might have to check signs here)

# Check:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \left[\begin{array}{c} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{array}\right] = -d \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{array}\right]$$

Set $p(z) = \det \left[\begin{array}{cccc|cc|cc} a_0 & 0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & 0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & c_6 & c_5 \\ 1 & z & z^2 & z^3 & 0 & 0 & 0 & 0 \end{array}\right]$

# Check:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{bmatrix} = -d \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Set $q(z) = \det \left[\begin{array}{cccc|cc|cc} a_0 & 0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & 0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & c_6 & c_5 \\ 0 & 0 & 0 & 0 & 1 & z & 0 & 0 \end{array}\right]$

# Check:

$$\left[\begin{array}{ccc|cc|cc} a_0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & b_6 & b_5 & c_6 & c_5 \end{array}\right] \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ q_0 \\ q_1 \\ r_0 \\ r_0 \end{bmatrix} = -d \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Set $r(z) = \det \left[\begin{array}{cccc|cc|cc} a_0 & 0 & 0 & 0 & b_0 & 0 & c_0 & 0 \\ a_1 & a_0 & 0 & 0 & b_1 & b_0 & c_1 & c_0 \\ a_2 & a_1 & a_0 & 0 & b_2 & b_1 & c_2 & c_1 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_2 & c_3 & c_2 \\ a_4 & a_3 & a_2 & a_1 & b_4 & b_3 & c_4 & c_3 \\ a_5 & a_4 & a_3 & a_2 & b_5 & b_4 & c_5 & c_4 \\ a_6 & a_5 & a_4 & a_3 & b_6 & b_5 & c_6 & c_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \end{array}\right]$

# Remaining process

- Use cross multiplication to increase order of other columns
- Correct normalization of leading coefficent matrix of new $M$
- Increase order of pivot column and multiply this by $\det S_{(3,3,2)}$
- Implies that we have a new $M$
  - Leading coefficent matrix is diagonal
  - Leading coefficients are now $\mathrm{d}_{(3,2,2)} \cdot \mathrm{d}_{(3,3,2)}$. Need $\mathrm{d}_{(3,3,2)}$.
  - Uniqueness implies $\mathrm{d}_{(3,2,2)}$ divides every term exactly.

# Singular Jumps

Suppose at nonsingular submatrix : $\vec{n}^{(3)} := (1, 1, 1)$

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Determinants: (leading term of residual)

$d_{(1,1,1)} \neq 0$

# Singular Jumps

Suppose at nonsingular submatrix : $\vec{n}^{(3)} := (1,1,1)$

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Determinants: (leading term of residual)

$$d_{(2,1,1)} = 0$$

# Singular Jumps

Suppose at nonsingular submatrix : $\vec{n}^{(3)} := (1, 1, 1)$

$$\left[\begin{array}{ccc|ccc|ccc}
a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\
a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6
\end{array}\right]$$

Determinants: (leading term of residual)

$$d_{(1,2,1)} = 0$$

# Singular Jumps

Suppose at nonsingular submatrix : $\vec{n}^{(3)} := (1, 1, 1)$

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Determinants: (leading term of residual)

$$d_{(1,1,2)} = 0$$

Implies $c_3$-rd coefficient linear dependant on previous $c_0,\ c_1,\ c_2$

# Singular Jumps

$\vec{n}^{(3)} := (1, 1, 1)$

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Determinants: (leading term of residual)

Implies $c_3$-rd coefficient linear dependant on previous $c_0,\ c_1,\ c_2$

# Singular Jumps

$\vec{n}^{(3)} := (1,1,1),\ \sigma^{(3)} = \sigma^{(3)} + 1$

$$
\left[\begin{array}{ccc|ccc|ccc}
a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\
a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \\
a_9 & a_8 & a_7 & b_9 & b_8 & b_7 & c_9 & c_8 & c_7
\end{array}\right]
$$

Determinants: (leading term of residual)

$d_{(1,1,1)} \neq 0$

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc} \color{red}{a_0} & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Points:

(1,0,0)

Closest Normal Points:

(1,0,0)

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc} \color{red}{a_0} & 0 & 0 & \color{red}{b_0} & 0 & 0 & c_0 & 0 & 0 \\ \color{red}{a_1} & a_0 & 0 & \color{red}{b_1} & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Points:

(1,0,0) , (1,1,0)

Closest Normal Points:

(1,0,0) , (1,1,0)

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$
\left[\begin{array}{ccc|ccc|ccc}
{\color{red}a_0} & 0 & 0 & {\color{red}b_0} & 0 & 0 & {\color{red}c_0} & 0 & 0 \\
{\color{red}a_1} & a_0 & 0 & {\color{red}b_1} & b_0 & 0 & {\color{red}c_1} & c_0 & 0 \\
{\color{red}a_2} & a_1 & a_0 & {\color{red}b_2} & b_1 & b_0 & {\color{red}c_2} & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6
\end{array}\right]
$$

Points:

(1,0,0) , (1,1,0) , (1,1,1)

Closest Normal Points:

(1,0,0) , (1,1,0) , (1,1,1)

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1)

Closest Normal Points:

(1,0,0) , (1,1,0) , (1,1,1)

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1) , (2,2,1)

Closest Normal Points:

(1,0,0) , (1,1,0) , (1,1,1) , (1,2,1)

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1) , (2,2,1) , (2,2,2)

Closest Normal Points:

(1,0,0) , (1,1,0) , (1,1,1) , (1,2,1) , (2,2,1)

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc}
a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\
a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6
\end{array}\right]$$

Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1) , (2,2,1) , (2,2,2) , (3,2,2)

Closest Normal Points:

(1,0,0) , (1,1,0) , (1,1,1) , (1,2,1) , (2,2,1) , (2,2,2)

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc} a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\ a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\ a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\ a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\ a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\ a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6 \end{array}\right]$$

Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1) , (2,2,1) , (2,2,2) , (3,2,2) , (3,3,2)

Closest Normal Points:

(1,0,0) , (1,1,0) , (1,1,1) , (1,2,1) , (2,2,1) , (2,2,2)

# Closest Normal Point

Intuition : Pick the appropriate columns/pivots.

$$\left[\begin{array}{ccc|ccc|ccc}
a_0 & 0 & 0 & b_0 & 0 & 0 & c_0 & 0 & 0 \\
a_1 & a_0 & 0 & b_1 & b_0 & 0 & c_1 & c_0 & 0 \\
a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\
a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\
a_4 & a_3 & a_2 & b_4 & b_3 & b_2 & c_4 & c_3 & c_2 \\
a_5 & a_4 & a_3 & b_5 & b_4 & b_3 & c_5 & c_4 & c_3 \\
a_6 & a_5 & a_4 & b_6 & b_5 & b_4 & c_6 & c_5 & c_4 \\
a_7 & a_6 & a_5 & b_7 & b_6 & b_5 & c_7 & c_6 & c_5 \\
a_8 & a_7 & a_6 & b_8 & b_7 & b_6 & c_8 & c_7 & c_6
\end{array}\right]$$

Points:

(1,0,0) , (1,1,0) , (1,1,1) , (2,1,1) , (2,2,1) , (2,2,2) , (3,2,2) , (3,3,2) , (3,3,3).

Closest Normal Points:

(1,0,0) , (1,1,0) , (1,1,1) , (1,2,1) , (2,2,1) , (2,2,2)

# Closed Normal Point

Pick pivot $\vec{e}_{v}$ as

$$v = \left\{ u \text{ s.t. } d(n^{(\vec{k})} + \vec{e}_u) \neq 0 |\ n_u^{(\vec{k})} - v_u^{(\vec{k})} = \max\{n_\ell^{(\vec{k})} - v_\ell^{(\vec{k})}\} \right\}$$

# Example

Closest normal points follow dots below.

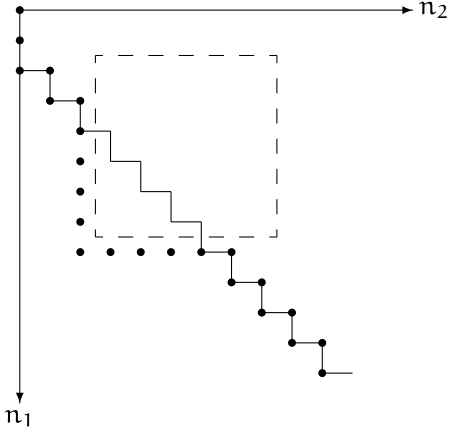


Figure: An example of singular Padé approximation.

# Fraction-Free Computation

Mahler Systems

Sylvester's Identity

# Big Picture

Given Mahler Systems $\mathbf{M}(\vec{v}^{\,(k)}, z)$, order $k$:

- $\vec{v}^{\,(0)}, \ \ldots \ , \ \vec{v}^{\,(k)}, \ldots,$ sequence of "closest normal points"
- Normal point $\quad \equiv \quad \underline{K}(\vec{v}^{\,(k)}, |\vec{v}^{\,(k)}|)$ nonsingular matrix
- compute "next" Mahler System $\mathbf{M}(\vec{v}^{\,(k+1)}, z)$, order $k+1$.

# Big Picture

Given Mahler Systems $\mathbf{M}(\vec{v}^{\,(k)}, z)$, order $k$:

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- compute "next" Mahler System $\mathbf{M}(\vec{v}^{\,(k+1)}, z)$, order $k+1$.

Issue : compute without fractions:

- Cross multiplier elimination in $\mathbf{M}(\vec{v}^{\,(k)}, z)$ gives order $k+1$
- Correction of degrees gives multiple of $\mathbf{M}(\vec{v}^{\,(k+1)}, z)$
- Sylvesters identity gives multiplier for free.

# The Procedure (Good Case)

Starting at $(n_1, \dots, n_i, \dots, n_m)$. Degree structure.

$$\text{degrees}: \quad \begin{bmatrix} n_1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ n_i - 1 & \cdots & n_i & \cdots & n_i - 1 & \cdots & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & n_j - 1 & \cdots & n_j & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ n_m - 1 & \cdots & n_m - 1 & \cdots & n_m - 1 & \cdots & n_m \end{bmatrix}$$

$$\text{residuals}: \quad [\; 0 \quad \cdots \quad \neq 0 \quad \cdots \quad \neq 0 \quad \cdots \quad \neq 0 \;]$$

$$\text{orders}: \quad \sigma_k + 1 \quad \cdots \quad \sigma_k \quad \cdots \quad \sigma_k \quad \cdots \quad \sigma_k$$

# The Procedure (Good Case)

Starting at $(n_1, \ldots, n_i, \ldots, n_m)$. Eliminate using column $i$

$$\text{degrees}: \quad \begin{bmatrix} n_1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ n_i & \cdots & n_i & \cdots & n_i & \cdots & n_i \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & n_j - 1 & \cdots & n_j & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ n_m - 1 & \cdots & n_m - 1 & \cdots & n_m - 1 & \cdots & n_m \end{bmatrix}$$

$$\text{residuals}: \quad \begin{bmatrix} 0 & \cdots & \neq 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

$$\text{orders}: \quad \sigma_k + 1 \quad \cdots \quad \sigma_k \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1$$

# The Procedure (Good Case)

Starting at $(n_1, \ldots, n_i, \ldots, n_m)$. Increase order of column $i$

$$\text{degrees}: \begin{bmatrix} n_1 & \cdots & n_1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ n_i & \cdots & n_i + 1 & \cdots & n_i & \cdots & n_i \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & n_j & \cdots & n_j & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ n_m - 1 & \cdots & n_m & \cdots & n_m - 1 & \cdots & n_m \end{bmatrix}$$

$$\text{residuals}: \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

$$\text{orders}: \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1$$

# The Procedure (Good Case)

Starting at $(n_1, \ldots, n_i, \ldots, n_m)$. Correct degrees in column $i$

$$\text{degrees} : \begin{bmatrix} n_1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ n_i & \cdots & n_i + 1 & \cdots & n_i & \cdots & n_i \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & n_j - 1 & \cdots & n_j & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ n_m - 1 & \cdots & n_m - 1 & \cdots & n_m - 1 & \cdots & n_m \end{bmatrix}$$

$$\text{residuals} : \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

$$\text{orders} : \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1$$

# The Procedure (Good Case)

Now at point $(n_1, \ldots, n_i + 1, \ldots, n_m)$. Degrees, orders are correct.

$$\text{degrees}: \begin{bmatrix} n_1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ n_i & \cdots & n_i + 1 & \cdots & n_i & \cdots & n_i \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & n_j - 1 & \cdots & n_j & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ n_m - 1 & \cdots & n_m - 1 & \cdots & n_m - 1 & \cdots & n_m \end{bmatrix}$$

$$\text{residuals}: \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

$$\text{orders}: \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1$$

# The Procedure (Good Case)

Now at point $(n_1, \ldots, n_i + 1, \ldots, n_m)$. Degrees, orders are correct.

$$\text{degrees}: \quad \begin{bmatrix} n_1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 & \cdots & n_1 - 1 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ n_i & \cdots & n_i + 1 & \cdots & n_i & \cdots & n_i \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & n_j - 1 & \cdots & n_j & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ n_m - 1 & \cdots & n_m - 1 & \cdots & n_m - 1 & \cdots & n_m \end{bmatrix}$$

$$\text{residuals}: \quad \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

$$\text{orders}: \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1$$

Question : Is normalization correct?

# The Procedure (Good Case)

Starting point : $\mathbf{M}_{\vec{n}}(z)$. Look at leading coefficients.

lcoeffs :

$$\begin{bmatrix} d_{(\vec{n})} & \cdots & d_{(\vec{n}-\vec{e}_1)} & \cdots & d_{(\vec{n}-\vec{e}_1)} & \cdots & d_{(\vec{n}-\vec{e}_1)} \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ & \cdots & d_{(\vec{n})} & \cdots & d_{(\vec{n}-\vec{e}_i)} & \cdots & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & d_{(\vec{n}-\vec{e}_j)} & \cdots & d_{(\vec{n})} & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ d_{(\vec{n}-\vec{e}_m)} & \cdots & d_{(\vec{n}-\vec{e}_m)} & \cdots & d_{(\vec{n}-\vec{e}_m)} & \cdots & d_{(\vec{n})} \end{bmatrix}$$

residuals :

$$\begin{bmatrix} 0 & \cdots & d_{(\vec{n}+\vec{e}_i)} & \cdots & d_{(\vec{n}+\vec{e}_j)} & \cdots & d_{(\vec{n}+\vec{e}_m)} \end{bmatrix}$$

orders :

$$\sigma_k + 1 \quad \cdots \quad \sigma_k \quad \cdots \quad \sigma_k \quad \cdots \quad \sigma_k$$

# The Procedure (Good Case)

Starting point : $\mathbf{M}(\vec{n}, z)$. Use column $i$ to increase orders

$$\text{lcoeffs}: \quad \begin{bmatrix} d^+{}_{(\vec{n})} & \cdots & d_{(\vec{n}-\vec{e}_1)} & \cdots & d^*{}_{(\vec{n}-\vec{e}_1)} & \cdots & d^*{}_{(\vec{n}-\vec{e}_1)} \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ & \cdots & d_{(\vec{n})} & \cdots & d^*{}_{(\vec{n}-\vec{e}_i)} & \cdots & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & d_{(\vec{n}-\vec{e}_j)} & \cdots & d^+{}_{(\vec{n})} & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ d^*{}_{(\vec{n}-\vec{e}_m)} & \cdots & d_{(\vec{n}-\vec{e}_m)} & \cdots & d^*{}_{(\vec{n}-\vec{e}_m)} & \cdots & d^+{}_{(\vec{n})} \end{bmatrix}$$

$$\text{residuals}: \quad \begin{bmatrix} 0 & \cdots & d_{(\vec{n}+\vec{e}_i)} & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

$$\text{orders}: \quad \sigma_k + 1 \quad \cdots \quad \sigma_k \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1$$

# The Procedure (Good Case)

Starting point : $\mathbf{M}_{(\vec{n})}(z)$. Increase order of column $i$. Correct column $i$

lcoeffs :

$$\begin{bmatrix} d^{+}{}_{(\vec{n})} & \cdots & d^{*}{}_{(\vec{n}-\vec{e}_1)} & \cdots & d^{*}{}_{(\vec{n}-\vec{e}_1)} & \cdots & d^{*}{}_{(\vec{n}-\vec{e}_1)} \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ & \cdots & d^{*}{}_{(\vec{n})} & \cdots & d^{*}{}_{(\vec{n}-\vec{e}_i)} & \cdots & \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ & \cdots & d^{*}{}_{(\vec{n}-\vec{e}_j)} & \cdots & d^{+}{}_{(\vec{n})} & \cdots & \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ d^{*}{}_{(\vec{n}-\vec{e}_m)} & \cdots & d^{*}{}_{(\vec{n}-\vec{e}_m)} & \cdots & d^{*}{}_{(\vec{n}-\vec{e}_m)} & \cdots & d^{+}{}_{(\vec{n})} \end{bmatrix}$$

residuals : $[\ 0 \quad \cdots \quad 0 \quad \cdots \quad 0 \quad \cdots \quad 0\ ]$

orders : $\sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1 \quad \cdots \quad \sigma_k + 1$

# Is normalization correct ?

Result of procedure is : $\mathrm{d}_{(\vec{n})} \cdot \mathbf{M}_{(\vec{n}+\vec{e}_i)}(z)$.

Why?

Elimination gives :

$$\begin{aligned} {\mathrm{d}^*}_{(\vec{n}-\vec{e}_k)} &= \mathrm{d}_{(\vec{n}-\vec{e}_k)} \cdot \mathrm{d}_{(\vec{n}+\vec{e}_j)} - \mathrm{d}_{(\vec{n}-\vec{e}_j)} \cdot \mathrm{d}_{(\vec{n}+\vec{e}_i)} \\ &= \mathrm{d}_{(\vec{n})} \cdot \mathrm{d}_{(\vec{n}+\vec{e}_i)} \end{aligned}$$

# Is normalization correct ?

Result of procedure is : $\color{red}{d_{(\vec{n})}} \cdot \mathbf{M}_{(\vec{n}+\vec{e}_i)}(z)$.

Why?

Elimination gives :

$$\begin{aligned} d^*_{(\vec{n}-\vec{e}_k)} &= d_{(\vec{n}-\vec{e}_k)} \cdot d_{(\vec{n}+\vec{e}_j)} - d_{(\vec{n}-\vec{e}_j)} \cdot d_{(\vec{n}+\vec{e}_i)} \\ &= d_{(\vec{n})} \cdot d_{(\vec{n}+\vec{e}_i)} \end{aligned}$$

Question : Why is second equation correct?

# Is normalization correct ?

Result of procedure is : $\mathrm{d}_{(\vec{n})} \cdot \mathbf{M}_{(\vec{n}+\vec{e}_i)}(z)$.

Why?

Elimination gives :

$$\begin{aligned} \mathrm{d}^*{}_{(\vec{n}-\vec{e}_k)} &= \mathrm{d}_{(\vec{n}-\vec{e}_k)} \cdot \mathrm{d}_{(\vec{n}+\vec{e}_j)} - \mathrm{d}_{(\vec{n}-\vec{e}_j)} \cdot \mathrm{d}_{(\vec{n}+\vec{e}_i)} \\ &= \mathrm{d}_{(\vec{n})} \cdot \mathrm{d}_{(\vec{n}+\vec{e}_i)} \end{aligned}$$

Question : Why is second equation correct?

Answer : Sylvester's identity!

# Sylvester's Identity

Let $A_{r,c}$ be $A$ with row $r$ and column $c$ removed.

⋆ Sylvester's identity:

$$A = \begin{bmatrix} a_{1\,1} & \cdots & a_{1\,c_1} & \cdots & a_{1\,c_2} & \cdots & a_{1\,m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_1\,1} & \cdots & a_{r_1\,c_1} & \cdots & a_{r_1\,c_2} & \cdots & a_{r_1\,m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_2\,1} & \cdots & a_{r_2\,c_1} & \cdots & a_{r_2\,c_2} & \cdots & a_{r_2\,m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m\,1} & \cdots & a_{m\,c_1} & \cdots & a_{m\,c_2} & \cdots & a_{m\,m} \end{bmatrix}$$

$$\det(A) =$$

# Sylvester's Identity

Let $A_{r,c}$ be $A$ with row $r$ and column $c$ removed.

⋆ Sylvester's identity:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1c_1} & \cdots & a_{1c_2} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_1 1} & \cdots & a_{r_1 c_1} & \cdots & a_{r_1 c_2} & \cdots & a_{r_1 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_2 1} & \cdots & a_{r_2 c_1} & \cdots & a_{r_2 c_2} & \cdots & a_{r_2 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mc_1} & \cdots & a_{mc_2} & \cdots & a_{mm} \end{bmatrix}$$

$$\det(A) = \det(A_{r_1,c_1})$$

# Sylvester's Identity

Let $A_{r,c}$ be $A$ with row $r$ and column $c$ removed.

$\star$ Sylvester's identity:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1c_1} & \cdots & a_{1c_2} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_1 1} & \cdots & a_{r_1 c_1} & \cdots & a_{r_1 c_2} & \cdots & a_{r_1 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_2 1} & \cdots & a_{r_2 c_1} & \cdots & a_{r_2 c_2} & \cdots & a_{r_2 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mc_1} & \cdots & a_{mc_2} & \cdots & a_{mm} \end{bmatrix}$$

$$\det(A) = \det(A_{r_1,c_1}) \det(A_{r_2,c_2})$$

# Sylvester's Identity

Let $A_{r,c}$ be $A$ with row $r$ and column $c$ removed.

⋆ Sylvester's identity:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1c_1} & \cdots & a_{1c_2} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_1 1} & \cdots & a_{r_1 c_1} & \cdots & a_{r_1 c_2} & \cdots & a_{r_1 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_2 1} & \cdots & a_{r_2 c_1} & \cdots & a_{r_2 c_2} & \cdots & a_{r_2 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mc_1} & \cdots & a_{mc_2} & \cdots & a_{mm} \end{bmatrix}$$

$$\det(A) = \det(A_{r_1,c_1}) \det(A_{r_2,c_2}) - \det(A_{r_1,c_2})$$

# Sylvester's Identity

Let $A_{r,c}$ be $A$ with row $r$ and column $c$ removed.

⋆ Sylvester's identity:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1c_1} & \cdots & a_{1c_2} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_1 1} & \cdots & a_{r_1 c_1} & \cdots & a_{r_1 c_2} & \cdots & a_{r_1 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_2 1} & \cdots & a_{r_2 c_1} & \cdots & a_{r_2 c_2} & \cdots & a_{r_2 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mc_1} & \cdots & a_{mc_2} & \cdots & a_{mm} \end{bmatrix}$$

$$\det(A) = \det(A_{r_1,c_1}) \det(A_{r_2,c_2}) - \det(A_{r_1,c_2}) \det(A_{r_2,c_1})$$

# Sylvester's Identity

Let $A_{r,c}$ be $A$ with row $r$ and column $c$ removed.

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$$A = \begin{bmatrix} a_{11} & \cdots & a_{1c_1} & \cdots & a_{1c_2} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_1 1} & \cdots & a_{r_1 c_1} & \cdots & a_{r_1 c_2} & \cdots & a_{r_1 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_2 1} & \cdots & a_{r_2 c_1} & \cdots & a_{r_2 c_2} & \cdots & a_{r_2 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mc_1} & \cdots & a_{mc_2} & \cdots & a_{mm} \end{bmatrix}$$

$$\det(A_{r_1,r_2,c_1,c_2}) \cdot \det(A) = \det(A_{r_1,c_1})\det(A_{r_2,c_2}) - \det(A_{r_1,c_2})\det(A_{r_2,c_1})$$

# Sylvester's Identity

Let $A_{r,c}$ be $A$ with row $r$ and column $c$ removed.

$\star$ Sylvester's identity:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1c_1} & \cdots & a_{1c_2} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_1 1} & \cdots & a_{r_1 c_1} & \cdots & a_{r_1 c_2} & \cdots & a_{r_1 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{r_2 1} & \cdots & a_{r_2 c_1} & \cdots & a_{r_2 c_2} & \cdots & a_{r_2 m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mc_1} & \cdots & a_{mc_2} & \cdots & a_{mm} \end{bmatrix}$$

$$\det(A_{r_1,r_2,c_1,c_2}) \cdot \det(A) = \det(A_{r_1,c_1}) \det(A_{r_2,c_2}) - \det(A_{r_1,c_2}) \det(A_{r_2,c_1})$$

$\star$ In our case we need :

$$d_{(\vec{n})} \cdot d_{(\vec{n}+2\vec{e}_i)} = d_{(\vec{n}+2\vec{e}_i)} \cdot d_{(\vec{n}+\vec{e}_j)} - d_{(\vec{n}+\vec{e}_i)} \cdot d_{(\vec{n}+2\vec{e}_j)}$$

# Mahler Systems

## Definition

Given $\sigma$ and $\vec{n}$ a Mahler System is

$$\mathbf{M}_\sigma = [\mathbf{M}_\sigma^{(\cdot,\ 1)}, \ldots, \mathbf{M}_\sigma^{(\cdot,\ m)}]$$

where the $i$th column is $\pm\, p(\vec{n} + \vec{e}_i, z)$.

**Note :** Mahler systems have degrees bounded by

$$\deg \mathbf{M}_\sigma = \begin{bmatrix} n_1 & n_1 - 1 & \cdots & n_1 - 1 \\ n_2 - 1 & n_2 & & n_2 - 1 \\ \vdots & & \ddots & \\ n_m - 1 & \cdots & n_m - 1 & n_m \end{bmatrix}$$

NOTE: Leading coefficents of diagonals : $d_{(\vec{n})}$.